Title: General Relativity for Cosmology Lecture - 110223

Speakers: Achim Kempf

Collection: General Relativity for Cosmology

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GR for Cosmology, Achim Kempf — Lecture 15

Recall: If we choose the bases $\{\frac{\partial}{\partial x^\nu}\}$, $\{dx^\nu\}$, then:

E.g.: $L_{EM} = -\frac{1}{16\pi} F_{\mu\nu}F^{\mu\nu}$

$$S[g_{\mu\nu},\psi] = \int \left(\frac{1}{16\pi G} R(g_{\mu\nu}(x)) + L_{matter}\left(g_{\mu\nu}(x), \psi^{(i)}(x), \psi^{(i)}_{,\mu}(x)\right)\right)\sqrt{g}\, d^4x$$

$\frac{\delta S}{\delta \psi^{(i)}} = 0 \Rightarrow$ Eqns. of motion of matter (Maxwell, Klein Gordon eqns. etc)

$\frac{\delta S}{\delta g_{\mu\nu}} = 0 \Rightarrow$ Einstein equation:

$$\boxed{R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = 8\pi G T^{\mu\nu}}$$

What is the Einstein equation when using a frame so that

$$g_{\mu\nu}(x) = \eta_{\mu\nu} \;?$$

Recall:

- Frames $\{\theta^\mu\}, \{e_\mu\}$:

Often, one uses as the bases of $T_p(M)$, and $T_p(M)'$ the canonical bases $\{dx^\mu\}$ and $\{\frac{\partial}{\partial x^\mu}\}$ respectively, which suggest themselves when one chooses coordinates, say $(x^0, \dots, x^3)$. Thus, when changing coordinate system, $x \to \tilde{x}$, one also usually automatically changes basis in $T_p(M)$, $T_p(M)'$.

Important: The only reason why the components of a tensor can change when we change coordinates is that we can change basis in the (co-) tangent spaces, namely from one canonical basis to another canonical basis, when we change coord. system.

Recall: a fixed vector has different coefficients in different bases:

$$\xi^\mu \frac{\partial}{\partial x^\mu} = \xi^\mu \frac{\partial \tilde{x}^\nu}{\partial x^\mu}\frac{\partial}{\partial \tilde{x}^\nu} = \tilde{\xi}^\nu \frac{\partial}{\partial \tilde{x}^\nu} \Rightarrow \tilde{\xi}^\nu = \frac{\partial \tilde{x}^\nu}{\partial x^\mu}\xi^\mu$$

$$\longrightarrow \quad \xi = \xi^\mu \frac{\partial}{\partial x^\mu} = \tilde{\xi}^\nu \frac{\partial}{\partial \tilde{x}^\nu}$$

Conversely: Even staying with one coordinate system, we can freely change our choice of basis in the (co-) tangent spaces:

scalar functions.

$$\theta'^\mu = A^\mu{}_\nu \theta^\nu$$
$$e'_\mu = (A^{-1})_\mu{}^\nu e_\nu$$

So we have e.g.:

$$\xi = \xi^\mu e_\mu = \xi^\mu A^\nu{}_\mu e'_\nu = \xi'^\nu e'_\nu$$

I.e.: $\xi'^\nu = A^\nu{}_\mu \xi^\mu$

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$$\frac{\delta S'}{\delta \psi^{(i)}} = 0 \quad \Rightarrow \text{ Eqns. of motion of matter}$$

(Maxwell, Klein Gordon eqns. etc)

$$\frac{\delta S'}{\delta g_{\mu\nu}} = 0 \quad \Rightarrow \text{ Einstein equation:}$$

$$\boxed{R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = 8\pi G T^{\mu\nu}}$$

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$$\xi = \xi^\mu \frac{\partial}{\partial x^\mu} = \bar{\xi}^\nu \frac{\partial}{\partial \bar{x}^\nu}$$

We notice: If we choose a fixed basis, say $\{\theta^\mu\}, \{e_\mu\}$ then the coefficients of tensors no longer depend on the choice of coordinates!

E.g.: $\xi = \tilde{\xi}^\mu e_\mu$ the same numbers in every coordinate system.

How to specify frames?

Often, one uses as the bases of $T_p(M)$, and $T_p(M)'$ the canonical bases $\{dx^\mu\}$ and $\{\frac{\partial}{\partial x^\mu}\}$ respectively, which suggest themselves when one chooses coordinates, say $(x^0, \ldots, x^3)$. Thus, when changing coordinate system, $x \to \bar{x}$, one also usually automatically changes basis in $T_p(M), T_p(M)'$.

