Title: General Relativity for Cosmology Lecture - 110223

Speakers: Achim Kempf

Collection: General Relativity for Cosmology

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GR for Cosmology, Achim Kempf — Lecture 15

Recall: If we choose the bases $\{\frac{\partial}{\partial x^\mu}\}$, $\{dx^\nu\}$, then:

E.g. $L_{EM} = -\frac{1}{16\pi} F_{\mu\nu}F^{\mu\nu}$

$$S'[g_{\mu\nu}, \Psi] = \int \left(\frac{1}{16\pi G} R(g_{\mu\nu}(x)) + L_{matter}(g_{\mu\nu}(x), \Psi^{(i)}(x), \Psi^{(i)}_{;\mu}(x)) \right) \sqrt{g}\, d^4x$$

$\frac{\delta S'}{\delta \Psi^{(i)}} = 0 \Rightarrow$ Eqns. of motion of matter (Maxwell, Klein Gordon eqns. etc)

$\frac{\delta S'}{\delta g_{\mu\nu}} = 0 \Rightarrow$ Einstein equations:

$$\boxed{R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = 8\pi G T^{\mu\nu}}$$

Important: The only reason why the components of a tensor can change when we change coordinates is that we can change basis in the (co-) tangent spaces, namely from one canonical basis to another canonical basis, when we change coord. system.

Recall: a fixed vector has different coefficients in different bases:

$\xi^\mu \frac{\partial}{\partial x^\mu} = \xi^\mu \frac{\partial \bar{x}^\nu}{\partial x^\mu} \frac{\partial}{\partial \bar{x}^\nu} = \bar{\xi}^\nu \frac{\partial}{\partial \bar{x}^\nu} \Rightarrow \bar{\xi}^\nu = \frac{\partial \bar{x}^\nu}{\partial x^\mu} \xi^\mu$

$$\xi = \xi^\mu \frac{\partial}{\partial x^\mu} = \bar{\xi}^\nu \frac{\partial}{\partial \bar{x}^\nu}$$

What is the Einstein equation when using a frame so that

$$g_{\mu\nu}(x) = \eta_{\mu\nu} \; ?$$

Recall:

- **Frames $\{\theta^\mu\}$, $\{e_\mu\}$:**

  Often, one uses as the bases of $T_p(M)$, and $T_p(M)'$ the canonical bases $\{dx^\mu\}$ and $\{\frac{\partial}{\partial x^\mu}\}$ respectively, which suggest themselves when one chooses coordinates, say $(x^0, \dots, x^3)$. Thus, when changing coordinate system, $x \to \bar{x}$, one also usually automatically changes basis in $T_p(M)$, $T_p(M)'$.

Conversely: Even staying with one coordinate system, we can freely change our choice of basis in the (co-) tangent spaces:

$\theta'^\mu = A^\mu{}_\nu \theta^\nu$ ($A^\mu{}_\nu$: scala functions.)

$e'_\mu = (A^{-1})_\mu{}^\nu e_\nu$

So we have e.g.:

$$\xi = \xi^\mu e_\mu = \xi^\mu A^\nu{}_\mu e'_\nu = \xi'^\nu e'_\nu$$

I.e.:

$$\xi'^\nu = A^\nu{}_\mu \xi^\mu$$

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$\frac{\delta \mathcal{L}}{\delta \psi^{(i)}} = 0 \Rightarrow$ Eqns. of motion of matter (Maxwell, Klein Gordon eqns. etc)

$\frac{\delta \mathcal{L}'}{\delta g_{\mu\nu}} = 0 \Rightarrow$ Einstein equations:

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Often, one uses as the bases of $T_p(M)$, and $T_p(M)'$ the canonical bases $\{dx^\mu\}$ and $\{\frac{\partial}{\partial x^\mu}\}$ respectively, which suggest themselves when one chooses coordinates, say $(x^0, \dots, x^3)$. Thus, when changing coordinate system, $x \to \bar{x}$, one also usually automatically changes basis in $T_p(M)$, , $T_p(M)'$.

Important: The only reason why the components of a tensor can change when we change coordinates is that we can change basis in the (co-) tangent spaces, namely from one canonical basis to another canonical basis, when we change coord. system.

(Recall: a fixed vector has different coefficients in different bases: $\xi^\mu \frac{\partial}{\partial x^\mu} = \xi^\mu \frac{\partial \bar{x}^\nu}{\partial x^\mu} \frac{\partial}{\partial \bar{x}^\nu} = \bar{\xi}^\nu \frac{\partial}{\partial \bar{x}^\nu} \Rightarrow \bar{\xi}^\nu = \frac{\partial \bar{x}^\nu}{\partial x^\mu} \xi^\mu$) $\longrightarrow$ $\xi = \xi^\mu \frac{\partial}{\partial x^\mu} = \bar{\xi}^\nu \frac{\partial}{\partial \bar{x}^\nu}$

We notice: If we choose a fixed basis, say $\{\theta^\mu\}, \{e_\mu\}$ then the coefficients of tensors no longer depend on the choice of coordinates!

