Title: Kazhdan-Lusztig correspondence for a class of Lie superalgebras

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Abstract: For a simple Lie algebra \mathfrak{g}, Kazhdan-Lusztig correspondence states that for certain values of the level k, there is an equivalence between two braided tensor categories: the category of modules of the affine Lie algebra of \mathfrak{g} at level k and the category of modules of the quantum group of \mathfrak{g} at q=e^{\pi i/k}. I will report on recent work to appear with T. Creutzig and T. Dimofte proving such a statement for a class of Lie superalgebras. These Lie superalgebras and their affine VOAs arise from the study of boundary conditions in 3d \mathcal{N}=4 abelian gauge theories. I will also explain how the corresponding supergroups act on the category of matrix factorizations.

Zoom link https://pitp.zoom.us/j/92338197336?pwd=OG40V2p6V0FmalNCZnV2ZFF1NHAzZz09

Kazhdan-Lusztig Correspondence for a Class of Lie Superalgebras

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(in prep with T. Creutzig and T. Dimofte)

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Kazhdan-Lusztig Correspondence

- Braided tensor categories are important objects in the study of 3d TQFT.
- There are three ways of obtaining a braided tensor ($\mathbb{E}_2$-) category:
  - Representation of a quasi-triangular Hopf algebra.
  - Representation of a vertex operator algebra.
  - Trace of a monoidal 2-category.
- Kazhdan-Lusztig correspondence is an equivalence between the first two.

Example: Chern-Simons Theory

- Famous example: 3d Chern-Simons Theory with simple Lie group G and level k.
  - VOA $L_k(\mathfrak{g})$, integral highest weight quotient of $V_k(\mathfrak{g})$. Kazhdan-Lusztig category KL_k.
  - Quantum group $U_q(\mathfrak{g})$ with $q = e^{\pi i/k}$. Lusztig's integral representations $\mathrm{Rep}_q(G)$.
- Witten, Reshetikhin–Turaev: uses semi-simplification of the category of representations.
- Kazhdan-Lusztig shows the following:

Theorem (Kazhdan-Lusztig)

When $k \notin \mathbb{Q}_+$ avoiding small roots of unity, there is an equivalence of BTC:

$$KL_k \simeq \mathrm{Rep}_q(G).$$

Extension to Abelian Lie Superalgebras

- Our goal: extending this to Lie superalgebras associated to B twist of 3d $\mathcal{N} = 4$ abelian gauge theories.
- The definition involves $\rho : \mathbb{Z}^r \to \mathbb{Z}^n$, determining a faithful action of $U(1)^r$ on $V = \mathbb{C}^n$. $\Rightarrow$ 3d $\mathcal{N} = 4$ theory.
- Costello-Gaiotto: Dirichlet-type boundary condition supporting a VOA. Purterbatively the affine Lie superalgebra of $\mathfrak{g}_\rho$ (Π is parity shift):

$$\mathfrak{g}_\rho = T^*\mathfrak{g} \oplus T^*\Pi V \ni (N_a, E^a, \psi^i_\pm)$$

$$[N_a, \psi^i_\pm] = \pm {\rho^i}_a \psi^i_\pm, \qquad \{\psi^i_+, \psi^i_-\} = \sum {\rho^i}_a E^a$$

- This allows us to give a quantum group description to the category of line operators.

Result

- Associated to this are:
  - A VOA $V_k(\mathfrak{g}_\rho)$. Kazhdan-Lsztig category $KL_{k,\rho}$.
  - A quasi-triangular Hopf algebra $U_q(\mathfrak{g}_\rho)$, where $\{\psi^i_+, \psi^i_-\} = q^{\sum \rho_{ia} E^a} - 1$. Finite-dimensional reps $\mathrm{Rep}(U_q(\mathfrak{g}_\rho))$.
- We prove:

Theorem (Creutzig-Dimofte-N)

There is an equivalence of BTC:

$$KL_{k,\rho} \simeq \mathrm{Rep}(U_q(\mathfrak{g}_\rho))$$

- Proof using two steps, outlined in the work of Creutzig-Lentner-Rupert:
  - Recognize both categories as relative centers of some simpler categories.
  - Show that these simpler categories are equivalent using screening operators.

