Title: A theory of Inaccessible Information

Speakers: Jacopo Surace

Series: Quantum Foundations

Date: October 26, 2023 - 11:00 AM

URL: https://pirsa.org/23100110

Abstract: Out of the many lessons quantum mechanics seems to teach us, one is that it seems there are things we cannot experimentally have access to. There is, indeed, a fundamental limit to our ability to experimentally explore the world. In this work we accept this lesson as a fact and we build a general theory based on this principle. We start by assuming the existence of statements whose truth value is not experimentally accessible. That is, there is no way, not even in theory, to directly test if these statements are true or false. We further develop a theory in which experimentally accessible statements are a union of a fixed minimum number of inaccessible statements. For example, the value of truth of the statements a and b is not accessible, but the value of truth of the statement "a or b" is accessible. We do not directly assume probability theory, we exclusively define experimentally accessible and inaccessible statements and build on these notions using the rules of classical logic. We find that an interesting structure emerges. Developing this theory, we relax the logical structure, naturally obtaining a derivation of a constrained quasi-probabilistic theory rich in structure that we name theory of inaccessible information. Surprisingly, the simplest model of theory of inaccessible information is the qubit in quantum mechanics. Along the path for the construction of this theory, we characterise and study a family of multiplicative information measures that we call inaccessibility measures. arXiv:https://arxiv.org/abs/2305.05734

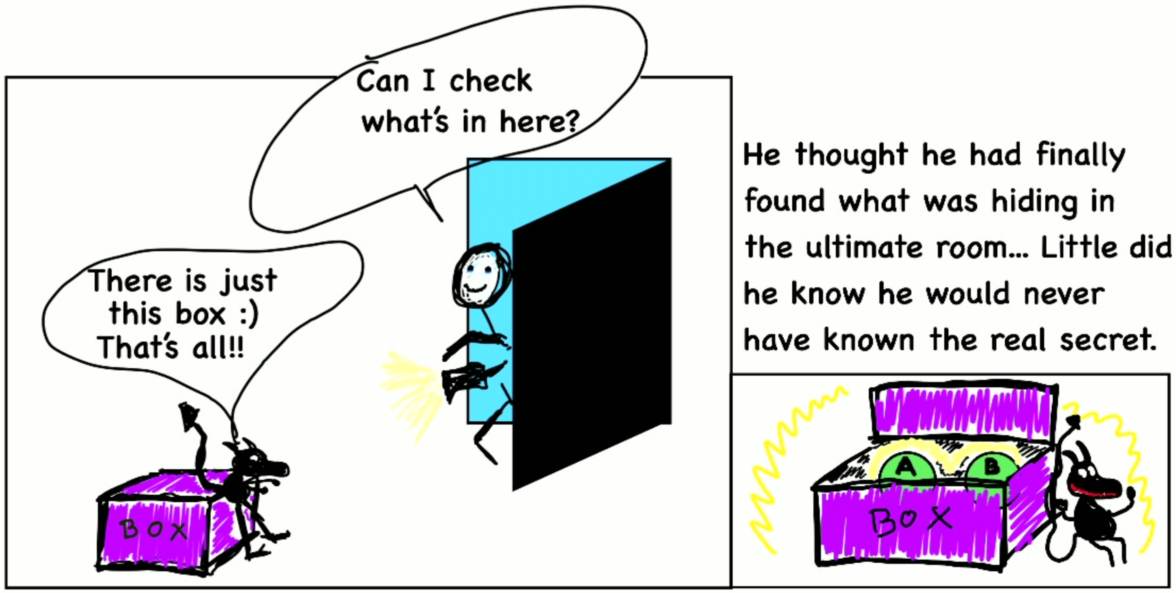
Zoom link https://pitp.zoom.us/j/91350754706?pwd=V1dVdGM3Zk9MNkp4VlpCYUoxbXg3UT09

A Theory of Inaccessible Information

The consequences of seriously accepting that some statements cannot be directly proven or disproven experimentally.

Jacopo Surace

October 2023



Quantum mechanics → There is something we cannot experimentally know

There is something we cannot experimentally know → ?

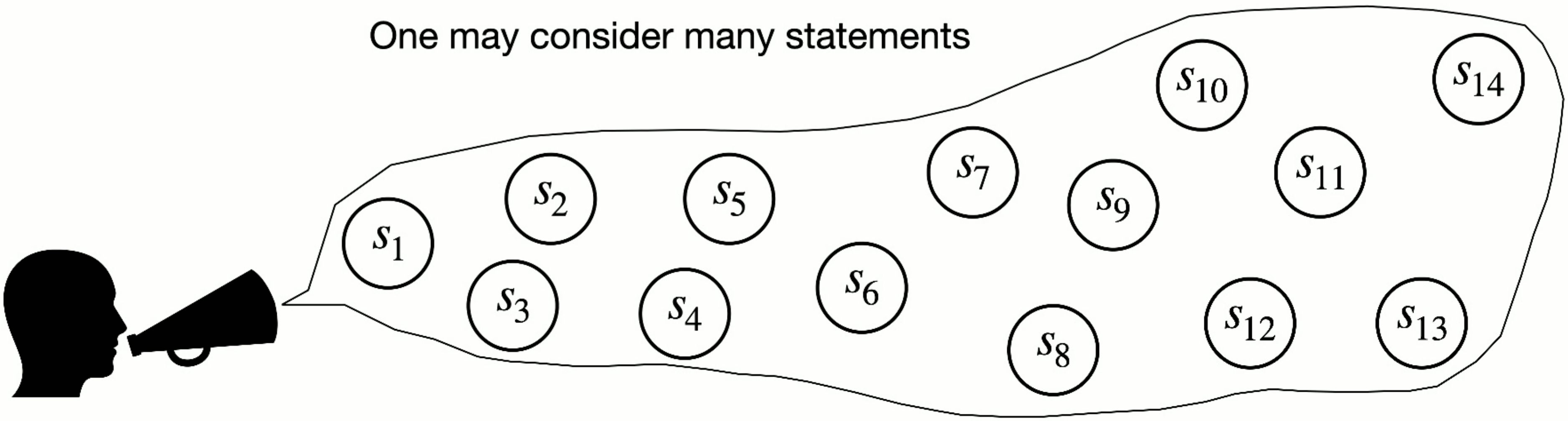
Lattices of statements

Everything we can talk about can be expressed as a statement.

Statement s_1 = "Everything we can talk about can be expressed with statements."

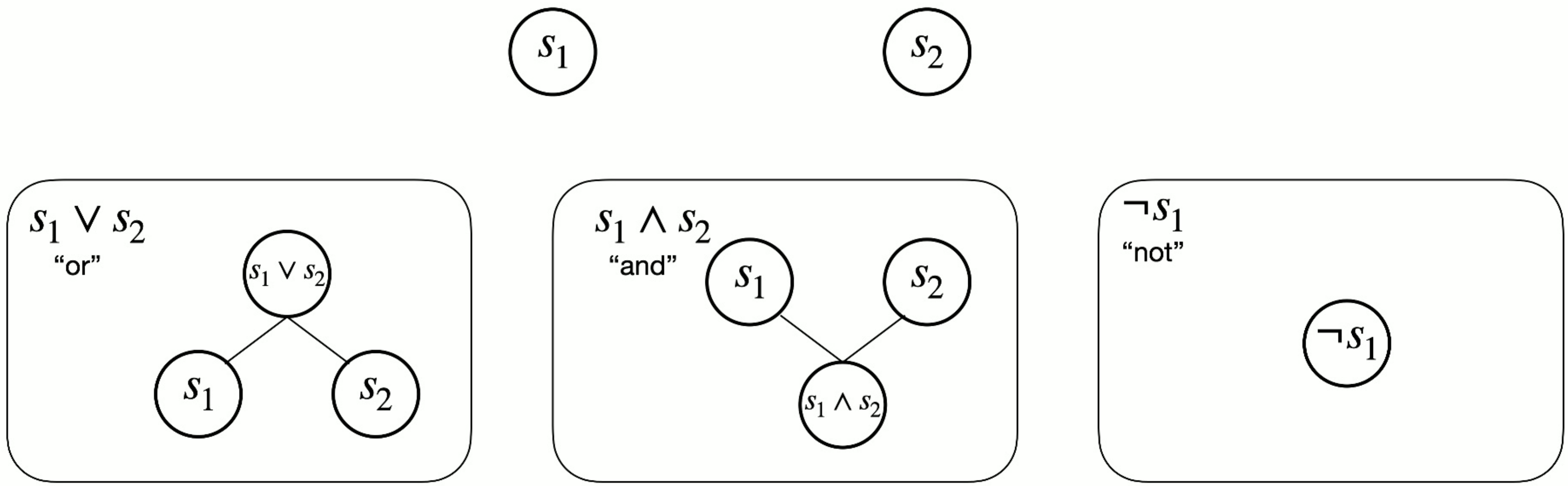
We denote statements with letters in circles s_1

One may consider many statements



Lattices of statements

One can play with statements to form new ones



$s_1 \vee s_2 = s_1 \vee s_2 \vee s_2 = s_1 \vee s_2 \vee s_2 = \ldots$

$s_1 \wedge s_2 = s_1 \wedge s_2 \wedge s_1 = s_1 \wedge s_2 \wedge s_2 = \ldots$

$\neg\neg s_1 = s_1$

A world with just two statements

s_1 = "The apple is blue",

s_2 = "The dog is red",

$s_3 := \neg s_1$ = "The apple is not blue",

$s_4 := \neg s_2$ = "The dog is not red",

$s_5 := s_1 \vee s_2$ = "The apple is blue or the dog is red",

$s_6 := s_1 \vee \neg s_2$ = "The apple is blue or the dog is not red",

$s_7 := \neg s_1 \vee s_2$ = "The apple is not blue or the dog is red",

$s_8 := \neg s_1 \vee \neg s_2$ = "The apple is not blue or the dog is not red",

$s_9 := s_1 \wedge s_2$ = "The apple is blue and the dog is red",

$s_{10} := s_1 \wedge \neg s_2$ = "The apple is blue and the dog is not red",

$s_{11} := \neg s_1 \wedge s_2$ = "The apple is not blue and the dog is red",

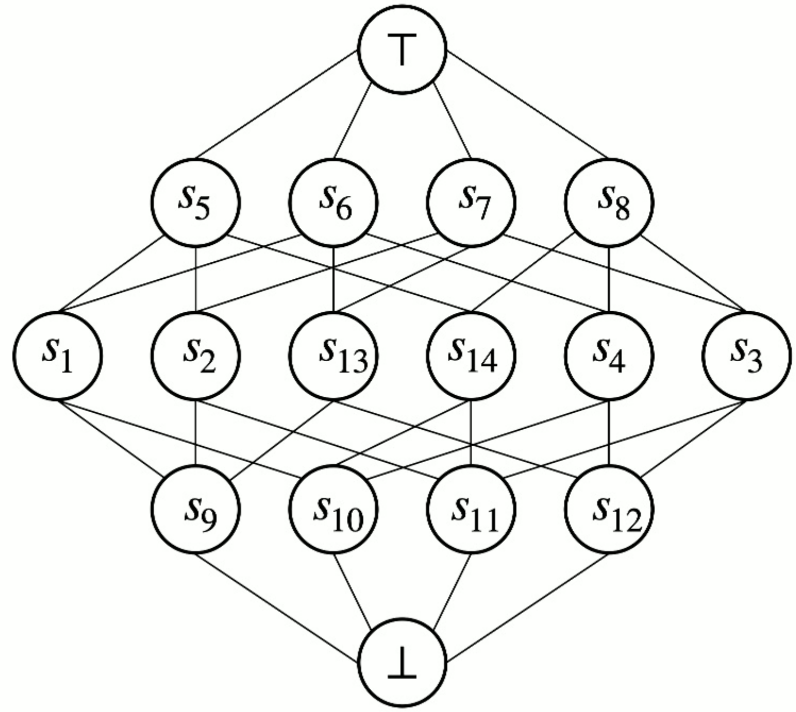
$s_{12} := \neg s_1 \wedge \neg s_2$ = "The apple is not blue and the dog is not red",