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Examples:

- The curvature form: $\Omega'^\mu{}_\nu = A^\mu{}_a (A^{-1})_\nu{}^b \Omega^a{}_b$
- But: the connection form $\omega^\mu{}_\nu(\xi) = \xi^\kappa \Gamma^\mu_{\kappa\nu}$ obeys:

$$\omega'^\mu{}_\nu = A^\mu{}_a \omega^a{}_b (A^{-1})_\nu{}^b - (dA)^\mu{}_c (A^{-1})_\nu{}^c$$

- Existence? Always: At each $p \in M$ may choose e.g.

1/28

1

Recall:

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canonical basis, when we change coord. system.

$$\xi = \xi^\mu \frac{\partial}{\partial x^\mu} = \bar{\xi}^\nu \frac{\partial}{\partial \bar{x}^\nu}$$

We notice: If we choose a fixed basis, say $\{\theta^\mu\}, \{e_\mu\}$ then the coefficients of tensors no longer depend on the choice of coordinates!

E.g.: $\xi = \tilde{\xi}^\mu e_\mu$ — the same numbers in every coordinate system.

So we have e.g.:

$$\xi = \xi^\mu e_\mu = \xi^\mu A^\nu{}_\mu e'_\nu = \xi'^\nu e'_\nu$$

I.e.: $\xi'^\nu = A^\nu{}_\mu \xi^\mu$

Examples:

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- But: the connection form $\omega^\mu{}_\nu(\xi) = \xi^\kappa \Gamma^\mu_{\kappa\nu}$ obeys:

$$\omega'^\mu{}_\nu = A^\mu{}_a \, \omega^a{}_b (A^{-1})_\nu{}^b - (dA)^\mu{}_c (A^{-1})_\nu{}^c$$

How to specify frames?

In an arbitrary coordinate system, we may specify the bases in terms of the canonical bases:

$$\theta^i(x) = A^i{}_j(x)\, dx^j$$

(Another possibility: Take n scalar functions $f^{(1)}, \dots, f^{(n)}$ and define $\theta^i := df^{(i)}$. For generic functions these $\{\theta^i\}$ will be linearly independent almost everywhere)

Note: the $A^i{}_j(x)$ change nontrivially when changing the coordinate system!

Our choice now: orthonormal frames, or "Tetrads"

- We say that a frame $\{\theta^\mu\}, \{e_\mu\}$ is orthonormal if in this frame, for all $p \in M$:

$$g(e_\mu, e_\nu) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}_{\mu\nu} = \eta_{\mu\nu} \quad \text{i.e. if:} \quad g = -\theta^0 \otimes \theta^0 + \sum_{i=1}^{3} \theta^i \otimes \theta^i$$

- Existence? Always: At each $p \in M$ may choose e.g. $\theta^\mu = dx^\mu$ where dx^μ are canonical ON basis at centre of a geodesic cds.
- Uniqueness?

For a given space-time, (M, g), any ON frame yields a new ON frame by transforming the bases through

$$\theta'^\mu(x) = \Lambda(x)^\mu{}_\nu \, \theta^\nu(x),$$

if the linear maps $\Lambda(x)$ preserve the orthonormality:

$$\eta_{\mu\nu}\, \theta'^\mu \otimes \theta'^\nu = \eta_{ab}\, \theta^a \otimes \theta^b$$

i.e. if: $\Lambda^\mu{}_a \Lambda^\nu{}_b \eta_{\mu\nu} = \eta_{ab}$ — recall: this is the defining equation for Lorentz transformations. **(*)**

$\Rightarrow$ Frames are unique up to local Lorentz transformations.

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recall: this is the defining equation for Lorentz transformations.

$\Rightarrow$ Frames are unique up to local Lorentz transformations.

Re-express the degrees of freedom:

- We used to specify space-times through these data: $(\mathcal{M}, g)$
- Now, let us specify space-times, equivalently, through data $(\mathcal{M},\{\theta^\mu\})$:

Namely:

Assume the $\{\theta^i\}$ are given w. resp. to a basis $\{dx^\mu\}$. through functions $A^\mu{}_\nu$,

$$\theta^\mu(x) = A^\mu{}_\nu(x)\,dx^\nu$$

so that: $g_{\mu\nu} = \begin{pmatrix} -1 & & 0 \\ & 1 & \\ 0 & & 1 \end{pmatrix} = \eta_{\mu\nu}$ in the basis $\{\theta^i\}$!