E.g.: $\xi = \tilde{\xi}^\mu e_\mu$ ← the same numbers in every coordinate system.

Conversely: Even staying with one coordinate system, we can freely change our choice of basis's in the (co-) tangent spaces:

$$\theta'^\mu = A^\mu{}_\nu \theta^\nu \quad \text{(scalar functions.)}$$
$$e'_\mu = (A^{-1})_\mu{}^\nu e_\nu$$

So we have e.g.:

$$\xi = \xi^\mu e_\mu = \xi^\mu A^\nu{}_\mu e'_\nu = \xi'^\nu e'_\nu$$

I.e.: $\xi'^\nu = A^\nu{}_\mu \xi^\mu$

Examples:

- The curvature form: $\Omega'^\mu{}_\nu = A^\mu{}_a (A^{-1})_\nu{}^b \Omega^a{}_b$
- But: the connection form $\omega^\mu{}_\nu(\xi) = \xi^k \Gamma^\mu{}_{k\nu}$ obeys:

$$\omega'^\mu{}_\nu = A^\mu{}_a \omega^a{}_b (A^{-1})_\nu{}^b - (dA)^\mu{}_c (A^{-1})_\nu{}^c$$

How to specify frames?

- **Existence?** Always: At each $p \in M$ may choose e.g.

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Recall:
a field vector has different coefficients in different bases:
$\xi^\mu \frac{\partial}{\partial x^\mu} = \xi^\mu \frac{\partial \bar{x}^\nu}{\partial x^\mu} \frac{\partial}{\partial \bar{x}^\nu} = \bar{\xi}^\nu \frac{\partial}{\partial \bar{x}^\nu} \Rightarrow \bar{\xi}^\nu = \frac{\partial \bar{x}^\nu}{\partial x^\mu} \xi^\mu$

canonical basis, when we change coord. system.

$$\xi = \xi^\mu \frac{\partial}{\partial x^\mu} = \bar{\xi}^\nu \frac{\partial}{\partial \bar{x}^\nu}$$

We notice: If we choose a fixed basis, say $\{\theta^\mu\}, \{e_\mu\}$ then the coefficients of tensors no longer depend on the choice of coordinates!

E.g.: $\xi = \xi^\mu e_\mu$ ← the same numbers in every coordinate system.

So we have e.g.:

$$\xi = \xi^\mu e_\mu = \xi^\mu A^\nu{}_\mu e'_\nu = \xi'^\nu e'_\nu$$

I.e.:

$$\xi'^\nu = A^\nu{}_\mu \xi^\mu$$

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$$\omega'^\mu{}_\nu = A^\mu{}_a \omega^a{}_b (A^{-1})_\nu{}^b - (dA)^\mu{}_c (A^{-1})_\nu{}^c$$

How to specify frames?

In an arbitrary coordinate system, we may specify the bases in terms of the canonical bases:

$$\theta^i(x) = A^i{}_j(x)\, dx^j$$

(Another possibility? Take n scalar functions $f^1, \dots, f^n$ and define $\theta^i := df^i$. For generic functions these θ^i's will be linearly independent almost everywhere)

Note: the $A^i{}_j(x)$ change non-trivially when changing the coordinate system!

Our choice now: orthonormal frames, or "Tetrads":

- We say that a frame $\{\theta^\mu\}, \{e_\mu\}$ is orthonormal if in this frame, for all $p \in M$:

$$g(e_\mu, e_\nu) = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}_{\mu\nu} = \eta_{\mu\nu} \quad \text{i.e. if:} \quad g = -\theta^0 \otimes \theta^0 + \sum_{i=1}^{3} \theta^i \otimes \theta^i$$

- Existence? Always: At each $p \in M$ may choose e.g. $\theta^\mu = dx^\mu$ where dx^μ are canonical ON basis at centre of a geodesic cds.
- Uniqueness?

For a given space-time, (M, g), any ON frame yields a new ON frame by transforming the bases through

$$\theta'^\mu(x) = \Lambda(x)^\mu{}_\nu \theta^\nu(x),$$

if the linear maps $\Lambda(x)$ preserve the orthonormality:

$$\eta_{\mu\nu} \theta'^\mu \otimes \theta'^\nu = \eta_{ab} \theta^a \otimes \theta^b$$

recall: this is the defining equation for Lorentz transformations.

i.e. if: $\Lambda^\mu{}_a \Lambda^\nu{}_b \eta_{\mu\nu} = \eta_{ab}$ (*)

⇒ Frames are unique up to *local* Lorentz transformations.