$s_{13} := (s_1 \wedge s_2) \vee (\neg s_1 \wedge \neg s_2)$ = "(The apple is blue and the dog is red) or (The apple is not blue or the dog is not red)",

$s_{14} := (s_1 \wedge \neg s_2) \vee (\neg s_1 \wedge s_2)$ = "(The apple is blue and the dog is not red) or (The apple is not blue or the dog is red)",

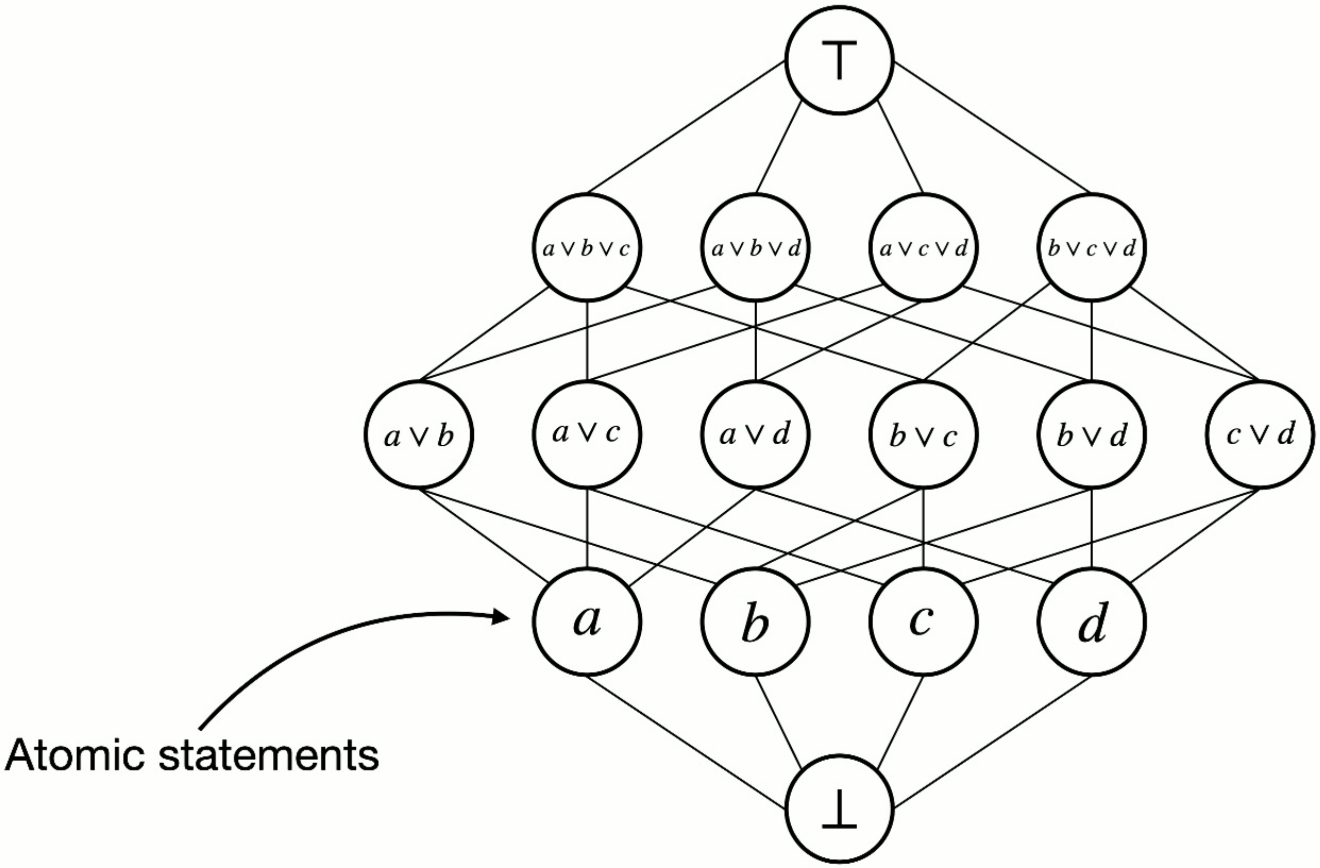
$s_{15} := s_1 \vee \neg s_1$ = "The apple is blue or the apple is not blue",

$s_{16} := s_1 \wedge \neg s_1$ = "The apple is blue and the apple is not blue" .



Lattices of propositions

Build or discover nice structures within of the algebra of statements



Boolean lattice of statements

Truth or false

Every statement has a **truth value label** attached.

A statements can be true or false

Tables of truth are the rule of propagation of truth values when composing statements.

a	b	$a \wedge b$
T	**T**	**T**
T	**F**	**F**
F	**T**	**F**
F	**F**	**F**

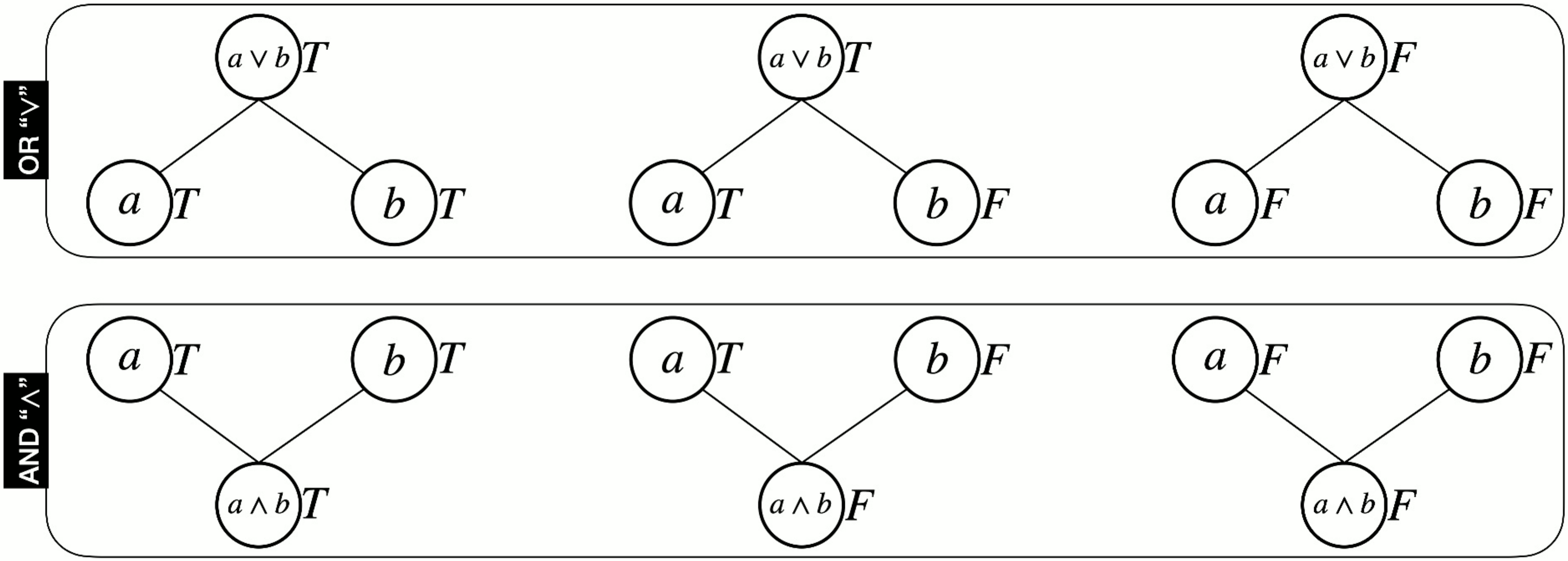
a	b	$a \vee b$
T	**T**	**T**
T	**F**	**T**
F	**T**	**T**
F	**F**	**F**

a	$\neg a$
T	**F**
F	**T**

Table I. Truth tables.

We talk about propagation because the truth value assignement to a lattice is completely specified by the truth value assigned to its atomic statements.

Composition of statements

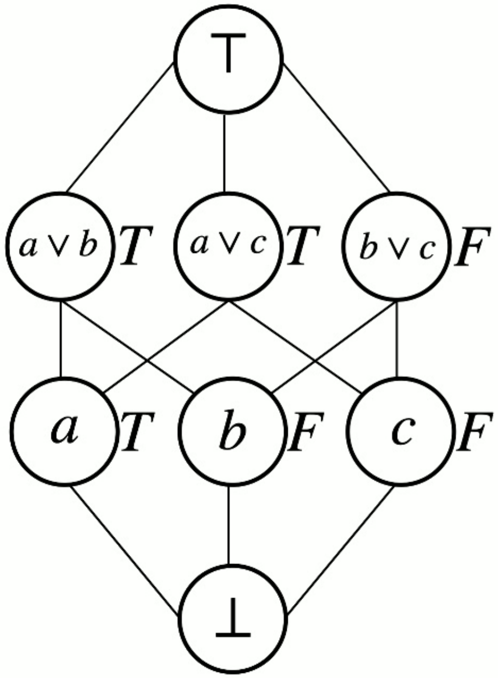


a	b	a∧ b
T	**T**	**T**
T	**F**	**F**
F	**T**	**F**
F	**F**	**F**

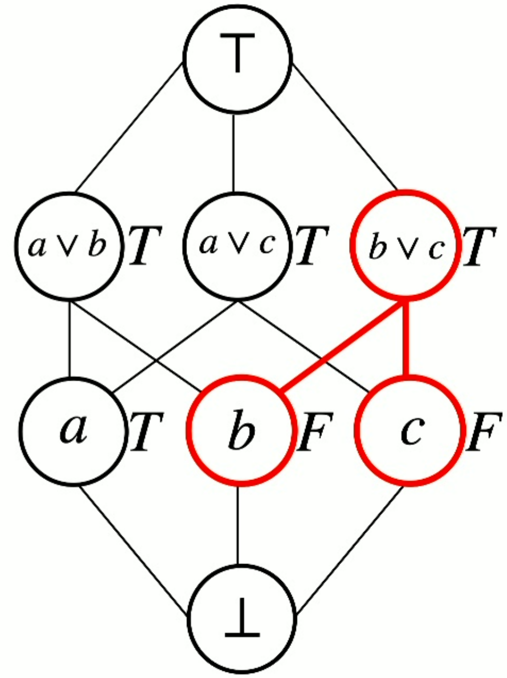
a	b	a∨ b
T	**T**	**T**
T	**F**	**T**
F	**T**	**T**
F	**F**	**F**

a	¬ a
T	**F**
F	**T**

Truth assignments compatible with the lattice



Admissible

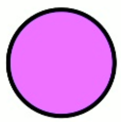


Not Admissible

Accessibility value

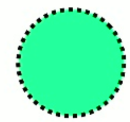
We differentiate between two kinds of statements

Accessible statements (A)



These represent statements which we can formulate and we can verify experimentally.