Notice: knowing the $A^\mu{}_\nu(x)$, we can reconstruct $g_{\mu\nu}(x)$ in basis $\{dx^\mu\}$:

We use that the abstract g is the same in every basis:

$$g = \overbrace{\eta_{\mu\nu}\,\theta^\mu\otimes\theta^\nu}^{\text{because it's tetrad}} = \eta_{\mu\nu}\,\overbrace{A^\mu{}_a\,A^\nu{}_b}^{= g_{ab}(x)}\,dx^a\otimes dx^b = g_{\mu\nu}(x)\,dx^\mu\otimes dx^\nu$$

$$\Rightarrow \quad \boxed{g_{ab}(x) = \eta_{\mu\nu}\,A^\mu{}_a(x)\,A^\nu{}_b(x)}$$

$\Rightarrow$ $\{\theta^i(x)\}$ indeed determines $g_{\mu\nu}(x)$:

$\Rightarrow$ The $A^\mu{}_\nu(x)$ carry all physical (here shape) info!

5 / 28

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$$\Rightarrow \quad \boxed{g_{ab}(x) = \eta_{\mu\nu} A^\mu{}_a(x) A^\nu{}_b(x)}$$

$\Rightarrow$ $\{\theta^i(x)\}$ indeed determines $g_{\mu\nu}(x)$:

$\Rightarrow$ The $A^\mu{}_\nu(x)$ carry all physical (here shape) info!

How then does $A^\mu{}_\nu(x)$ encode $C^i{}_{jk}, \omega^i{}_j, \Omega^i{}_j$?

□ Start with orthonormal frame: $\boxed{\theta^i(x) = A^i{}_j(x)\, dx^j}$ $(*)$

1.) How do the $A^i{}_j(x)$ determine the $C^i{}_{jk}(x)$?

Recall from lecture 11:

$$d\theta^i(x) = -\frac{1}{2} C^i{}_{jk}(x)\, \theta^j(x) \wedge \theta^k(x)$$

Here: $d\theta^i(x) = A^i{}_{j,k}(x)\, dx^k \wedge dx^j$ because of $(*)$

$$= -\frac{1}{2} C^i{}_{ab}\, \theta^a \wedge \theta^b = -\frac{1}{2} C^i{}_{ab} A^a{}_k A^b{}_j\, dx^k \wedge dx^j$$

$$\Rightarrow \quad A^i{}_{j,k} = -\frac{1}{2} C^i{}_{ab} A^a{}_k A^b{}_j$$

$$\Rightarrow \quad \boxed{C^i{}_{ab}(x) = -2 A^i{}_{j,k}(x) \left(A^{-1}(x)\right)^j{}_a \left(A^{-1}(x)\right)^k{}_b}$$

2.) The $C^i{}_{jk}(x)$ yield the $\Gamma^i{}_{jk}(x)$ through:

$$\Gamma^l{}_{ki} := \frac{1}{2}\left(C^l{}_{ki} - g_{is} g^{lj} C^s{}_{kj} - g_{ks} g^{lj} C^s{}_{ij}\right) \qquad \text{(lecture 11)}$$

$$+ \frac{1}{2} g^{lj} \cancel{\left(g_{ij,k} + g_{jk,i} - g_{ki,j}\right)} \leftarrow \text{These all vanish because } g_{\mu\nu,\alpha} = 0 \text{ now}$$

Notice: This simplifies for orthonormal frames with $g_{\mu\nu}(x) = \eta_{\mu\nu}$!

3.) The $\Gamma^i{}_{kj}(x)$ yield the $\omega^i{}_j(x)$:

$$\omega^i{}_j(x) := \Gamma^i{}_{kj}(x)\, \theta^k(x)$$

4.) Recall the 2nd structure equation:

$$\Omega^i{}_j(x) := d\omega^i{}_j + \omega^i{}_k \wedge \omega^k{}_j$$

$\Rightarrow$ We have: $A^i{}_j \to \theta^i \to C^i{}_{jk} \to \Gamma^i{}_{jk} \to \omega^i{}_j \to \Omega^i{}_j$

Recall important identities: (torsionless case)

Tetrad formulation of GR:

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$$+\frac{1}{2} g^{ij}\left(\cancel{g_{ij,k} + g_{jk,i}} - g_{ki,j}\right)$$

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Recall important identities: (torsionless case)