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⇒ Frames are unique up to *local* Lorentz transformations.

Re-express the degrees of freedom:

- We used to specify space-times through these data: (M, g)
- Now, let us specify space-times, equivalently, through data $(M, \{\theta^\mu\})$:

Namely:

Assume the $\{\theta^i\}$ are given w. resp. to a basis $\{dx^\mu\}$. through functions $A^\mu_{\ \nu}$,

$$\theta^\mu(x) = A^\mu_{\ \nu}(x)\, dx^\nu$$

so that: $g_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} = \eta_{\mu\nu}$ in the basis $\{\theta^i\}$!

Notice: knowing the $A^\mu_{\ \nu}(x)$, we can reconstruct $g_{\mu\nu}(x)$ in basis $\{dx^\mu\}$:

We use that the abstract g is the same in every basis:

$$g = \underbrace{\eta_{\mu\nu}\, \theta^\mu \otimes \theta^\nu}_{\text{because it's abstract}} = \eta_{\mu\nu}\, A^\mu_{\ a}\, A^\nu_{\ b}\, dx^a \otimes dx^b = \overbrace{g_{ab}(x)}\, dx^\mu \otimes dx^\nu$$

$$\Rightarrow \quad \boxed{g_{ab}(x) = \eta_{\mu\nu}\, A^\mu_{\ a}(x)\, A^\nu_{\ b}(x)}$$

$$\Rightarrow \quad \{\theta^i(x)\} \text{ indeed determines } g_{\mu\nu}(x):$$

$$\Rightarrow \quad \text{The } A^\mu_{\ \nu}(x) \text{ carry all physical (here shape) info!}$$

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Namely:

Assume the $\{\theta^i\}$ are given w. resp. to a basis $\{dx^\mu\}$ through functions $A^\mu_{\ \nu}$,

$$\theta^\mu(x) = A^\mu_{\ \nu}(x)\, dx^\nu$$

so that: $g_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} = \eta_{\mu\nu}$ in the basis $\{\theta^i\}$!

$$\Rightarrow \quad \boxed{g_{ab}(x) = \eta_{\mu\nu} A^\mu_{\ a}(x) A^\nu_{\ b}(x)}$$

$\Rightarrow$ $\{\theta^i(x)\}$ indeed determines $g_{\mu\nu}(x)$:

$\Rightarrow$ The $A^\mu_{\ \nu}(x)$ carry all physical (here shape) info!

How then does $A^\mu_{\ \nu}(x)$ encode C^i_{jk}, $\omega^i_{\ j}$, $\Omega^i_{\ j}$?

Start with orthonormal frame: $\boxed{\theta^i(x) = A^i_{\ j}(x)\, dx^j}$ (*)

1.) How do the $A^i_{\ j}(x)$ determine the $C^i_{jk}(x)$?

Recall from lecture II:

$$d\theta^i(x) = -\frac{1}{2} C^i_{jk}(x)\, \theta^j(x) \wedge \theta^k(x)$$

Here: $d\theta^i(x) = A^i_{j,k}(x)\, dx^k \wedge dx^j$ because of (*)

$$= -\frac{1}{2} C^i_{ab}\, \theta^a \wedge \theta^b = -\frac{1}{2} C^i_{ab} A^a_{\ k} A^b_{\ j}\, dx^k \wedge dx^j$$

$$\Rightarrow \quad A^i_{j,k} = -\frac{1}{2} C^i_{ab} A^a_{\ k} A^b_{\ j}$$

$$\Rightarrow \quad \boxed{C^i_{ab}(x) = -2 A^i_{j,k}(x) \left(A^{-1}(x)\right)^j_{\ a} \left(A^{-1}(x)\right)^k_{\ b}}$$

2.) The $C^i_{jk}(x)$ yield the $\Gamma^i_{jk}(x)$ through:

$$\Gamma^\ell_{ki} := \frac{1}{2}\left(C^\ell_{ki} - g_{is} g^{\ell j} C^s_{kj} - g_{ks} g^{\ell j} C^s_{ij}\right) \qquad \text{(Lecture II)}$$

$$+ \frac{1}{2} g^{\ell j}\left(g_{js,ik} + g_{is,jk} - g_{ki,js}\right)$$ ← These all vanish because $g_{\mu\nu}(x) = \eta_{\mu\nu}$ now

Notice: This simplifies for orthonormal frames with $g_{\mu\nu}(x) = \eta_{\mu\nu}$!