Non-accessible statements (N)

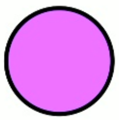


These represent statements which we can formulate, but we cannot directly verify experimentally.

Accessibility value

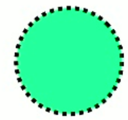
We differentiate between two kinds of statements

Accessible statements (A)



These represent statements which we can formulate and we can verify experimentally.

Non-accessible statements (N)



These represent statements which we can formulate, but we cannot directly verify experimentally.

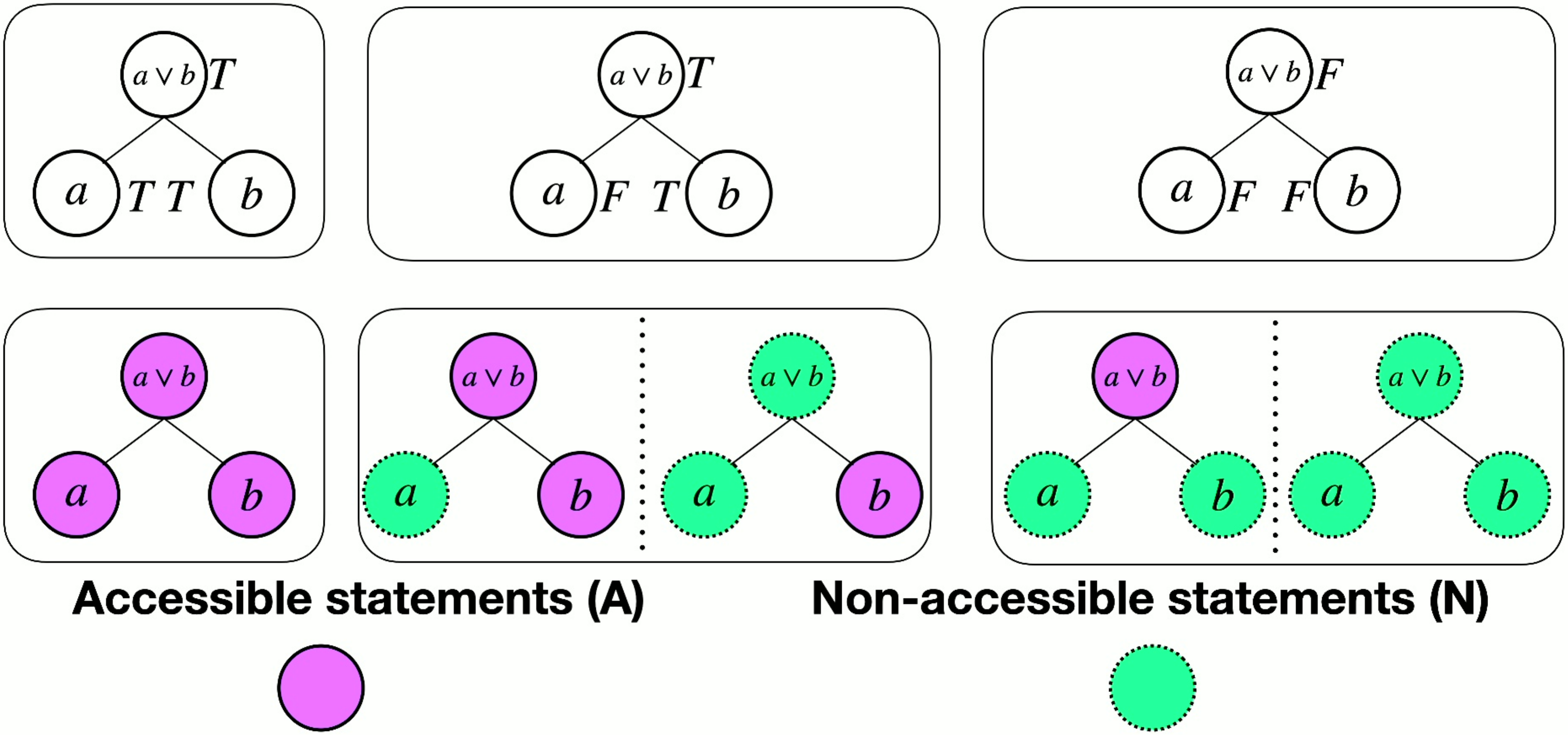
Every statement has an **accessibility value label** attached.

a	b	$a \wedge b$
A	**A**	**A**
A	**N**	**A/N**
N	**A**	**A/N**
N	**N**	**A/N**

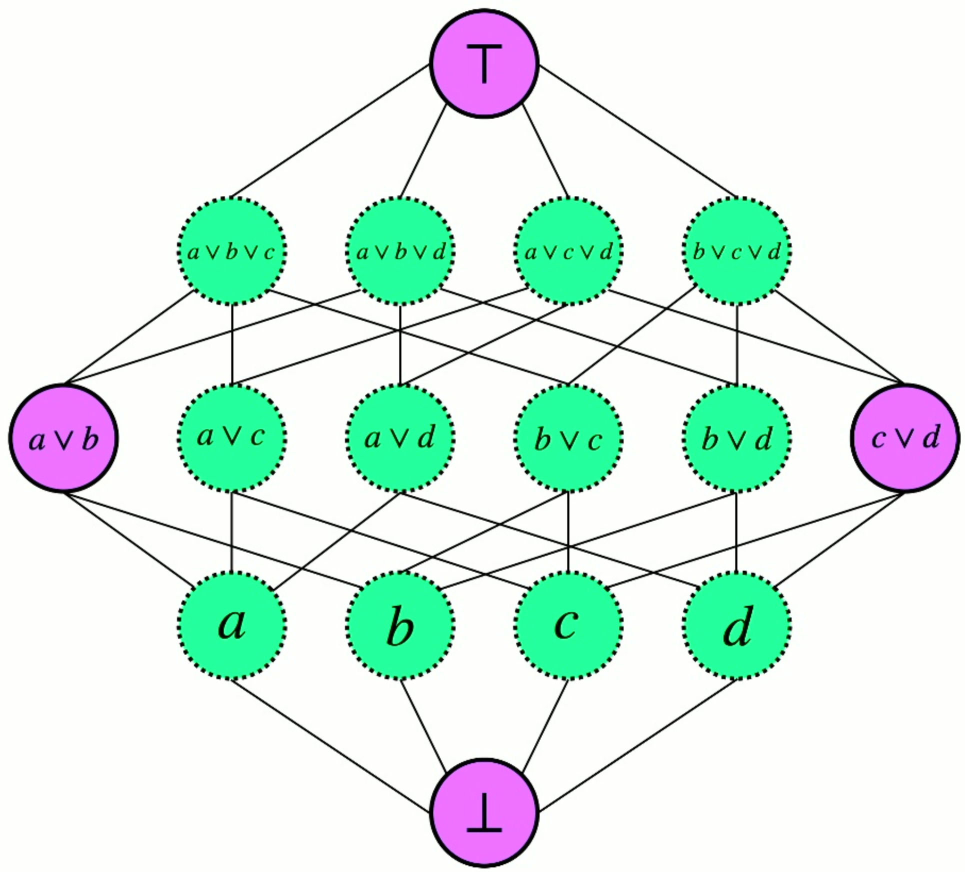
a	b	$a \vee b$
A	**A**	**A**
A	**N**	**A/N**
N	**A**	**A/N**
N	**N**	**A/N**

a	$\neg a$
A	**A**
N	**N**

Composition of inaccessible statements



Inaccessibility models



TIF Assumption 1: Maximum resolution

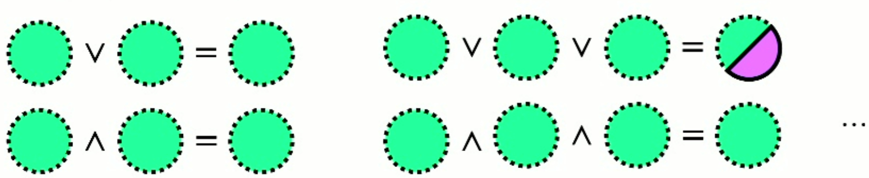
Our experimental knowledge is limited to be always uncertain about a fixed number of statements d.
We call d the *resolution* of the model.

➡ Accessible statements are constituted by a minimum number d of inaccessible statements.

TIF assumption 1 example for d=2:

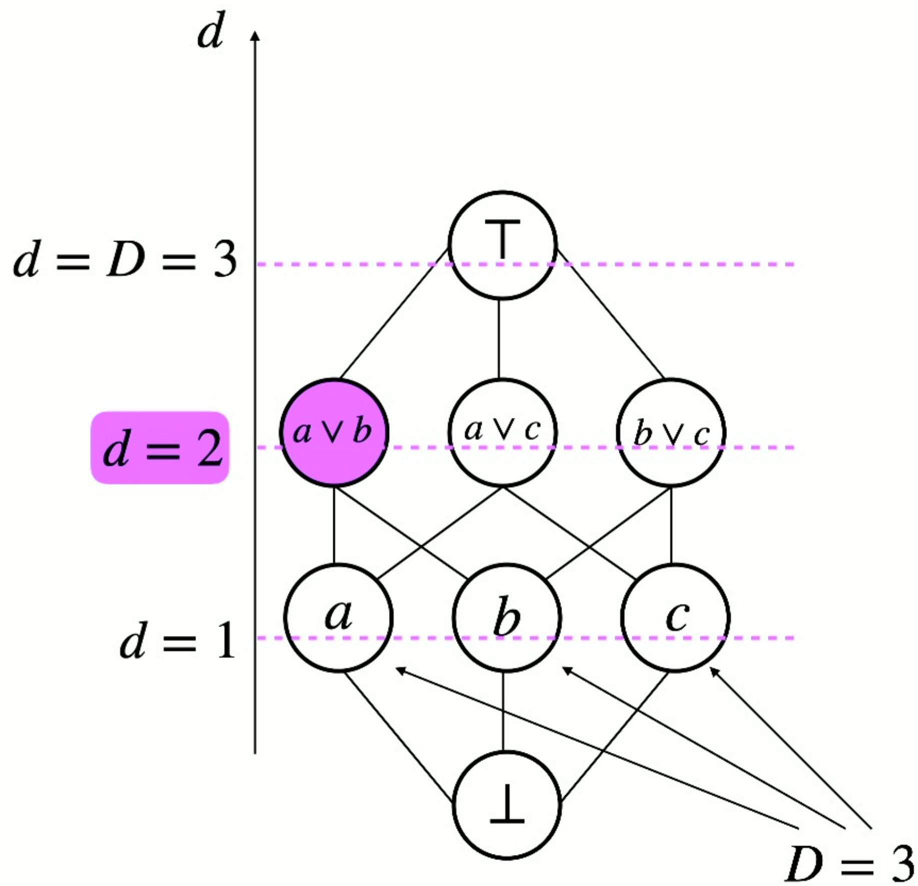
Statements composed by less than d atomic statements are not directly experimentally accessible. For example, for d=2 the value of truth of the statements **a** and **b** is not accessible, but the value of truth of the statement "**a** or **b**" is accessible