- Structure eqn. I:

$$\Theta^i = D\theta^i = d\theta^i + \omega^i{}_j \wedge \theta^j = 0$$

(Ordinarily: $\theta^i = dx^i \Rightarrow d\theta = 0$ and $\omega^i{}_j \wedge \theta^j = 0$ is $\Gamma^i{}_{jk} = \Gamma^i{}_{kj}$)

- Structure eqn. II:

$$\Omega^i{}_j = d\omega^i{}_j + \omega^i{}_k \wedge \omega^k{}_j$$

(Recall: $R^{\cdot}{}_{\cdots} = \Gamma^{\cdot}{}_{\cdot,\cdot} + \Gamma^{\cdot}{}_{\cdot\cdot,\cdot} + \Gamma\Gamma + \Gamma\Gamma$)

- Bianchi identity I:

$$\Omega^i{}_j \wedge \theta^j = 0$$

- Bianchi identity II: (From diffeomorphism invariance)

$$D\Omega^i{}_j = 0$$

And, in the case of ON frames:

Tetrad formulation of GR:

Consider the action, for now, without cosmological constant and without matter:

$$S_{grav} := \frac{1}{16\pi G} \int_B \underbrace{R}_{\text{0-form}} \sqrt{g}\, d^4x$$

Recall Hodge $*$: If $\quad v = \frac{1}{p!} v_{i_1 \ldots i_p}\, \theta^{i_1} \wedge \ldots \wedge \theta^{i_p}$

then $\quad *v = \frac{1}{p!} \sqrt{g}\, \underbrace{\epsilon_{i_1 \ldots i_n}}_{= \pm 1,\ \text{totally anti-symmetric}} v^{i_1 \ldots i_p}\, \theta^{i_{p+1}} \wedge \ldots \wedge \theta^{i_n}$

i.e. $\quad * : \Lambda^p \to \Lambda^{n-p}$

Thus:

$$S_{grav} = \frac{1}{16\pi G} \int *R$$

9 / 28

9

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through functions $A^\mu{}_\nu$:

$$\theta^\mu(x) = A^\mu{}_\nu(x)\, dx^\nu$$

so that: $g_{\mu\nu} = \begin{pmatrix} -1 & \\ & \ddots & \\ & & 1\end{pmatrix} = \eta_{\mu\nu}$ in the basis $\{\theta^i\}$!

$\Rightarrow$ $\{\theta^i(x)\}$ indeed determines $g_{\mu\nu}(x)$:

$\Rightarrow$ The $A^\mu{}_\nu(x)$ carry all physical (here shape) info!

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Here: $d\theta^i(x) = A^i{}_{j,k}(x)\, dx^k \wedge dx^j$ because of $(*)$

$$= -\tfrac{1}{2} C^i{}_{ab}\, \theta^a \wedge \theta^b = -\tfrac{1}{2} C^i{}_{ab} A^a{}_k A^b{}_j\, dx^k \wedge dx^j$$

$\Rightarrow$ $A^i{}_{j,k} = -\tfrac{1}{2} C^i{}_{ab} A^a{}_k A^b{}_j$

$\Rightarrow$ $\boxed{C^i{}_{ab}(x) = -2 A^i{}_{j,k}(x) \left(A^{-1}(x)\right)^j{}_a \left(A^{-1}(x)\right)^k{}_b}$

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$$+ \tfrac{1}{2}\, g^{\ell j}\left(\cancel{g_{ij,k} + g_{jk,i} - g_{ki,j}}\right)$$

← These all vanish because $g_{\mu\nu,\alpha} = 0$ now

Notice: This simplifies for orthonormal frames with $g_{\mu\nu}(x) = \eta_{\mu\nu}$!

3.) The $\Gamma^i{}_{kj}(x)$ yield the $\omega^i{}_j(x)$:

$$\omega^i{}_j(x) := \Gamma^i{}_{kj}(x)\, \theta^k(x)$$

4.) Recall the 2nd structure equation:

$$\Omega^i{}_j(x) := d\omega^i{}_j + \omega^i{}_k \wedge \omega^k{}_j$$

$\Rightarrow$ We have: $A^i{}_j \to \theta^i \to C^i{}_{jk} \to \Gamma^i{}_{jk} \to \omega^i{}_j \to \Omega^i{}_j$

Recall important identities: (torsionless case)

- Structure eqn. I:

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Consider the action, for now, without cosmological constant and without matter:

0-form

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(Recall: $R^{\cdot}{}_{\cdots} = \Gamma^{\cdot}_{\cdot,\cdot} + \Gamma^{\cdot}{}_{\cdot,\cdot} + \Gamma\Gamma + \Gamma\Gamma$)