3.) The $\Gamma^i_{kj}(x)$ yield the $\omega^i_{\ j}(x)$:

$$\omega^i_{\ j}(x) := \Gamma^i_{kj}(x)\, \theta^k(x)$$

4) Recall the 2nd structure equation:

$$\Omega^i_{\ j}(x) := d\omega^i_{\ j} + \omega^i_{\ k} \wedge \omega^k_{\ j}$$

$\Rightarrow$ We have: $A^i_{\ j} \to \theta^i \to C^i_{jk} \to \Gamma^i_{jk} \to \omega^i_{\ j} \to \Omega^i_{\ j}$

Recall important identities: (torsionless case)

Tetrad formulation of GR:

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$$+\frac{1}{2} g^{il}\left(g_{lj,k} + g_{lk,j} - g_{jk,l}\right)$$ ← These all vanish because $g_{ij,k} = 0$ now

Notice: This simplifies for orthonormal frames with $g_{\mu\nu}(x) = \eta_{\mu\nu}$!

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Recall important identities: (torsionless case)

- Structure eqn. I:
$$\Theta^i = D\theta^i = d\theta^i + \omega^i_{\ j} \wedge \theta^j = 0$$
(Ordinarily: $\theta^i = dx^i \Rightarrow d\theta = 0$ and $\omega^i_{\ j} \wedge \theta^j = 0$ is $\Gamma^i_{jk} = \Gamma^i_{kj}$)
- Structure eqn II:
$$\Omega^i_{\ j} = d\omega^i_{\ j} + \omega^i_{\ k} \wedge \omega^k_{\ j}$$
(Recall: $R^i_{\ jkl} = \Gamma^i_{jl,k} + \Gamma^i_{jk,l} + \Gamma\Gamma + \Gamma\Gamma$)
- Bianchi identity I:
$$\Omega^i_{\ j} \wedge \theta^j = 0$$
- Bianchi identity II: (From diffeomorphism invariance)
$$D\Omega^i_{\ j} = 0$$

And, in the case of ON frames:

Tetrad formulation of GR:

Consider the action, for now, without cosmological constant and without matter:

$$S_{grav} := \frac{1}{16\pi G} \int_B R \sqrt{g}\, d^4x$$ ← 0-form

Recall Hodge *: If $\nu = \frac{1}{p!} \nu_{i_1 \ldots i_p}\, \theta^{i_1} \wedge \ldots \wedge \theta^{i_p}$

then $*\nu = \frac{1}{p!} \sqrt{g}\, \varepsilon_{i_1 \ldots i_n}\, \nu^{i_{n-p+1} \ldots i_n}\, \theta^{i_1} \wedge \ldots \wedge \theta^{i_{n-p}}$ ← $\varepsilon = \pm 1$, totally anti-symmetric

i.e. $* : \Lambda^p \to \Lambda^{n-p}$

Thus:

$$S_{grav} = \frac{1}{16\pi G} \int *R$$

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through pullbacks A^{-1}:

$$\theta^{\mu}(x) = A^{\mu}{}_{\nu}(x)\, dx^{\nu}$$

so that: $g_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} = \eta_{\mu\nu}$ in the basis $\{\theta^i\}$!

$\Rightarrow$ $\{\theta^i(x)\}$ indeed determines $g_{\mu\nu}(x)$:

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Here: $d\theta^i(x) = A^i_{j,k}(x)\, dx^k \wedge dx^j$ because of (*)

$$= -\frac{1}{2} C^i_{ab}\, \theta^a \wedge \theta^b = -\frac{1}{2} C^i_{ab} A^a_k A^b_j\, dx^k \wedge dx^j$$

$\Rightarrow$ $A^i_{j,k} = -\frac{1}{2} C^i_{ab} A^a_k A^b_j$

$\Rightarrow$ $\boxed{C^i_{ab}(x) = -2 A^i_{j,k}(x) \left(A^{-1}(x)\right)^j_a \left(A^{-1}(x)\right)^k_b}$

2.) The $C^i_{jk}(x)$ yield the $\Gamma^i_{jk}(x)$ through:

$$\Gamma^{\ell}_{ki} := \frac{1}{2}\left(C^{\ell}_{ki} - g_{is} g^{\ell r} C^s_{kr} - g_{ks} g^{\ell r} C^s_{ir}\right) \quad \text{(Lecture II)}$$

$$+ \frac{1}{2} g^{\ell r}\left(g_{ri,k} + g_{rk,i} - g_{ki,r}\right)$$ ← These all vanish because $g_{\mu\nu,\rho} = 0$ now

Notice: This simplifies for orthonormal frames with $g_{\mu\nu}(x) = \eta_{\mu\nu}$!