TIF assumption 1 graphical representation example for d=3:



Two parameters for TIF models:

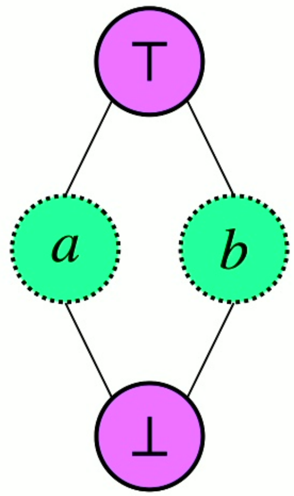
d: the resolution.
Grafically it is the level in the graph where we can find the first accessible (purple) statement and it can take values from 1 to D.



D: the number of atomic statements or fundamental statements is a free parameter.

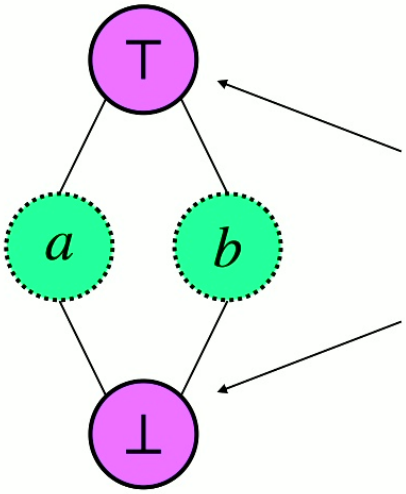
Examples: $D = 2$

$d = 2$, a useless model



Examples: $D = 2$

$d = 2$, a useless model



Only two experimentally accessible propositions.

A lattice of this form does not contain any experimentally relevant information

$d =$ (Number atomic statements) corresponds ALWAYS to a useless model

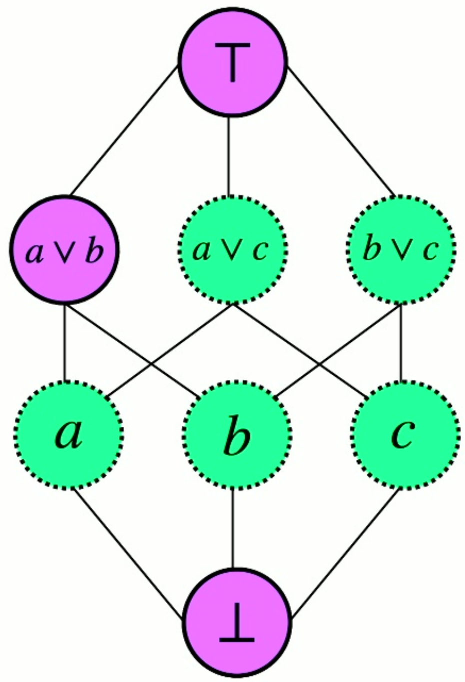
Examples: $D = 3$

We choose d=2

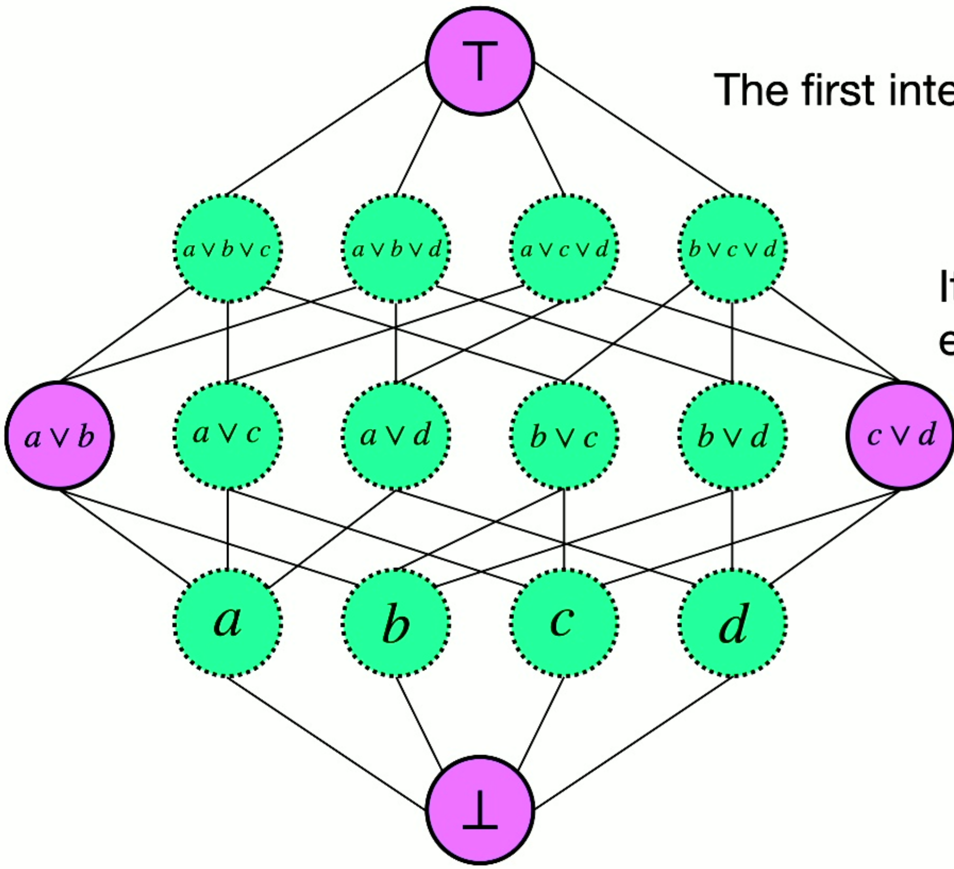
$d = 3 \; \rightarrow$ useless

$d = 2 \; \rightarrow$?

$d = 1 \; \rightarrow$ classical

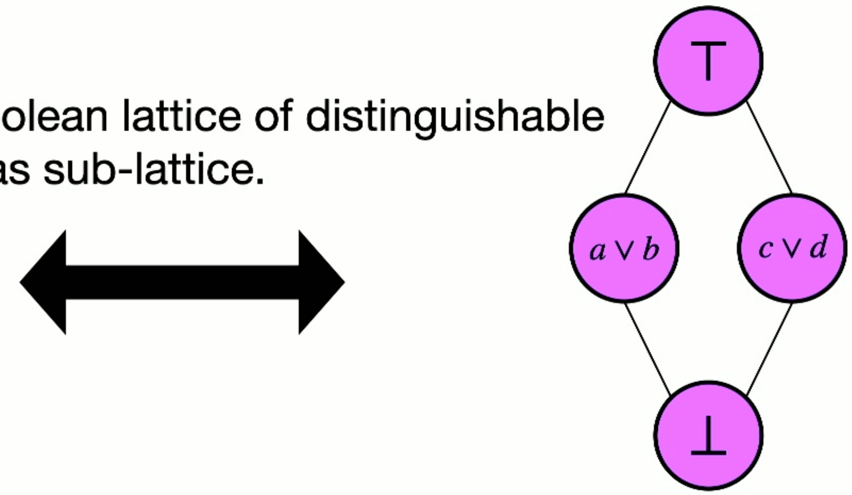


Examples: $D = 4$



The first interesting lattice for $d = 2$ has 4 atomic elements.

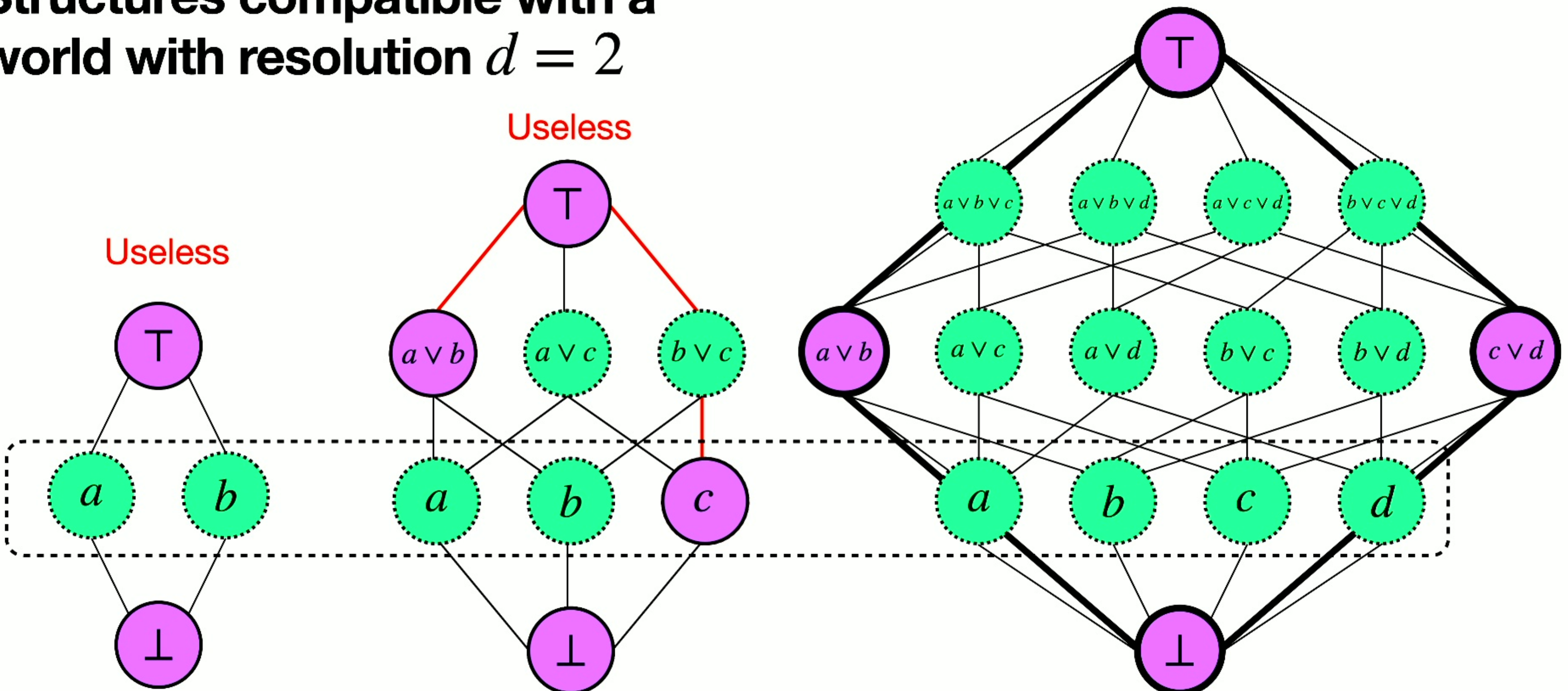
It has a boolean lattice of distinguishable elements as sub-lattice.