- Bianchi identity I:

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- Bianchi identity II: (From diffeomorphism invariance)

$$D\Omega^i{}_j = 0$$

And, in the case of ON frames:

$$\omega_{\mu\nu} + \omega_{\nu\mu} = 0$$

Recall Hodge $*$: If $v = \frac{1}{p!} v_{i_1 \ldots i_p} \theta^{i_1} \wedge \ldots \wedge \theta^{i_p}$

then $*v = \frac{1}{p!} \sqrt{g}\, \underbrace{\varepsilon_{i_1 \ldots i_n}}_{= \pm 1,\ \text{totally anti-symmetric}} v^{i_1 \ldots i_p}\, \theta^{i_{p+1}} \wedge \ldots \wedge \theta^{i_n}$

i.e. $* : \Lambda^p \to \Lambda^{n-p}$

Thus:

$$S'_{grav} = \frac{1}{16\pi G} \int_B \underbrace{*R}_{4\text{-form}}$$

Aim now: Re-express S'_{grav} in terms of θ^μ and $\Omega^\mu{}_\nu$.

- **Define:** "capital eta" is a $(0,2)$ tensor-valued 2-form

$$H_{\alpha\beta} := *\left(\theta^\alpha \wedge \theta^\beta\right) = \frac{1}{2} \overset{1}{\overset{\|}{\sqrt{g}}}\, \varepsilon_{\alpha\beta\gamma\delta}\, \theta^\gamma \wedge \theta^\delta$$

$$H_{\alpha\beta\gamma} := *\left(\theta^\alpha \wedge \theta^\beta \wedge \theta^\gamma\right) = \frac{1}{2} \overset{1}{\overset{\|}{\sqrt{g}}}\, \varepsilon_{\alpha\beta\gamma\delta}\, \theta^\delta$$

a $(0,3)$ tensor-valued 1-form.

- **Proposition:**

$$*R = H_{\mu\nu} \wedge \Omega^{\mu\nu}$$

(it is a $(0,0)$ tensor-valued 4-form)

i.e.: $$\boxed{S_{grav}(\theta^\mu) = \int H_{\mu\nu} \wedge \Omega^{\mu\nu}}$$

- **Proof:**

Use $\Omega^\mu{}_\nu = \frac{1}{2} R^\mu{}_{\nu\kappa\lambda}\, \theta^\kappa \wedge \theta^\lambda \Rightarrow$

$$H_{\mu\nu} \wedge \Omega^{\mu\nu} = \frac{1}{2\cdot 2} \varepsilon_{\mu\nu\gamma\delta} R^{\mu\nu}{}_{\kappa\lambda}\, \underbrace{\theta^\gamma \wedge \theta^\delta \wedge \theta^\kappa \wedge \theta^\lambda}_{= \varepsilon_{\gamma\delta\kappa\lambda}\, \theta^\gamma \otimes \theta^\delta \otimes \theta^\kappa \otimes \theta^\lambda}$$

Use also: $\varepsilon_{\mu\nu\gamma\delta}\, \varepsilon_{\gamma\delta\kappa\lambda} = 2\left(\delta_{\kappa\mu}\delta_{\lambda\nu} - \delta_{\kappa\nu}\delta_{\lambda\mu}\right) \Rightarrow$

$$H_{\mu\nu} \wedge \Omega^{\mu\nu} = \frac{4}{4} R^{\mu\nu}{}_{\mu\nu}\, \theta^1 \wedge \theta^2 \wedge \theta^3 \wedge \theta^4 = *R \ ✓$$

(need later for derivation of the Einstein equation)

- **Proposition:** $D H_{\mu\nu} = 0$

Recall the "first structure equation": $D\theta^\rho = 0$

- **Proof:** $D H_{\mu\nu} = D\Big(\underbrace{\tfrac{1}{2} \overset{\|}{\underset{1}{\sqrt{g}}}\, \varepsilon_{\mu\nu\varsigma\sigma}}_{\text{constant because ON basis}}\, \theta^\varsigma \wedge \theta^\sigma\Big) = \frac{1}{2} \varepsilon_{\mu\nu\varsigma\sigma}\Big(\overset{0}{\overset{\|}{\overbrace{D\theta^\varsigma}}} \wedge \theta^\sigma + \theta^\varsigma \wedge \overset{0}{\overset{\|}{\overbrace{D\theta^\sigma}}}\Big)$

The main proposition: variation, not co-derivative

$\Rightarrow$ The equation of motion, i.e., the Einstein equatio 11 / 28

11

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$$H_{\alpha\beta\gamma} := *(\theta^\alpha \wedge \theta^\beta \wedge \theta^\gamma) = \tfrac{1}{2}\overset{1}{\overset{\shortparallel}{\sqrt{g}}}\,\varepsilon_{\alpha\beta\gamma\delta}\,\theta^\delta$$

↖ a (0,3) tensor-valued 1-form.