3.) The $\Gamma^i_{kj}(x)$ yield the $\omega^i_j(x)$:

$$\omega^i_j(x) := \Gamma^i_{kj}(x)\, \theta^k(x)$$

4) Recall the 2nd structure equation:

$$\Omega^i_j(x) := d\omega^i_j + \omega^i_k \wedge \omega^k_j$$

$\Rightarrow$ We have: $A^i_j \to \theta^i \to C^i_{jk} \to \Gamma^i_{jk} \to \omega^i_j \to \Omega^i_j$

Recall important identities: (torsionless case)

- Structure eqn. I:

$$\Theta^i = D\theta^i = d\theta^i + \omega^i_j \wedge \theta^j = 0$$

Tetrad formulation of GR:

Consider the action, for now, without cosmological constant and without matter:

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(Recall: $R^{\cdots}{}_{\cdots} = \Gamma_{,} + \Gamma_{,} + \Gamma\Gamma + \Gamma\Gamma$)

- **Bianchi identity I:**
$$\Omega^{i}{}_{j} \wedge \theta^{j} = 0$$
- **Bianchi identity II:** ⇒ (From diffeomorphism invariance)
$$D\Omega^{i}{}_{j} = 0$$

And, in the case of ON frames:
$$\omega_{\mu\nu} + \omega_{\nu\mu} = 0$$

Recall Hodge $*$: If $\nu = \frac{1}{p!}\nu_{i_1\cdots i_p}\theta^{i_1}\wedge\cdots\wedge\theta^{i_p}$

then $*\nu = \frac{1}{p!}\sqrt{g}\,\varepsilon_{i_1\cdots i_n}\nu^{i_{p+1}\cdots i_n}\theta^{i_1}\wedge\cdots\wedge\theta^{i_{n-p}}$ ($\varepsilon = \pm 1$, totally anti-symmetric)

i.e. $*: \Lambda^p \to \Lambda^{n-p}$

Thus:
$$S'_{\mu\nu} = \frac{1}{16\pi G}\int_{\mathcal{B}} \underbrace{*R}_{4\text{-form}}$$

Aim now: Re-express $S'_{\mu\nu}$ in terms of θ^{μ} and $\Omega^{\mu}{}_{\nu}$.

- **Define:** ("capital η" is a $(0,2)$ tensor-valued 2-form)
$$H_{\alpha\beta} := *(\theta^{\alpha}\wedge\theta^{\beta}) = \frac{1}{2}\underbrace{\sqrt{g}}_{=1}\varepsilon_{\alpha\beta\gamma\delta}\theta^{\gamma}\wedge\theta^{\delta}$$
$$H_{\alpha\beta\gamma} := *(\theta^{\alpha}\wedge\theta^{\beta}\wedge\theta^{\gamma}) = \frac{1}{2}\underbrace{\sqrt{g}}_{=1}\varepsilon_{\alpha\beta\gamma\delta}\theta^{\delta}$$
(a $(0,3)$ tensor-valued 1-form.)

- **Proposition:**
$$*R = H_{\mu\nu}\wedge\Omega^{\mu\nu}$$
($\mathcal{A}$ is a $(0,0)$ tensor-valued 4-form)

i.e.: $\boxed{S'_{\mu\nu}(\theta^{\mu}) = \int H_{\mu\nu}\wedge\Omega^{\mu\nu}}$

- **Proof:**
Use $\Omega^{\mu}{}_{\nu} = \frac{1}{2}R^{\mu}{}_{\nu\kappa\lambda}\theta^{\kappa}\wedge\theta^{\lambda}$ ⇒
$$H_{\mu\nu}\wedge\Omega^{\mu\nu} = \frac{1}{2\cdot 2}\varepsilon_{\mu\nu\rho\delta}R^{\mu\nu}{}_{\kappa\lambda}\underbrace{\theta^{\rho}\wedge\theta^{\delta}\wedge\theta^{\kappa}\wedge\theta^{\lambda}}_{=\varepsilon^{\rho\delta\kappa\lambda}\theta^{\rho}\otimes\theta^{\delta}\otimes\theta^{\kappa}\otimes\theta^{\lambda}}$$
Use also: $\varepsilon_{\mu\nu\rho\delta}\varepsilon^{\rho\delta\kappa\lambda} = 2(\delta_{\nu\rho}\delta_{\lambda\omega} - \delta_{\nu\omega}\delta_{\lambda\rho})$ ⇒
$$H_{\mu\nu}\wedge\Omega^{\mu\nu} = \frac{4}{4}R^{\mu\nu}{}_{\mu\nu}\,\theta^{1}\wedge\theta^{2}\wedge\theta^{3}\wedge\theta^{4} = *R \;\checkmark$$