Example

- When $\rho = 1$, the Lie superalgebra is $\mathfrak{gl}(1|1)$.
- The quantum group $U_q^{N,E}(\mathfrak{gl}(1|1))$, the unrolled quantum group, with braiding:
$$R = e^{2\pi i(N \otimes E)}(1 - \psi_+ \otimes \psi_-)$$
- We have an equiavlence of BTC:
$$KL_k \simeq \mathrm{Rep}(U_q^{N,E}(\mathfrak{gl}(1|1))).$$

Relative Center: Quantum Group

- The algebra $U_q(\mathfrak{g}_\rho)$ is a relative Drinfeld double. Let B_ρ be the Borel (N, E, ψ^+) and H_ρ the torus (N and E).

$$U_q(\mathfrak{g}_\rho) = D_{H_\rho}(B_\rho) = B_\rho \bowtie_{H_\rho} B_\rho^*$$

- Categorically this means:

$$\mathrm{Rep}(U_q(\mathfrak{g}_\rho)) \simeq \mathcal{Z}_{\mathrm{Rep}(H_\rho)}(\mathrm{Rep}(B_\rho)) .$$

The relative center of the category.

- The relative center:

Definition

Let $\mathcal{B}$ be a monoidal category.

- *The center $\mathcal{Z}(\mathcal{B})$ is the subcategory of objects of $\mathcal{B}$, say M, together with a functorial $c_{-,M} : - \otimes M \to M \otimes -$.*
- *$\mathcal{H}$ be a subcategory of $\mathcal{Z}(\mathcal{B})$. Then the relative center $\mathcal{Z}_{\mathcal{H}}(\mathcal{B})$ consists of those objects M such that $c_{M,N}c_{N,M} = 1$ for all $N \in \mathcal{H}$.*

Relative Center: VOA

- The VOA $V_k(\mathfrak{g}_\rho)$ embeds into a lattice VOA W_ρ (Ballin-Creutzig-Dimofte-N).
- $W_\rho \Rightarrow A_\rho$ an algebra in $KL_{k,\rho}$.
- We have categories $A_\rho - \mathrm{Mod}$ and $A_\rho - \mathrm{Mod}_{loc}$.
- The work of Creutzig-Lentner-Rupert:

Proposition (Creutzig-Lentner-Rupert)

Let V be a VOA and $\mathcal{U}$ a category of V modules, that is a BTC. Let W be a VOA containing V, defining an algebra object A in $\mathcal{U}$. Assuming a list of easy-to-check technical conditions, there is a fully-faithful functor:

$$\mathcal{S} : \mathcal{U} \to \mathcal{Z}_{A-\mathrm{Mod}_{loc}}(A - \mathrm{Mod})$$

- This is satisfied by $V_k(\mathfrak{g}_\rho)$ and $KL_{k,\rho}$.

Example

- In the case of $\mathfrak{gl}(1|1)$, there is an embedding of $V(\mathfrak{gl}(1|1))$ into $H_{X,Y} \otimes V_{bc}$, where $\partial X \partial Y = \frac{1}{(z-w)^2}$.

$$N \mapsto \partial X + :bc:, \qquad E \mapsto \partial Y, \qquad \psi_+ \mapsto b, \qquad \psi_- \mapsto \partial c + \partial Y c$$

- The image is the kernel of the screening operator $S = \oint \exp(Z - Y)$.

Relative Center: comparison

- We have two pictures:

$$\begin{array}{ccccc} \text{BTC} & & \text{tensor category} & & \text{BTC} \\ \mathrm{Rep}(H_\rho) & \hookrightarrow & \mathrm{Rep}(B_\rho) & \mapsto & \mathrm{Rep}(U_q) \\ A-\mathrm{Mod}_{loc} & \hookrightarrow & A-\mathrm{Mod} & \mapsto & KL_{k,\rho} \end{array}$$

- If these easier categories are equivalent, then a fully-fatihful functor $KL_{k,\rho} \to \mathrm{Rep}(U_q)$.
- The BTC equivalence $\mathrm{Rep}(H_\rho) \simeq A - \mathrm{Mod}_{loc}$ is easy.

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Relative Center: comparison

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- If these easier categories are equivalent, then a fully-fatihful functor $KL_{k,\rho} \to \mathrm{Rep}(U_q)$.
- The BTC equivalence $\mathrm{Rep}(H_\rho) \simeq A - \mathrm{Mod}_{loc}$ is easy. The equivalence $\mathrm{Rep}(B_\rho) \simeq A - \mathrm{Mod}$ is achieved via screening operators.

Screening Operators

- Let W be a VOA and M be a module, and intertwining operator $Y : M \times W \to M$. A screening operator is determined by $m \in M$ with:

$$S_m := \mathrm{Res}_{z=0} Y(m, z) : W \to M$$

  $V = \mathrm{Ker}(S_m)$ is a sub-VOA. The VOA $V_k(\mathfrak{g}_\rho)$ and W_ρ fits into this situation.