Proposition:

$$*R = H_{\mu\nu} \wedge \Omega^{\mu\nu}$$

(is a (0,0) tensor-valued 4-form)

i.e.: $\boxed{S_{grav}(\theta^\mu) = \int H_{\mu\nu} \wedge \Omega^{\mu\nu}}$

$\varepsilon_{\gamma\delta\kappa\lambda}\, \theta^\gamma \otimes \theta^\delta \otimes \theta^\kappa \otimes \theta^\lambda$

Use also: $\varepsilon_{\mu\nu\gamma\delta}\, \varepsilon_{\gamma\delta\kappa\lambda} = 2(\delta_{\kappa\mu}\delta_{\lambda\nu} - \delta_{\kappa\nu}\delta_{\lambda\mu}) \Rightarrow$

$$H_{\mu\nu} \wedge \Omega^{\mu\nu} = \tfrac{4}{4} R^{\mu\nu}{}_{\mu\nu}\, \theta^1 \wedge \theta^2 \wedge \theta^3 \wedge \theta^4 = *R \checkmark$$

(need later for derivation of the Einstein equation)

Proposition: $DH_{\mu\nu} = 0$

Recall the "first structure equation": $D\theta^\sigma = 0$

Proof: $DH_{\mu\nu} = D\big(\overbrace{\tfrac{1}{2}\sqrt{g}\,\varepsilon_{\mu\nu\rho\sigma}}^{\text{constant because ON basis}}\,\theta^\rho \wedge \theta^\sigma\big) = \tfrac{1}{2}\,\varepsilon_{\mu\nu\rho\sigma}\big(\overset{0}{D\theta^\rho} \wedge \theta^\sigma + \theta^\rho \wedge \overset{0}{D\theta^\sigma}\big)$

The main proposition:

Variation of the action with respect to $\delta\theta^\mu(x)$ yields:

variation, not co-derivative

i.e., we vary the $A^\mu{}_\nu(x)$ by local Lorentz transformations

$$\delta(*R) = (\delta\theta^\mu) \wedge H_{\mu\nu\rho} \wedge \Omega^{\nu\rho} + d(\text{something})$$

Stokes: $\int_B df = \int_{\partial B} f$

It implies:

$$16\pi G\, \delta S'_{grav} = \int_B \delta\theta^\mu \wedge H_{\mu\nu\rho} \wedge \Omega^{\nu\rho} + \int_{\partial B} (\text{something})$$

← require variation to vanish at boundary ∂B, so: $= 0$

Definition: The "energy-momentum 1-form" T_μ is defined as the solution to:

$$\delta S_{matter} =: \int_B \delta\theta^\mu \wedge (*T_\mu)$$

⇒ The equation of motion, i.e., the **Einstein equation**,

$$\frac{\delta(S'_{grav} + S'_{matter})}{\delta\theta^\mu} = 0$$

becomes: $\boxed{-\tfrac{1}{2} H_{\mu\nu\rho} \wedge \Omega^{\nu\rho} = 8\pi G *T_\mu}$

Exercise: add the cosmological constant.

(is a (0,1) tensor-valued 1 form)

Remark: The Einstein form $G_\mu := G_{\mu\nu}\theta^\nu$ obeys

$$*G_\mu = -\tfrac{1}{2} H_{\mu\nu\rho} \wedge \Omega^{\nu\rho}$$

$$\Rightarrow \boxed{G_\mu = 8\pi G\, T_\mu}$$

13 / 28

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General Relativity as a "gauge theory"

Recall:

$$S_{grav}(\theta^\mu) = \int H_{\mu\nu} \wedge \Omega^{\mu\nu} \qquad \text{Einstein action}$$

$$-\frac{1}{2} H_{\mu\nu\rho} \wedge \Omega^{\nu\rho} = 8\pi G * T_\mu \qquad \text{Einstein equation}$$

are now the same in all coordinate systems.