(need lemma for derivation of the Einstein equation) →

- **Proposition:** $DH_{\mu\nu} = 0$ (Recall the "first structure equation": $D\theta^{\alpha} = 0$)
- **Proof:** $DH_{\mu\nu} = D\left(\frac{1}{2}\underbrace{\sqrt{g}}_{=1}\varepsilon_{\mu\nu\rho\sigma}\theta^{\rho}\wedge\theta^{\sigma}\right) = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}\left(\underbrace{D\theta^{\rho}}_{=0}\wedge\theta^{\sigma} + \theta^{\rho}\wedge\underbrace{D\theta^{\sigma}}_{=0}\right)$ (constant because ON basis)

The main proposition: variation, not co-derivative

⇒ The equation of motion, i.e., the Einstein equation

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$$H_{\mu\rho\nu} := *\left(\theta^\mu \wedge \theta^\rho \wedge \theta^\nu\right) = \frac{1}{2}\underbrace{\sqrt{g}}_{1}\,\varepsilon_{\mu\rho\nu\delta}\,\theta^\delta$$

↑ a (0,3) tensor-valued 1-form.

Proposition:

$$*R = H_{\mu\nu} \wedge \Omega^{\mu\nu}$$

(it is a (0,0) tensor-valued 4-form)

i.e.:

$$\int_{M_{spacetime}} (\theta^\mu) = \int H_{\mu\nu} \wedge \Omega^{\mu\nu}$$

The main proposition:

variation, not co-derivative

Variation of the action with respect to $\delta\theta^\mu(x)$ yields:

i.e., we vary the $A^{ab}{}_\mu(x)$ by local Lorentz transformations

$$\delta(*R) = (\delta\theta^\mu) \wedge H_{\mu\nu\rho} \wedge \Omega^{\nu\rho} + d(\text{something})$$

Stokes: $\int_B df = \int_{\partial B} f$

It implies:

$$16\pi G\,\delta S^{grav}_{Palat} = \int_B \delta\theta^\mu \wedge H_{\mu\nu\rho} \wedge \Omega^{\nu\rho} + \int_{\partial B}(\text{something})$$

← require variation to vanish at boundary ∂B, so $= 0$

Definition: The "energy-momentum 1-form" T_μ is defined as the solution to:

$$\delta S_{matter} =: \int_B \delta\theta^\mu \wedge (*T_\mu)$$

$\varepsilon_{\rho\delta\kappa\lambda}\,\theta^\rho \otimes \theta^\delta \otimes \theta^\kappa \otimes \theta^\lambda$

Use also: $\varepsilon_{\mu\nu\rho\delta}\,\varepsilon^{\rho\delta\kappa\lambda} = 2\left(\delta_{\nu\mu}\delta^{\kappa\lambda} - \delta_{\nu}{}^{\kappa}\delta^{\lambda}{}_{\mu}\right) \Rightarrow$

$$H_{\mu\nu} \wedge \Omega^{\mu\nu} = \frac{4}{4!} R^{\mu\nu}{}_{\mu\nu}\,\theta^1 \wedge \theta^2 \wedge \theta^3 \wedge \theta^4 = *R \;\checkmark$$

(need Palatini for derivation of the Einstein equation)

Proposition: $D H_{\mu\nu} = 0$

constant because ON basis

Recall the "first structure equation": $D\theta^\mu = 0$

Proof: $D H_{\mu\nu} = D\left(\frac{1}{2}\underbrace{\sqrt{g}}_{1}\,\varepsilon_{\mu\nu\rho\sigma}\,\theta^\rho \wedge \theta^\sigma\right) = \frac{1}{2}\,\varepsilon_{\mu\nu\rho\sigma}\left(\overset{0}{D\theta^\rho} \wedge \theta^\sigma + \theta^\rho \wedge \overset{0}{D\theta^\sigma}\right)$

⇒ The equation of motion, i.e., the **Einstein equation**,

$$\frac{\delta(S^{grav}_{Palat} + S_{matter})}{\delta\theta^\mu} = 0$$

becomes:

$$-\frac{1}{2} H_{\mu\nu\rho} \wedge \Omega^{\nu\rho} = 8\pi G\,*T_\mu$$

Exercise: add the cosmological constant.

Remark: The Einstein form $G_\mu := G_{\mu\nu}\,\theta^\nu$ (it is a (0,1) tensor-valued 1-form) obeys

$$*G_\mu = -\frac{1}{2} H_{\mu\nu\rho} \wedge \Omega^{\nu\rho}$$

$$\Rightarrow \quad G_\mu = 8\pi G\,T_\mu$$

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General Relativity as a "gauge theory"

Recall:

$$S_{EH}(\theta^\mu) = \int H_{\mu\nu} \wedge \Omega^{\mu\nu} \qquad \text{Einstein action}$$

$$-\tfrac{1}{2} H_{\mu\nu\rho} \wedge \Omega^{\nu\rho} = 8\pi G \, {}^*T_\mu \qquad \text{Einstein equation}$$

are now the same in all coordinate systems.