This is a strong property as other configurations are not allowed by compositional rules.

Try to transform any other statement to accessible and the configuration becomes not admissible.

Structures compatible with a world with resolution $d = 2$



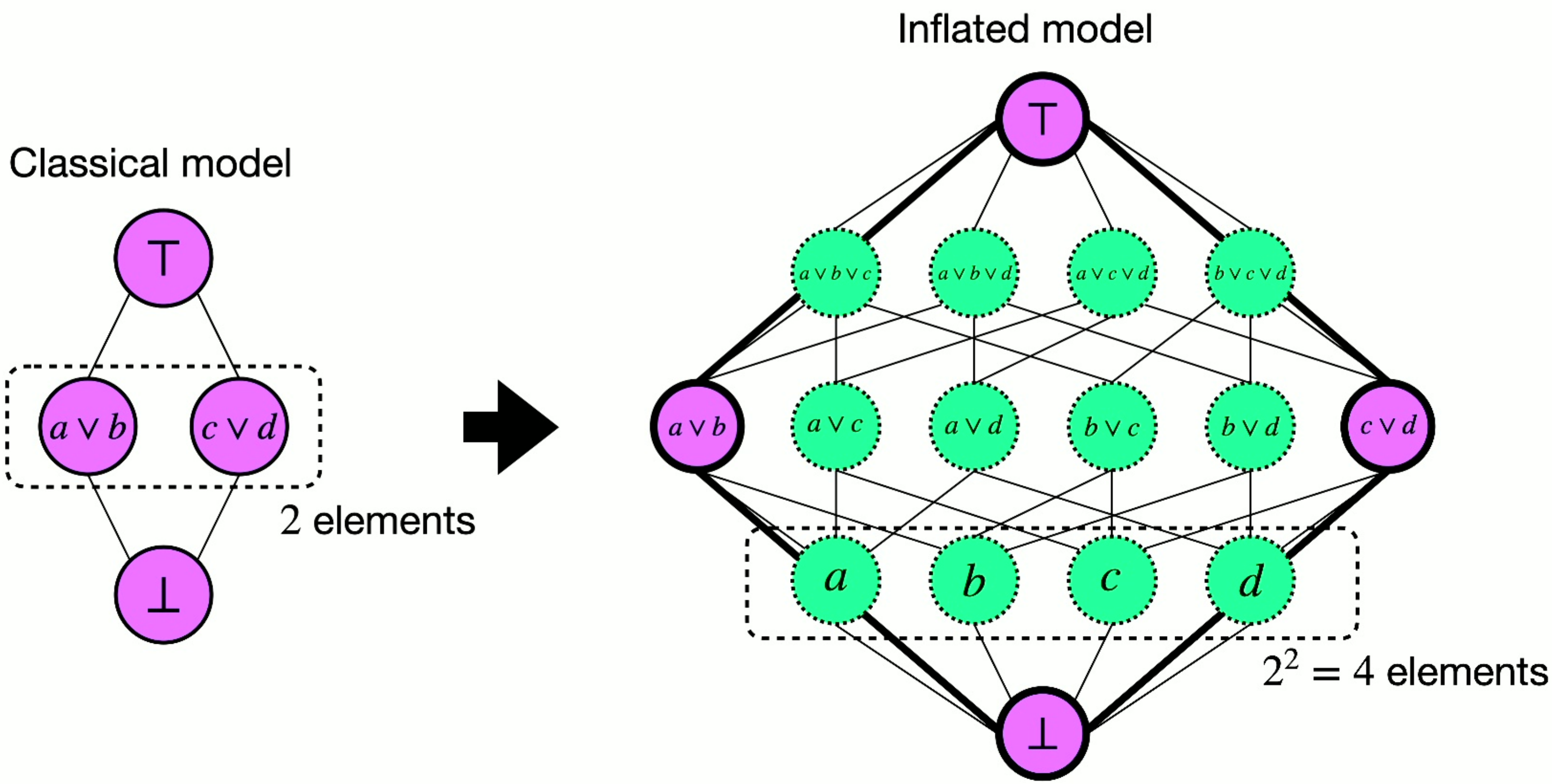
A lattice of this form does not contain any experimentally relevant information

Forbidden!

Four atomic elements is the minimum number of atomic elements for a lattice with resolution d=2 to convey meaningful physical information

With similar methods and nice tricks other properties can be proved one of these properties is:

Every classical model with dimension D inflates to an inaccessible information model with dimension D^2

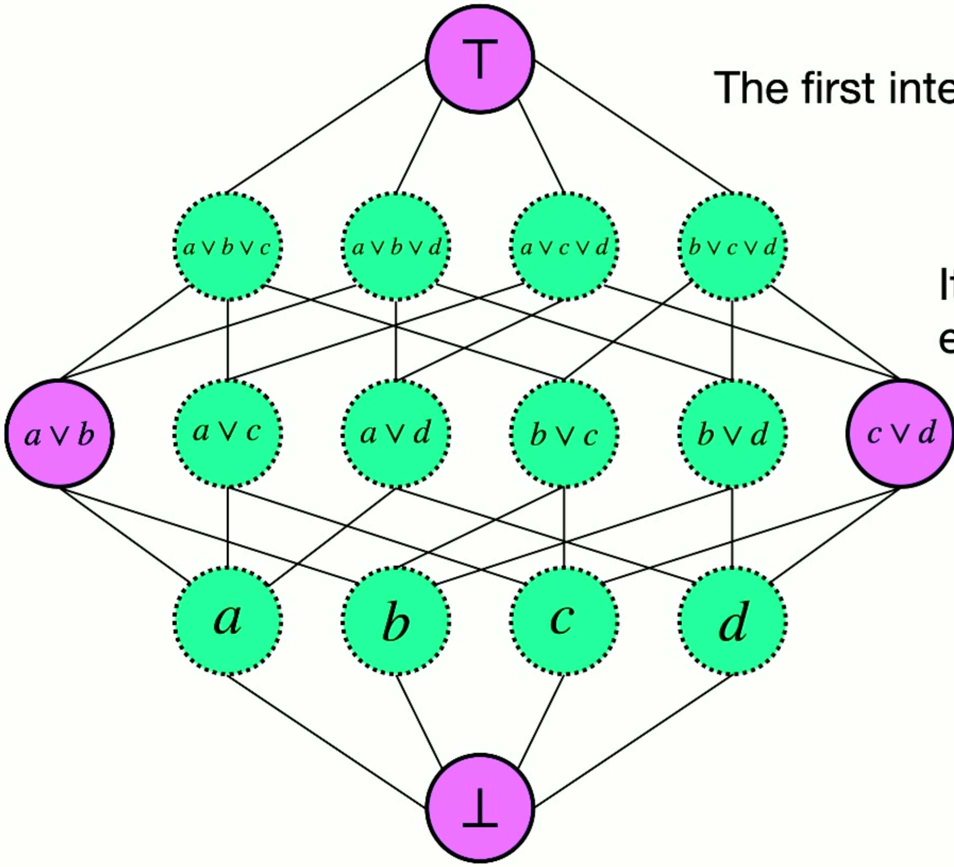


More or less

now you should know about logical lattices

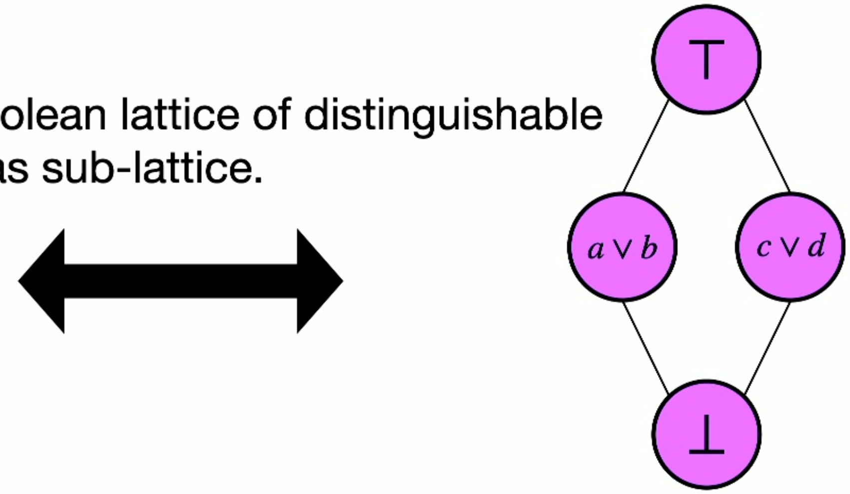
Let's relax a bit

Examples: $D = 4$



The first interesting lattice for $d = 2$ has 4 atomic elements.

It has a boolean lattice of distinguishable elements as sub-lattice.



This is a strong property as other configurations are not allowed by compositional rules.

Try to transform any other statement to accessible and the configuration becomes not admissible.