In addition:

They are the same also with any choice of ON bases in the tangent spaces, i.e., we have a local symmetry under:

$$\theta^\mu(x) \longrightarrow \tilde{\theta}^\mu(x) = \Lambda^\mu{}_\nu(x)\,\theta^\nu(x)$$

The $\Lambda^\mu{}_\nu(x)$ are local Lorentz transformations.

Upshot: We can start with any matter theory that is invariant under global Lorentz transformations and, through general relativity, turn it into a theory that is invariant under local Lorentz transformations.

Thereby:

Derivatives become covariant derivatives.

A new field is introduced: gravity's Γ.

⇝ This is analogous to the gauge principle of particle physics:

- A global symmetry is "gauged" to become local.
- Derivatives become covariant derivatives
- A new field is introduced.

The gauge principle:

Action for a Dirac field (electrons, quarks etc):

$$S[\psi] = \int \bar{\Psi}\left(i\gamma^\mu \partial_\mu - m\right)\Psi \, d^4x$$

It has a global symmetry:

$$\psi(x) \to \tilde{\Psi}(x) := e^{i\alpha}\psi(x), \text{ i.e., } \bar{\Psi}(x) \to \bar{\tilde{\Psi}}(x) = e^{-i\alpha}\bar{\Psi}(x)$$

Gauge principle: Introduce a new field $A_\mu(x)$ that transforms so as to absorb the extra term:

$$S[\psi, A] := \int \bar{\Psi}(x)\Big(i\gamma^\mu \underbrace{(\partial_\mu + iA_\mu(x))}_{\text{"Covariant derivative"}} - m\Big)\,\psi(x)\, d^4x$$

Now under

$$\psi(x) \to \tilde{\Psi}(x) := e^{i\alpha(x)}\psi(x)$$

$$A_\mu(x) \to \tilde{A}_\mu(x) := A_\mu(x) - i\,\partial_\mu \alpha(x)$$

19 / 28

19

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$$\Rightarrow S[\psi] \to S[\tilde\psi] = S[\psi]$$

However, no local symmetry:

$$\psi(x) \to \tilde\psi(x) := e^{i\alpha(x)}\psi(x) \qquad \bar\psi(x) \to \bar{\tilde\psi}(x) = e^{-i\alpha(x)}\bar\psi(x)$$

$$S[\psi] \to S[\tilde\psi] \neq S[\psi] \; !$$

$$A_\mu(x) \to \tilde A_\mu(x) := A_\mu(x) - i\partial_\mu\alpha(x)$$

the action obeys:

$$S[\psi, A] \to S[\tilde\psi, \tilde A]$$

$$= \int \bar\psi(x) e^{-i\alpha(x)} \left(i\gamma^\mu(\partial_\mu + iA_\mu - i\partial_\mu\alpha - m\right) e^{i\alpha(x)}\psi(x)\, d^4x$$

$$= S[\psi, A]$$

Generalization to Yang-Mills theory

Gauging $\psi(x) \to e^{i\alpha(x)}\psi(x)$ introduced $A_\mu(x)$.

and $A_\mu(x)$ turns out to exist: The EM 4-potential.

We "derived" the electromagnetic force!

Notice: $e^{i\alpha(x)} \in U(1)$

$$U(1) = \{G \in \mathbb{C} \mid G^+ = G^{-1}\}$$

Now give the Dirac particles an extra index (isospin bundle)

$$S[\psi] = \int \bar\psi_a \left(i\gamma^\mu \delta_{ab}\partial_\mu - m\delta_{ab}\right)\psi_b \, d^4x \qquad \left(\sum_{ab} \text{implied}\right)$$

It's invariant under:

$$\psi(x) \to G_{ab}\psi_b(x) \qquad \left(\sum_{b=1}^{N} \text{implied}\right)$$

where $G \in SU(N)$

$$SU(N) = \{G \in M_n(\mathbb{C}) \mid G^+ = G^{-1}, \det(G) = 1\}$$

Now, we gauge, i.e., require invariance under:

$$\psi(x) \to G_{ab}(x)\psi_b(x) \qquad \text{where } G \in SU(N)$$

⇝ Invariance of the action now requires new field $B_\mu(x)$:

$$S[\psi] = \int \bar\psi_a \Big(i\gamma^\mu \underbrace{\left(\delta_{ab}\partial_\mu + iB_\mu(x)_r T^r_{ab}\right)}_{\text{"covariant derivative"}} - m\delta_{ab}\Big)\psi_b \, d^4x$$

and $B_\mu(x)_r \to \tilde B_\mu(x)_r = B_\mu(x)_r + \text{complicated}$

Here: $T^r_{ab} \in su(N)$ are a Lie algebra basis, i.e. they are generators of infinitesimal $SU(N)$ transformations.