In addition:

They are the same also with any choice of ON bases in the tangent spaces, i.e., we have a local symmetry under:

$$\theta^\mu(x) \longrightarrow \tilde{\theta}^\mu(x) = A^\mu{}_\nu(x)\, \theta^\nu(x)$$

The $A^\mu{}_\nu(x)$ are **local Lorentz transformations**.

Upshot:

- We can start with any matter theory that is invariant under global Lorentz transformations and, through general relativity, turn it into a theory that is invariant under local Lorentz transformations.
- Thereby:
  Derivatives become covariant derivatives.
  A new field is introduced: gravity's Γ.

⇝ This is analogous to the gauge principle of particle physics:

- A global symmetry is "gauged" to become local.
- Derivatives become covariant derivatives
- A new field is introduced.

The gauge principle:

Action for a Dirac field (electrons, quarks etc):

$$S[\psi] = \int \bar{\psi}\left(i\gamma^\mu \partial_\mu - m\right)\psi \, d^4x$$

It has a global symmetry:

$$\psi(x) \to \tilde{\psi}(x) := e^{id}\psi(x), \text{ i.e., } \bar{\psi}(x) \to \bar{\tilde{\psi}}(x) = e^{-id}\bar{\psi}(x)$$

Gauge principle: Introduce a new field $A_\mu(x)$ that transforms so as to absorb the extra term:

$$S[\psi, A] := \int \bar{\psi}(x)\left(i\gamma^\mu(\partial_\mu + iA_\mu(x)) - m\right)\psi(x)\, d^4x$$

"covariant derivative"

Now under

$$\psi(x) \to \tilde{\psi}(x) := e^{id(x)}\psi(x)$$

$$A_\mu(x) \to \tilde{A}_\mu(x) := A_\mu(x) - i\partial_\mu d(x)$$

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$\psi(x) \to \tilde{\psi}(x) := e^{i\alpha}\psi(x)$, i.e., $\bar{\psi}(x) \to \bar{\tilde{\psi}}(x) = e^{-i\alpha}\bar{\psi}(x)$

$\Rightarrow S[\psi] \to S[\tilde{\psi}] = S[\psi]$

However, no local symmetry:

$\psi(x) \to \tilde{\psi}(x) := e^{i\alpha(x)}\psi(x)$ $\quad \bar{\psi}(x) \to \bar{\tilde{\psi}}(x) = e^{-i\alpha(x)}\bar{\psi}(x)$

$S[\psi] \to S[\tilde{\psi}] \neq S[\psi]$!

$A_\mu(x) \to \tilde{A}_\mu(x) := A_\mu(x) - i\partial_\mu\alpha(x)$

the action obeys:

$S[\psi, A] \to S[\tilde{\psi}, \tilde{A}]$

$= \int \bar{\psi}(x) e^{-i\alpha(x)} \left(i\gamma^\mu(\partial_\mu + iA_\mu - i\partial_\mu\alpha) - m \right) e^{i\alpha(x)}\psi(x) d^4x$

$= S[\psi, A]$

Generalization to Yang-Mills theory

Gauging $\psi(x) \to e^{i\alpha(x)}\psi(x)$ introduced $A_\mu(x)$.

and $A_\mu(x)$ turns out to exist: The EM 4-potential.

We "derived" the electromagnetic force!

Notice: $e^{i\alpha(x)} \in U(1)$

$U(1) = \{ G \in \mathbb{C} \mid G^+ = G^{-1} \}$

Now give the Dirac particles an extra index (isospin bundle)

$S[\psi] = \int \bar{\psi}_a \left(i\gamma^\mu \delta_{ab}\partial_\mu - m\delta_{ab} \right) \psi_b \, d^4x$ $\quad (\sum_{ab}$ implied)

It's invariant under:

$\psi_a(x) \to G_{ab}\psi_b(x)$ $\quad (\sum_{b=1}^{N}$ implied)

where $G \in SU(N)$

$SU(N) = \{ G \in M_N(\mathbb{C}) \mid G^+ = G^{-1}, \det(G) = 1 \}$

Now, we gauge, i.e., require invariance under:

$\psi_a(x) \to G_{ab}(x)\psi_b(x)$ $\quad$ where $G \in SU(N)$

$\rightsquigarrow$ Invariance of the action now requires new field $B_\mu(x)$:

$S[\psi] = \int \bar{\psi}_a \left(i\gamma^\mu \underbrace{\left(\delta_{ab}\partial_\mu + iB_\mu(x)_r T^r_{ab} \right)}_{\text{"covariant derivative"}} - m\delta_{ab} \right) \psi_b \, d^4x$

and $B_\mu(x)_r \to \tilde{B}_\mu(x)_r = B_\mu(x)_r +$ complicated

Here: $T^r_{ab} \in su(N)$ are a Lie algebra basis, i.e. they are generators of infinitesimal $SU(N)$ transformations.

