Title: Anomalies of Non-Invertible Symmetries in 3+1d

Speakers: Po-Shen Hsin

Series: Quantum Matter

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Abstract: Anomaly of global symmetry is an important tool to study dynamics of quantum systems. In recent years, new non-invertible global symmetries are discovered in many quantum systems such as the 2d Ising model, Standard Model like theories, and lattice models. I will discuss constraints on the dynamics in 3+1d systems using anomalies of non-invertible symmetries from the perspective of bulk-boundary correspondence. The discussion is based on the work https://arxiv.org/abs/2308.11706 with Clay Cordova and Carolyn Zhang.

Zoom link https://pitp.zoom.us/j/99162815973?pwd=M01nZXJIN2tCRjhuZlljNU1id01XQT09

Anomalies of Non-Invertible Symmetry

Po-Shen Hsin

UCLA & Simons Collaboration on Global Categorical Symmetries, 2023

2308.11706 with Clay Córdova and Carolyn Zhang (U Chicago & Harvard)

Also 2111.01139 and 2204.09025 with Choi, Córdova, Lam, Shao





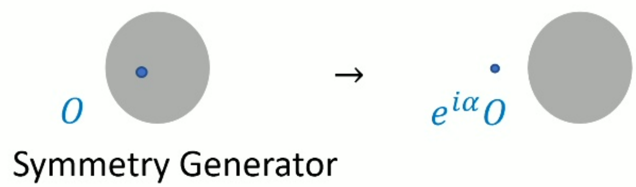
Symmetry as topological operators

[E.g. Gaiotto,Kapustin,Seiberg,Willett],[Wen],[McGreevy]...

- Symmetry: conserved charge Q of quantum system
- Conservation law: symmetry generator $e^{i\alpha Q}$ is topological

Deform symmetry generator from time 1 to time 2 preserves the total charge

- Symmetry = topological operators (topological defects)
- Transformation



- Topological operators of various dimensions: volume, surface, line, point...
  Codimension n ⇔ "(n-1)-form symmetry"

What structure does a symmetry have?

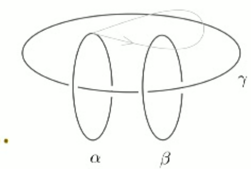
- Fusion rule. Invertible vs Non-invertible Symmetry

$$a \times b = c \quad \text{vs.} \quad a \times b = \sum c_i$$

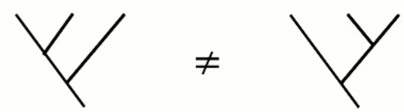
- Correlation functions of symmetry defects such as braiding

Eg: braiding of 3 loop excitations (surface oprs) in 3+1d

E.g. [Wang,Levin],[Else,Nayak],[Yoshida]...



- (Non) Associativity? Fusing the symmetry defects in different ways can produce different outcomes



- (Non) Commutativity? Fusing A with B $\neq$ fusing B with A

What are examples of non-invertible symmetries?

- 2d Ising model of spins: Kramers-Wannier duality

D:

- Symmetry with no inverse: “non-invertible”

$$D \times D = 1 + \eta, \qquad \eta\text{: spin flipping symmetry}$$

- Importance: non-invertible symmetries in many quantum systems

Example: Standard Model like theories, lattice gauge models, 4d Maxwell theory, supersymmetric Yang-Mills theory, …