More or less

now you should know about logical lattices

Let’s relax a bit

More or less

now you should know about logical lattices

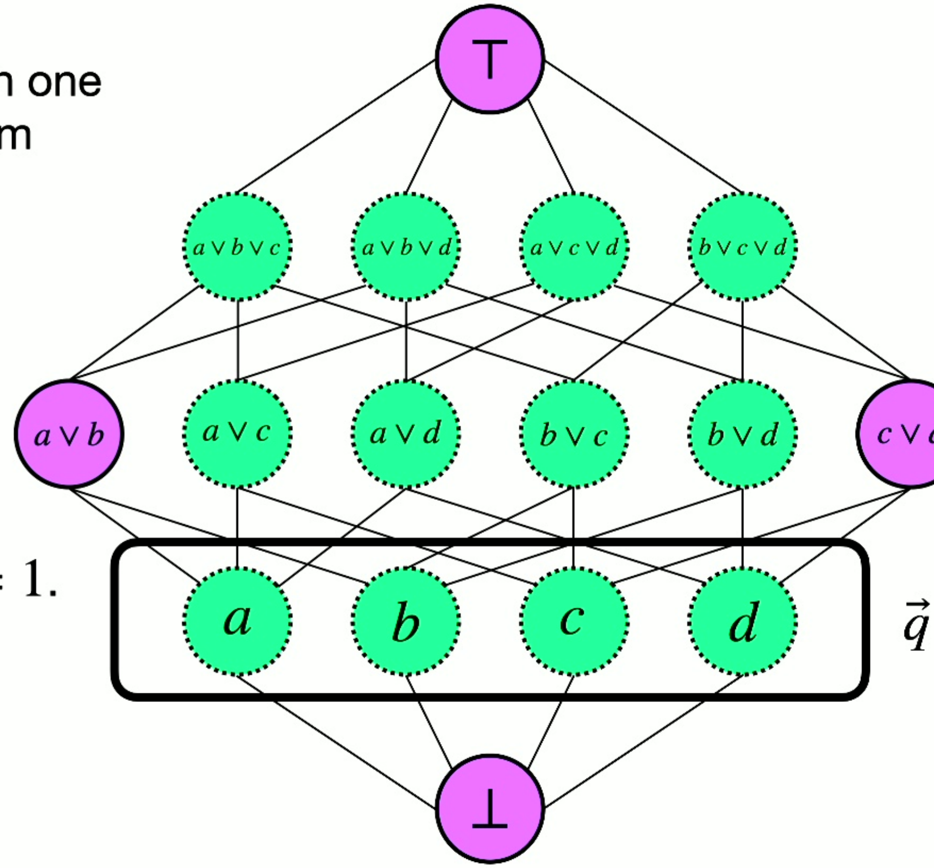
From logic to (quasi-)probability

From logic to (quasi-)probability

Using the methods of Cox, Jaynes and Knuth one relaxes the label association of the lattice from binary to continuous and naturally obtains a **(quasi-)probability model.**

Quasi-probabilities and probabilities are derived **from logical principles** and not assumed.

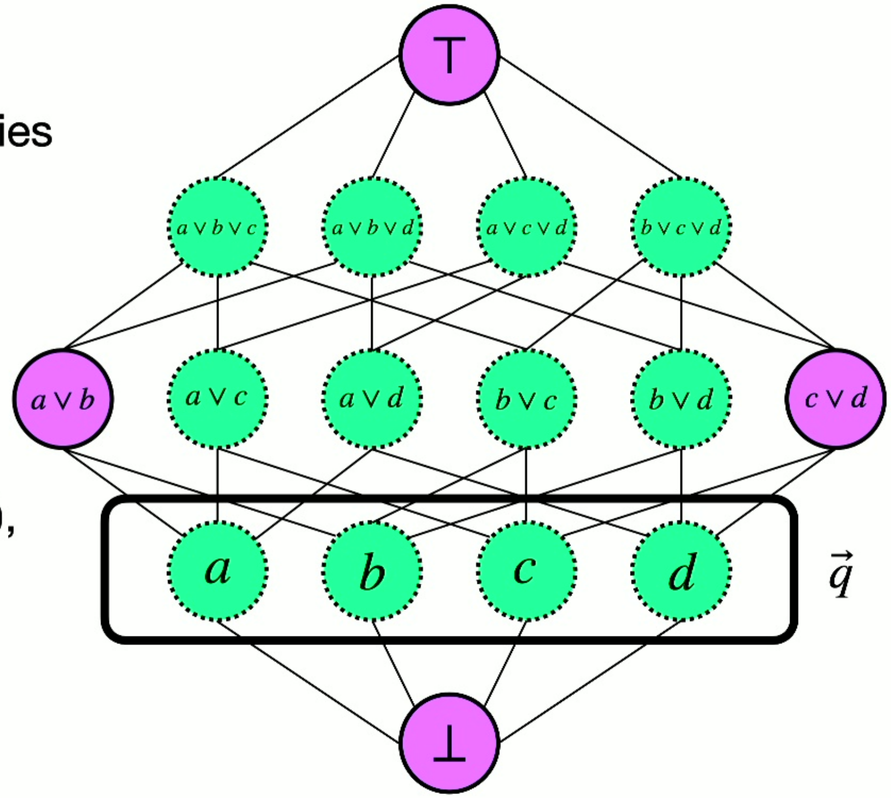
The systems are described by $\vec{q}$ s.t. $\sum_i q_i = 1.$



From logic to (quasi-)probability

A valuation Q of statements with the following properties

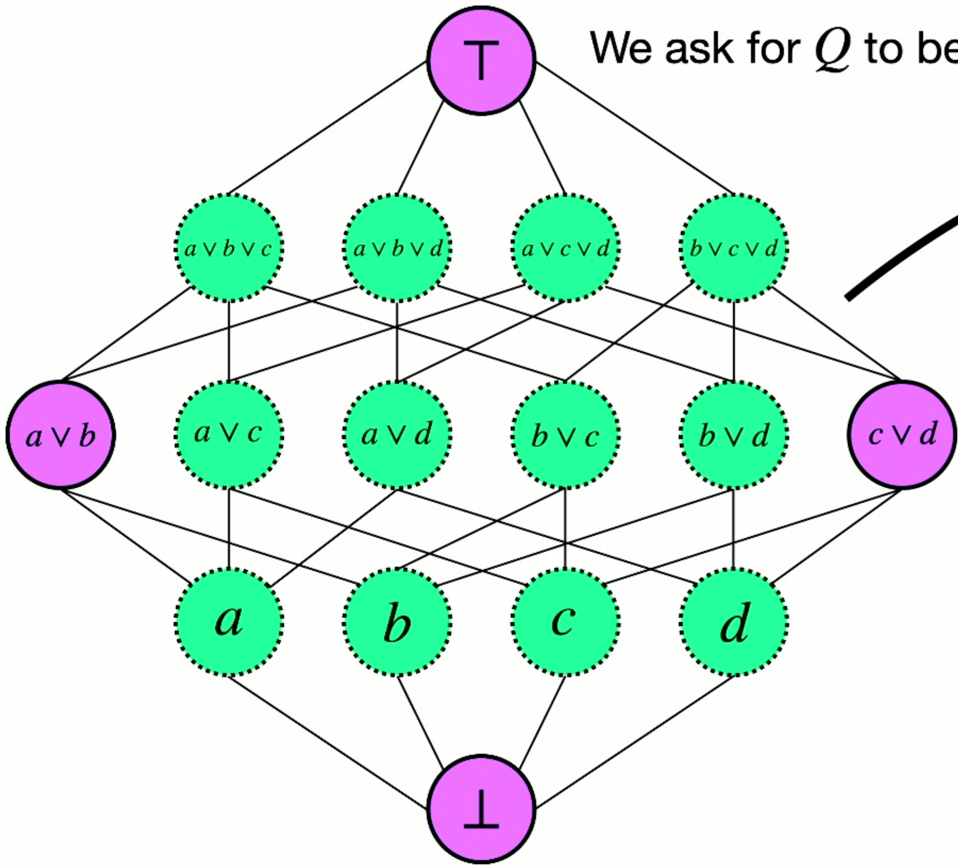
- $Q(\top) = 1,$
- $Q(\bot) = 0,$
- $Q(s) \in \mathbb{R},$
- $\sum_{s \in \mathscr{A}(\mathscr{S})} Q(s) = 1,$
- $Q(s_i \vee s_j) = Q(s_j \vee s_i) = Q(s_i) + Q(s_j) - Q(s_i \wedge s_j),$
- $Q(s_i \wedge s_j) = Q(s_j \wedge s_i),$
- $Q(s_i \wedge s_j) = Q(s_i)Q(s_i \mid s_j)$

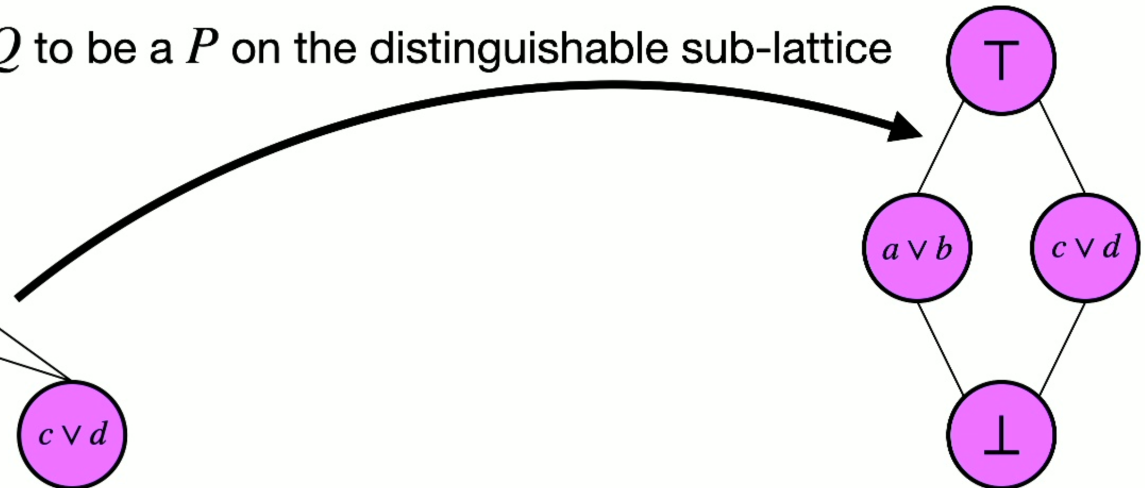


The values assumed by Q are completely characterised by the values that Q assumes on the atomic elements of the lattice. We denote these values with the vector $\vec{q}$.