Upshot: ☐ $N=2$ Weak force (though mixed with $N=1$ EM)

☐ $N=3$ Strong force QCD.

Recall:

$\rightsquigarrow$ Interpretation of the connection in ON frames:

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The $A^\mu{}_\nu(x)$ are local Lorentz transformations.

Our covariant derivative:

$$\nabla_{e_\mu} \left(v^\nu(x) e_\nu \right) = \left(\frac{\partial}{\partial x^\mu} v^\nu(x) \right) e_\nu + v^\nu(x) \, \omega^\rho{}_\nu(e_\mu) \, e_\rho$$

Do the $\omega^\rho{}_\nu$ indeed generate infinitesimal Lorentz transformations?

Plays rôle of A_μ, B_μ but is now gravity!

→ Interpretation of the connection in ON frames:

Q: The connection 1-forms $\omega^\mu{}_\nu$ are not, we know, tensor-valued 1-forms. Wherein do they take their values?

A: The connection 1-forms take values in the set of infinitesimal Lorentz transformations

Intuition?

The connection yields the change under infinitesimal parallel transport – and parallel transport preserves the metric, i.e. it preserves the lengths of vectors, i.e. the change can only be an infinitesimal "rotation", i.e. an infinitesimal Lorentz transformation.

Recall: "Lorentz transformations Λ^μ_a" are lin. maps obeying:

$$\Lambda^\mu_a \Lambda^\nu_b \eta_{\mu\nu} = \eta_{ab}$$

⇒ Infinitesimal Lorentz transformations

Proposition:

In orthonormal frames, the 1-form $\omega_{\mu\nu}$ obeys

$$\omega_{\mu\nu} + \omega_{\nu\mu} = 0$$

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Recall: "Lorentz transformations $\Lambda^\mu{}_a$" are lin. maps obeying:

$$\Lambda^\mu{}_a\,\Lambda^\nu{}_b\,\eta_{\mu\nu} = \eta_{ab}$$

$\Rightarrow$ Infinitesimal Lorentz transformations

$$\Lambda^\mu{}_a = \delta^\mu{}_a + \varepsilon^\mu{}_a \qquad \text{with } \left(\varepsilon^\mu{}_a\right)^2 = 0$$

obey:

$$\left(\delta^\mu{}_a + \varepsilon^\mu{}_a\right)\left(\delta^\nu{}_b + \varepsilon^\nu{}_b\right)\eta_{\mu\nu} = \eta_{ab}$$

i.e.: $\varepsilon^\mu{}_a\,\eta_{\mu b} + \varepsilon^\nu{}_b\,\eta_{a\nu} = 0$

$\Rightarrow$ Infinitesimal Lorentz transformations "JLT" are given by all $\Lambda^\mu{}_a = \delta^\mu{}_a + \varepsilon^\mu{}_a$ which obey: $\boxed{\varepsilon_{ba} + \varepsilon_{ab} = 0}$

Q: Are connection 1-forms JLT-valued?

Proposition:

In orthonormal frames, the 1-form $\omega_{\mu\nu}$ obeys

$$\omega_{\mu\nu} + \omega_{\nu\mu} = 0$$

i.e. it takes values that are infinitesimal Lorentz transformations.

Proof:

□ **Recall:** Absolute exterior derivative: (an anti-derivation)

$$Dt^{a\ldots b}{}_{c\ldots d} = dt^{a\ldots b}{}_{c\ldots d} + \omega^a{}_i \wedge t^{i\ldots b}{}_{c\ldots d} + \ldots - \omega^i{}_c \wedge t^{a\ldots b}{}_{i\ldots d} - \ldots$$

(t: any tensor-valued differential form; the ω-terms play the role of the $\Gamma^\alpha{}_{\beta\gamma}$)

Thus:

$$0 = \nabla g_{\mu\nu} = Dg_{\mu\nu} = \underbrace{dg_{\mu\nu}}_{=0 \text{ because } g_{\mu\nu} = \eta_{\mu\nu} = \text{const}} - \omega^i{}_\mu \wedge g_{i\nu} - \omega^i{}_\nu \wedge g_{\mu i}$$

($Dg_{\mu\nu}$: (0,2) tensor-valued 0-form; can drop the $\wedge$ because $g_{\mu\nu}$ is a 0-form)

i.e. $0 = \omega_{\nu\mu} + \omega_{\mu\nu}$ ✓

Recall that by using a tetrad, we achieved that $g_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} = \eta_{\mu\nu}$ everywhere!

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