Upshot:
- □ $N=2$ Weak force (though mixed with $N=1$ EM)
- □ $N=3$ Strong force QCD.

Recall:

$$S[\psi] = \int ...$$

→ Interpretation of the connection in ON frames:

21 / 28

21

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The $A^\mu{}_\nu(x)$ are local Lorentz transformations.

Our covariant derivative:

$$\nabla_{e_\mu} (v^\nu(x) e_\nu) = \left(\frac{\partial}{\partial x^\mu} v^\nu(x)\right) e_\nu + v^\nu(x)\, \omega^\rho{}_\nu(e_\mu)\, e_\rho$$

Do the $\omega^\rho{}_\nu$ indeed generate infinitesimal Lorentz transformations?

Plays rôle of A_μ, B_μ but is now gravity!

→ Interpretation of the connection in ON frames:

Q: The connection 1-forms $\omega^\mu{}_\nu$ are not, we know, tensor-valued 1-forms. Wherein do they take their values?

A: The connection 1-forms take values in the set of infinitesimal Lorentz transformations

Intuition?

The connection yields the change under infinitesimal parallel transport – and parallel transport preserves the metric, i.e. it preserves the lengths of vectors, i.e. the change can only be an infinitesimal "rotation", i.e. an infinitesimal Lorentz transformation.

Recall: "Lorentz transformations $\Lambda^\mu{}_a$" are lin. maps obeying:

$$\Lambda^\mu{}_a \Lambda^\nu{}_b \, \eta_{\mu\nu} = \eta_{ab}$$

⇒ Infinitesimal Lorentz transformations

Proposition:

In orthonormal frames, the 1-form $\omega_{\mu\nu}$ obeys

$$\omega_{\mu\nu} + \omega_{\nu\mu} = 0$$

23 / 28

23

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Recall: "Lorentz transformations $\Lambda^\mu{}_a$" are lin. maps obeying:

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$\Rightarrow$ Infinitesimal Lorentz transformations

$$\Lambda^\mu{}_a = \delta^\mu{}_a + \varepsilon^\mu{}_a \qquad \text{with } \left(\varepsilon^\mu{}_a\right)^2 = 0$$

obey:

$$\left(\delta^\mu{}_a + \varepsilon^\mu{}_a\right)\left(\delta^\nu{}_b + \varepsilon^\nu{}_b\right)\eta_{\mu\nu} = \eta_{ab}$$

i.e.: $\varepsilon^\mu{}_a\,\eta_{\mu b} + \varepsilon^\nu{}_b\,\eta_{a\nu} = 0$

$\Rightarrow$ Infinitesimal Lorentz transformations "ILT" are given by all $\Lambda^\mu{}_a = \delta^\mu{}_a + \varepsilon^\mu{}_a$ which obey: $\boxed{\varepsilon_{ba} + \varepsilon_{ab} = 0}$

Q: Are connection 1-forms ILT-valued?

Proposition:

In orthonormal frames, the 1-form $\omega_{\mu\nu}$ obeys

$$\omega_{\mu\nu} + \omega_{\nu\mu} = 0$$

i.e. it takes values that are infinitesimal Lorentz transformations.

Proof:

Recall: Absolute exterior derivative: (an anti-derivation)

$$Dt^{a..b}{}_{c..d} = dt^{a..b}{}_{c..d} + \omega^a{}_i \wedge t^{i..b}{}_{c..d} + \ldots - \omega^i{}_c \wedge t^{a..b}{}_{i..d} - \ldots$$

any tensor-valued differential form.

play the rôle of the $\Gamma^a{}_{bc}$

Thus:

(0,2) tensor-valued 0 form

= 0 because $g_{\mu\nu} = \eta_{\mu\nu} = \text{const}$

can drop the $\wedge$ because g is a 0 form.

$$0 = \nabla g_{\mu\nu} = Dg_{\mu\nu} = dg_{\mu\nu} - \omega^i{}_\mu \wedge g_{i\nu} - \omega^i{}_\nu \wedge g_{\mu i}$$

Recall that by using a tetrad, we achieved that $g_{\mu\nu} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \eta_{\mu\nu}$ everywhere!

i.e. $0 = \omega_{\nu\mu} + \omega_{\mu\nu}$ ✓