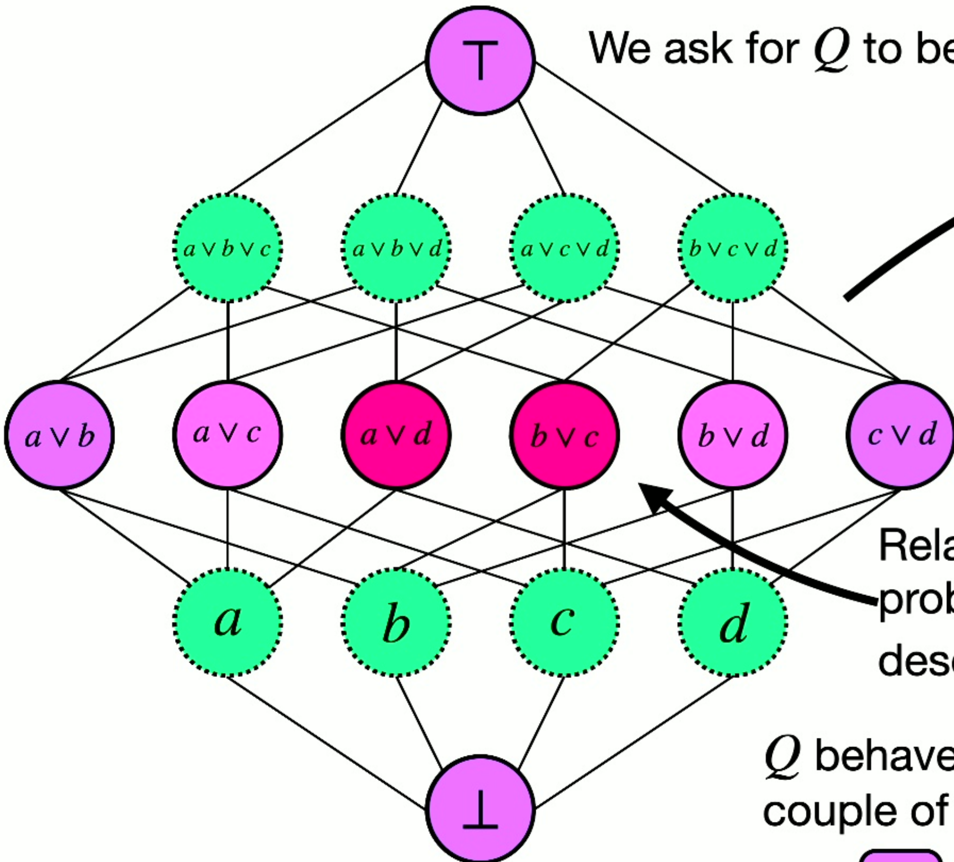
From logic to (quasi-)probability

We ask for Q to be a P on the distinguishable sub-lattice

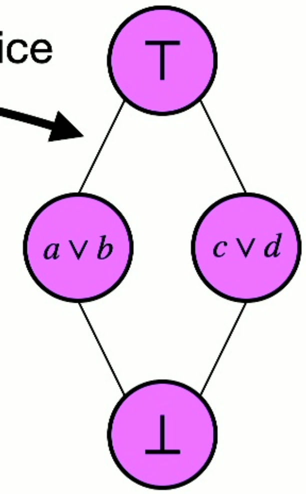




From logic to (quasi-)probability

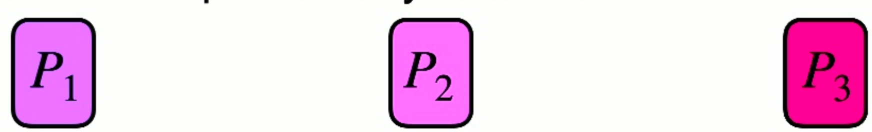


We ask for Q to be a P on the distinguishable sub-lattice



Relaxing the lattice from a logical lattice to a quasi-probability lattice we find out that these can be described by Ps function.

Q behaves as a different P function for every couple of complementary elements.



TIF Assumption 1*: Maximum resolution

Our experimental knowledge is limited to be always uncertain about a fixed number of statements d.

We call d the *resolution* of the model.

➡ The knowledge about d atomic statements is inaccessible.

If a set of atomic statements has associated probabilities $\vec{q}$, how do we quantify the number of inaccessible statements?

How to quantify the number of elements that are indistinguishable/that are inaccessible?

Accessibility and quasi-probability

The inaccessibility measure χ

If the probability of the atomic statements is $\vec{p}_{\frac{1}{d}} = (1/d, \ldots, 1/d)$, it means that we cannot distinguish between d of them: there are d inaccessible statements $\implies \chi((1/d, \ldots, 1/d)) = d$.

If the probability of the atomic statements is $\vec{p} = (1/2, 1/2, 0, \ldots, 0)$ it means that we cannot distinguish between 2 of them there are 2 inaccessible elements $\implies \chi((1/2, 1/2, 0, \ldots)) = 2$.

Request 1: Counting

$$\chi\Big((\underbrace{\frac{1}{d}, \ldots, \frac{1}{d}}_{d}, 0, \ldots, 0)\Big) = d$$

Request 2: Monotonicity

$$\forall \vec{p} \neq \vec{p}_{\frac{1}{d}} \in \Delta_d, \quad \chi\left(\vec{p}_{\frac{1}{d}}\right) > \chi(\vec{p})$$

Request 3: Symmetry

$$\chi(P_\pi \vec{p}) = \mathcal{X}(\vec{p})$$

Request 4: Multiplicativity

$$\chi(\vec{p} \otimes \vec{q}) = \chi(\vec{p})\chi(\vec{q})$$

Ansatz

$$\chi(\vec{p}) = \Big(\sum_{i=1}^{D} f_i(p_i)\Big)^z$$

f_i measurable functions

$z \in \mathbb{R}$

The inaccessibility measure χ

Lemma:

Given a probability vector $\vec{p} = (p_1, p_2, \ldots, p_n)$ with $n \geq 3$ the most general χ characterised by the requests form the 1-parameter family χ of uncertainty measure defined as

$$\chi_c(\vec{p}) = \frac{1}{\sqrt[c-1]{\sum_{i=1}^{D} p_i^c}},$$

with $c \in \mathbb{R}_+$ and $c \neq 1$.

Examples:

$$\chi_2\left((1,0)\right) = 1 \qquad \chi_2\left((\frac{1}{2},\frac{1}{2})\right) = 2 \qquad \chi_2\left((\frac{1}{3},\frac{2}{3})\right) = \frac{9}{5}$$

$$\chi_2\left((1,0,0)\right) = 1 \qquad \chi_2\left((\frac{1}{2},\frac{1}{2},0)\right) = 2 \qquad \chi_2\left((\frac{1}{2},0,\frac{1}{2})\right) = 2 \qquad \chi_2\left((\frac{1}{3},\frac{1}{3},\frac{1}{3})\right) = 3$$

The inaccessibility measure χ

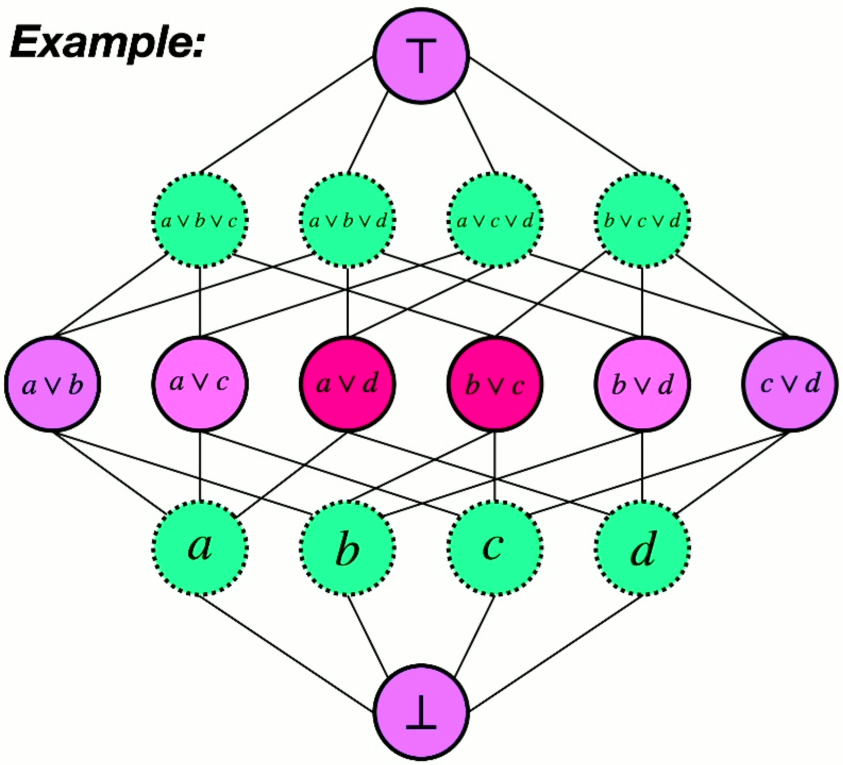
$$\chi(\vec{p}) = \frac{1}{\sum_{i=1}^{D} p_i^2},$$

* Quasi-probabilities of the Theory of Inaccessible Information

The inaccessibility measure χ

Assumption 1 can be recasted as asking for the inaccessibility of the $\vec{q}$ to be lower bounded by the resolution d.

Example:



Suppose $\vec{q}_\# = (1/3,2/3,0,0)$. In this case one would distinguish the non-distinguishable elements a and b.

$$\chi_2(\vec{q}_\#) = \frac{9}{5}$$

If $\vec{q}_v = (1/2,1/2,0,0)$. In this case one cannot distinguish a from b.

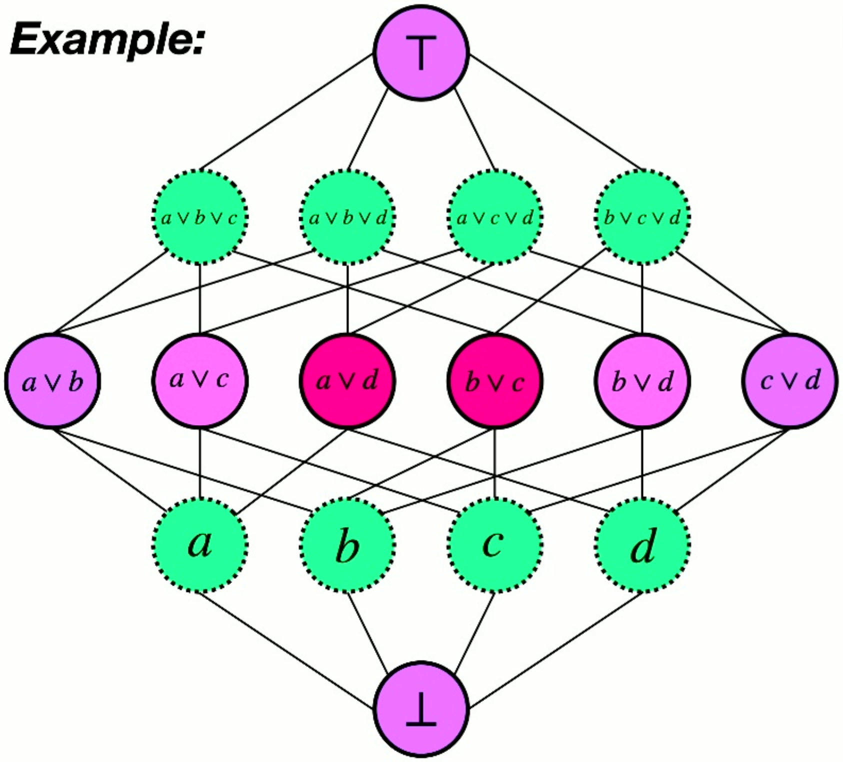
$$\chi_2(\vec{q}_v) = 2$$

$$Q(a \vee b) = q_1 + q_2,$$
$$Q(c \vee d) = q_3 + q_4,$$
$$\cdots$$

The inaccessibility measure χ

In order for **assumption 1** to be satisfied, the inaccessibility of the $\vec{q}$ has to be lower bounded by the dimension d.