- Let $\mathcal{U} = V - \mathrm{Mod}$ and A the algebra object defined by W. Semikhatov-Tipunin: conjectured relation between $A - \mathrm{Mod}$ and modules of Nichols algebra of screening operators.

Screening Operators and Nichols Algebras

- Precisely, let $T(M)$ be the tensor algebra of M in $A - \mathrm{Mod}_{loc}$, and define Nichols algebra $\mathcal{B}(M)$ to be the Hopf algebra quotient of $T(M)$ by the kernel of the braided symmetrizer maps. Conjecture of Semikhatov-Tipunin is that in good situations, $A - \mathrm{Mod} \simeq \mathrm{Rep}(\mathcal{B}(M))(A - \mathrm{Mod}_{loc})$.

Screening Operators and Nichols Algebras

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- Problem: $\mathcal{B}(M)$ is difficult to compute in general.
- For the purpose of applying to $V_k(\mathfrak{g}_\rho)$, we restrict to the case that M is a simple current with diagonal braiding (-1). In this case the Nichols algebra is simply $\mathbb{1} \oplus M$.

Main Statement on Screening

- We work with following assumptions:
  - W has two Heisenberg fields ∂A and ∂B where ∂A is level 1 and generates an action of $U(1)$ on W. Spectral flow σ_A.
  - V is the kernel of the screening $\mathrm{Res}_{z=0}e^{A(z)} : W \to \sigma_A W$.
  - A sufficiently unrolled condition: ∂B acts on the module M and $\sigma_A M$ with different generalized eigenvalues.
- Under these assumptions, we have the following:

Theorem (Creutzig-Dimofte-N)

The Nichols algebra $\mathcal{B}(\sigma_A W) \cong W \oplus \sigma_A W$. Moreover, there is an equivalence of tensor categories:

$$A - \mathrm{Mod} \simeq \mathrm{Rep}(\mathcal{B}(\sigma_A W))(A - \mathrm{Mod}_{loc}).$$

Application to our situation

- This applies to $V(\mathfrak{g}_\rho) \hookrightarrow W_\rho$, and leads to an equivalence:

$$A_\rho - \mathrm{Mod} \simeq \mathrm{Rep}(B_\rho)$$

- Combined with the functor $\mathcal{S}$, we obtain a fully-faithful functor of BTC:

$$KL_{k,\rho} \to \mathcal{Z}_{\mathrm{Rep}(H_\rho)}(\mathrm{Rep}(B_\rho)) \simeq \mathrm{Rep}(U_q(\mathfrak{g}_\rho)).$$

- Some simple algebraic computation shows that this must be an equivalence.

Action on Matrix Factorizations

- The quantum group $U_q(\mathfrak{g}_\rho)$ acts on matrix factorizations.
- Consider $X = T^*V \times G$, and function $W(x, y, g) = y(g-1)x$. Define matrix factorization

$$\mathrm{MF}_G(X, W)$$

- This category is a monoidal category, whose monoidal structure is given by the following diagram:

$$X \times X \longleftarrow T^*V \times G \times G \longrightarrow X$$

where the first map is $(x, y, g_1, g_2) \mapsto (g_2 x, y, g_1) \times (x, y, g_2)$ and the second is $(x, y, g_1, g_2) \mapsto (x, y, g_1 g_2)$.

- Since W is a quadratic potential on T^*V, Koszul duality asserts an equivalence:

$$\mathrm{MF}_G(X, W) \supseteq \langle \mathcal{O}_G \rangle \simeq \mathrm{Cliff} - \mathrm{Mod}^G$$

$\langle \mathcal{O}_G \rangle$ is the sub-category generated by $\mathcal{O}_G$. The algebra Cliff_G is generated over $\mathcal{O}_G$ by ψ_+, ψ_- such that $\{\psi_+, \psi_-\} = \frac{\partial^2 W}{\partial x \partial y} = \rho(g) - 1$.

Action on Matrix Factorizations

- By re-cognizing q^{E_a} as functions on G, this Cliff is a subalgebra of $U_q(\mathfrak{g}_\rho)$.
- The action of N_a is given by the derivative of G action.
- This means that every object in $\langle \mathcal{O}_G \rangle$ is canonically acted on by $U_q(\mathfrak{g}_\rho)$. The above equivalence is an equivalence of monoidal categories.

Theorem (Creutzig-Dimofte-N)

The equivalence:

$$\langle \mathcal{O}_G \rangle \simeq \mathrm{Cliff} - \mathrm{Mod}^G$$

is one of monoidal categories, and that the category $\mathrm{Cliff} - \mathrm{Mod}^G$ is braided, with braiding given by the R matrix from $U_q(\mathfrak{g}_\rho)$.

Thank you!