Example:



Examples of possible $\vec{q}$ with $\chi_2(\vec{q}) = 2$:

$$\vec{q} = (\frac{1}{2}, \frac{1}{2}0,0)$$

$$\vec{q} = (\frac{1}{2},0,\frac{1}{2},0)$$

$$\vec{q} = (0.678422,0.156921,0.0543703,0.110287)$$

$$\vec{q} = (0.390021, -0.181366,0.36504,0.426305)$$

The inaccessibility measure χ

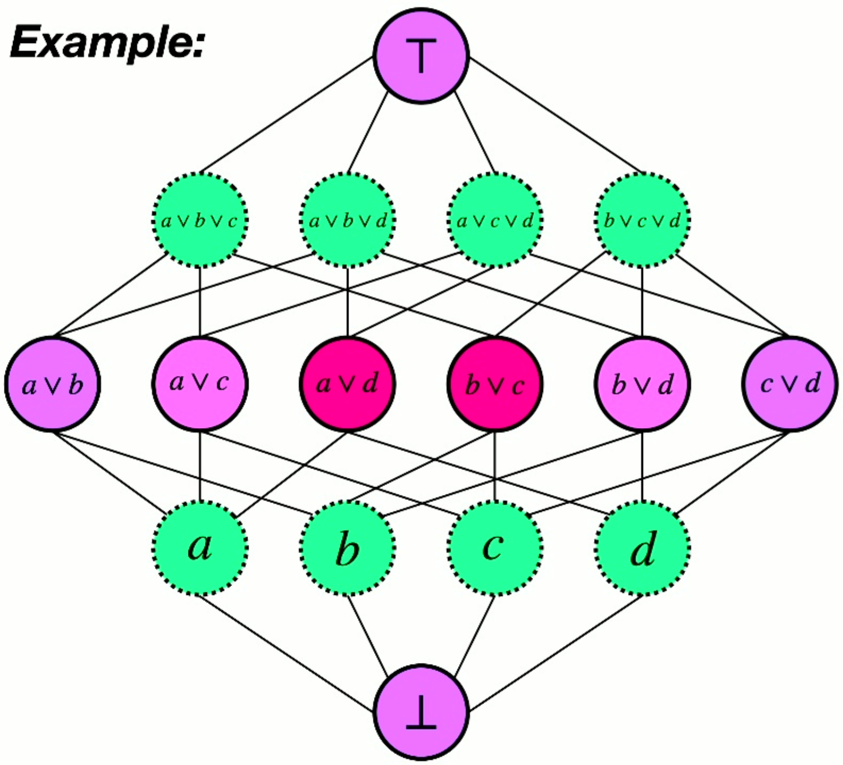
$$\chi(\vec{p}) = \frac{1}{\sum_{i=1}^{D} p_i^2},$$

* Quasi-probabilities of the Theory of Inaccessible Information

The inaccessibility measure χ

Assumption 1 can be recasted as asking for the inaccessibility of the $\vec{q}$ to be lower bounded by the resolution d.

Example:



Suppose $\vec{q}_{\#} = (1/3,2/3,0,0)$. In this case one would distinguish the non-distinguishable elements a and b.

$$\chi_2(\vec{q}_{\#}) = \frac{9}{5}$$

If $\vec{q}_{v} = (1/2,1/2,0,0)$. In this case one cannot distinguish a from b.

$$\chi_2(\vec{q}_{v}) = 2$$

$$Q(a \vee b) = q_1 + q_2,$$
$$Q(c \vee d) = q_3 + q_4,$$
$$\dots$$

Characterisation of allowed distributions

For $d = 2$ and $D = 4$ all the possible quasi-probability distributions compatible with assumption 1 are easily characterisable.

These are all the quasi-probability distribution $\vec{q} = (q_1, q_2, q_3, q_4)$ such that:

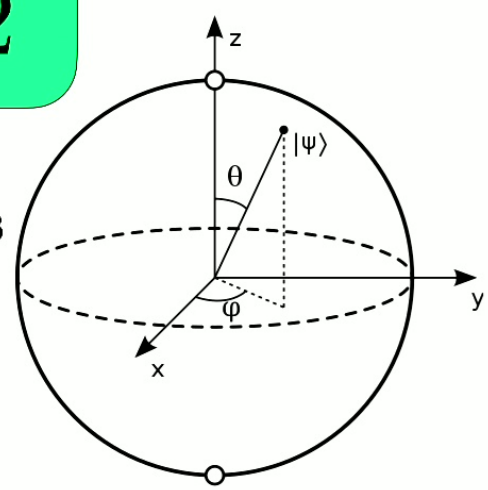
$$\begin{cases} \sum_i q_i = 1, \\ \chi(\vec{q}) \geq 2, \\ (q_i + q_j) \in [0,1] \ , \ \forall i \neq j, \end{cases}$$

Only for $d = 2$ and $D = 4$ →

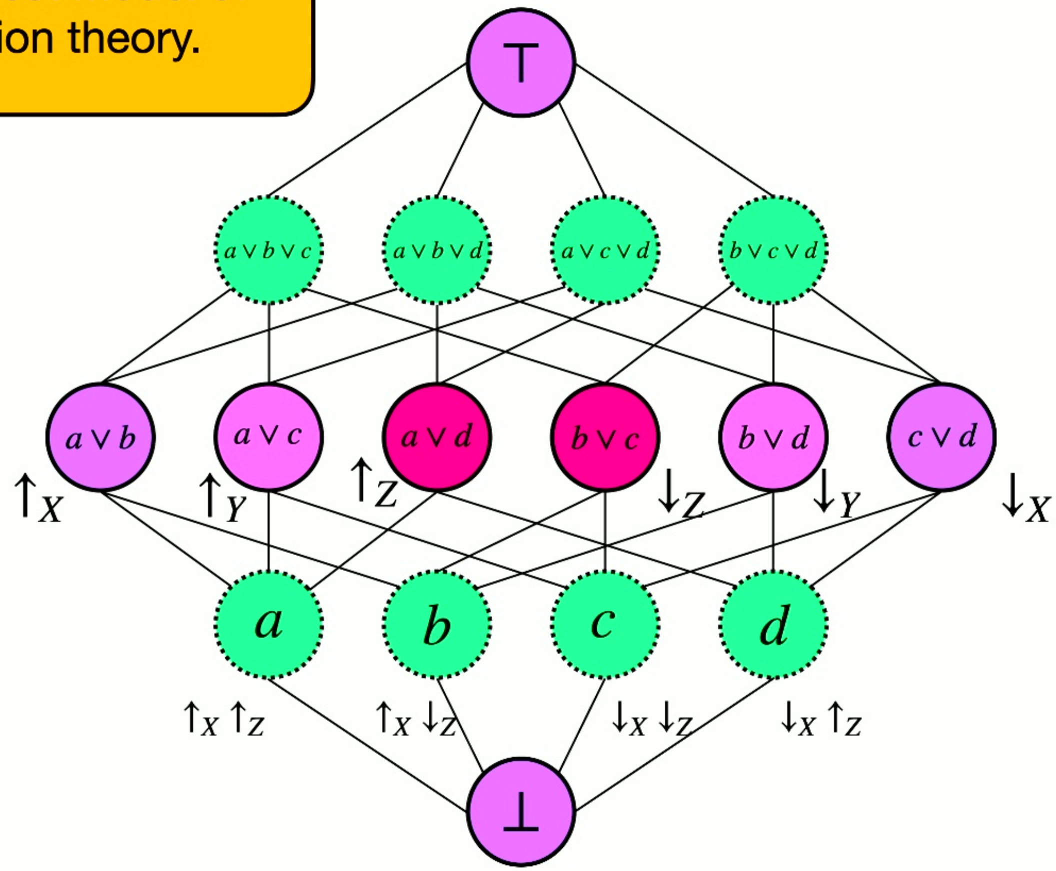
$$\begin{cases} \sum_i q_i = 1 \\ \mathcal{X}(\vec{q}) \geq 2 \end{cases}$$

$$\begin{cases} \sum_{i=1}^{4} q_i = 1 \text{ an hyperplane in } \mathbb{R}^4 \\ \chi(\vec{q}) \geq 2 \implies \sum_{i=1}^{4} q_i^2 \leq \frac{1}{2} \text{ an hyperball in } \mathbb{R}^4 \end{cases}$$

→ A ball in $\mathbb{R}^3$



The qubit is the simplest model of inaccessible information theory.



From quasi-probabilities to quantum states for $d = 2$

Theory of inaccessible information d=2

From the lattice properties

$$\vec{q} \; s.t. \begin{cases} \sum_i q_i = 1 \\ \mathcal{X}(\vec{q}) \geq 2 \end{cases}$$

Assumption 1.

$$\Longleftrightarrow$$

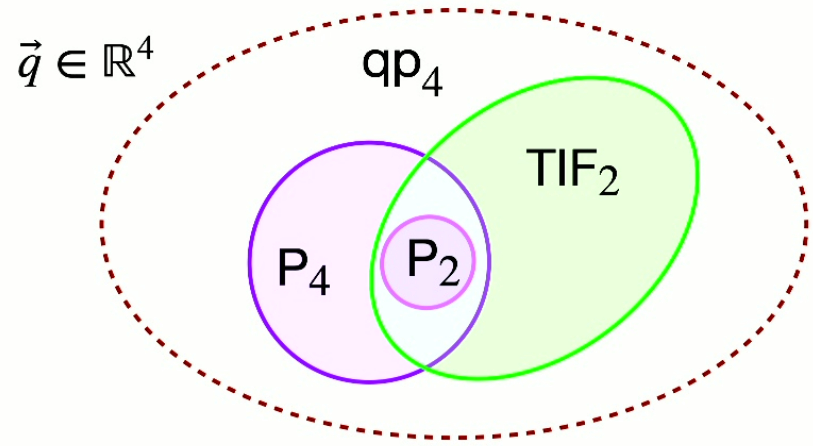
Quantum mechanic's qubit

$$\rho \; s.t. \begin{cases} \mathrm{Tr}[\rho] = 1 \\ \mathrm{Tr}[\rho^2] \leq 1 \end{cases}$$

Quantum as constrained probability

or

Probability as constrained quantum?



qp_4

$$\sum_{i=1}^{4} q_i = 1$$

add a constraint → P_4

$$\vec{q} \in [0,1]^4$$

add a constraint → TIF_2

$$\chi(\vec{q}) \geq 2$$

coarse graining →

(there is a lot of freedom in how to coarse-grain)

P_2

$$\begin{cases} \vec{q} = \left(\frac{p}{2}, \frac{p}{2}, \frac{1-p}{2}, \frac{1-p}{2}\right) \\ \vec{p} = (p, 1-p) \\ p \in [0,1] \end{cases} \quad \begin{cases} \vec{q} = \left(\frac{p}{2}, \frac{1-p}{2}, \frac{p}{2}, \frac{1-p}{2}\right) \\ \vec{p} = (p, 1-p) \\ p \in [0,1] \end{cases}$$

$$\begin{cases} \vec{q} = \left(\frac{1-p}{2}, \frac{1-p}{2}, \frac{p}{2}, \frac{p}{2}\right) \\ \vec{p} = (p, 1-p) \\ p \in [0,1] \end{cases} \quad \dots$$

Fine-完

Thank you