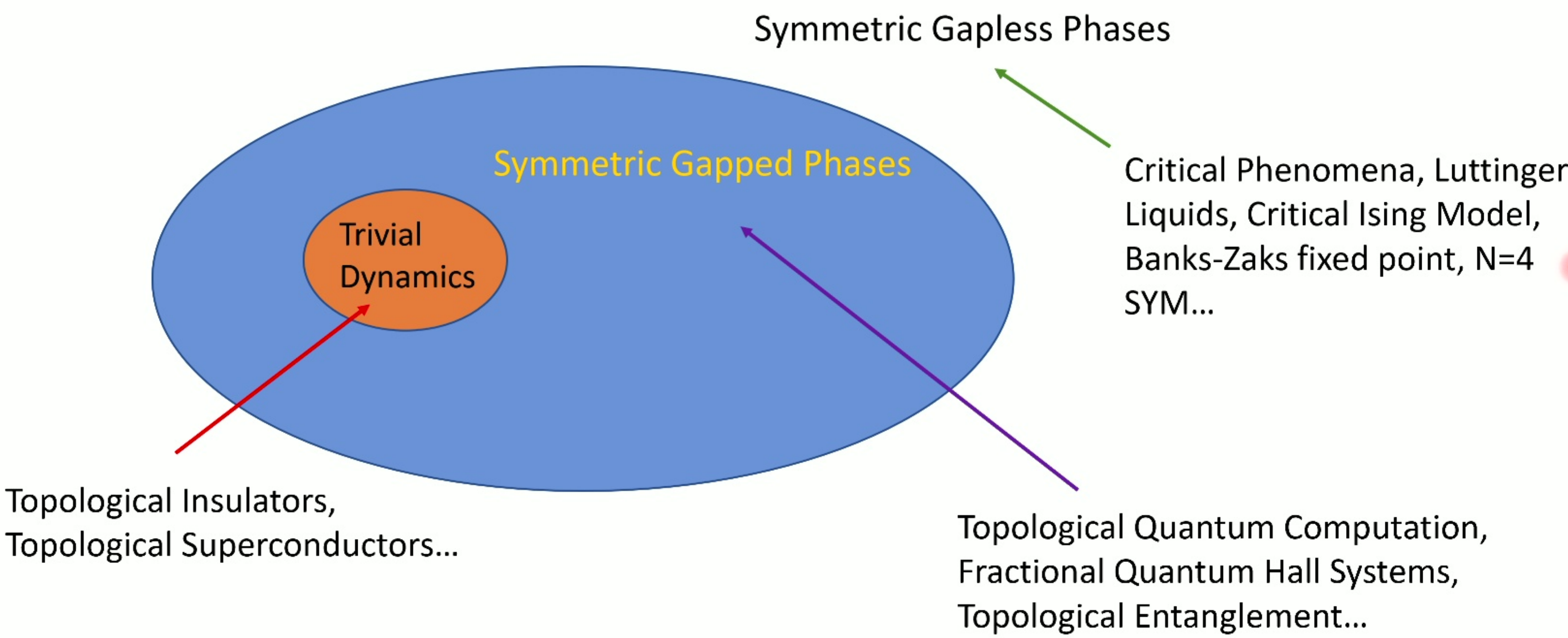
[Choi,Cordova, PH,Lam,Shao], [Kaidi, Ohmori, Zheng], [PH,Wang],[Choi,Lam,Shao]…

$$D \times \overline{D} = \sum \eta(S)$$

$\eta(S)$: supported on closed surfaces S on the domain wall

What can symmetry tell us about dynamics?

- Focus on unbroken symmetry



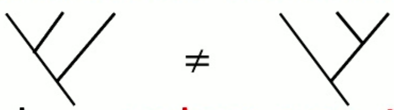
What does symmetry tell us about dynamics?

- Invertible symmetries: nontrivial symmetry structure (such as braiding or non-associativity) ⇒ nontrivial dynamics

E.g. [Callan,Harvey],['t Hooft],[Chen,Gu,Liu,Wen],[Nayak,Else]...

- Non-invertible symmetries: not true

Many counterexamples in 1+1d: e.g. $\mathbb{Z}_2 \times \mathbb{Z}_2$ Tambara-Yamagami symmetry with non-associativity realized in trivial dynamics [Thorngren,Wang]...



$$D \times D = 1 + \eta_1 + \eta_2 + \eta_1\eta_2, \qquad \eta_1^2 = \eta_2^2 = 1$$

- Hard problem: what non-invertible symmetry implies non-trivial dynamics? Previous studies: in 1+1d use complicated fiber functor

This talk: simple solution for general non-invertible symmetries

Symmetry as bulk TQFT

[Many references]

- All symmetry structures can be summarized into a bulk TQFT:
- Boundary symmetry = bulk symmetry (topological defects)

(1) ending on the boundary, or (2) restricted to the boundary

- Possible dynamics with unbroken symmetry = possible symmetric boundary conditions of bulk TQFT
- If bulk does not have symmetric gapped boundary ⇒ symmetry implies no symmetric gapped phase
- If bulk does not have trivial boundary ⇒ symmetry implies nontrivial dynamics

Gapped boundaries of bulk TQFT

[Kapustin],[Kapustin,Saulina],[Levin],[Barkeshli,Jian,Qi],[Kong],[Kong,Wen,Zheng]…

- Gapped boundaries: ``Lagrangian algebras" or Lagrangian subgroups for Abelian TQFTs
- Symmetric gapped boundaries: symmetry preserving ``Lagrangian subgroups"
- Relatively easy to study in bulk TQFTs that are Abelian, i.e. all excitations obey group-like fusion rule
- Lagrangian subgroup: boson excitation $b_1, \cdots, b_n$ (isotropic) such that condensing the excitations produces trivial theory (coisotropic)

Other excitations braid nontrivially with at least one b_i: removed after condensing $\{b\}$

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Example of gapped boundary: Z_2 gauge theory

[Kitaev,Kong],[Beigi,Shor,Whalen],...

- Example: Z_2 gauge theory in 2+1d $L = \frac{2}{2\pi} adb$

$\mathrm{Z}_2 \times \mathrm{Z}_2$ particles: electric charge $\mathrm{e}^{\mathrm{i}\int a}$, magnetic charge $\mathrm{e}^{\mathrm{i}\int \mathrm{b}}$

Statistics: general dyon (q_e, q_m) has statistics $q_e q_m/2$

- Two topological boundary conditions:

(1) $a| = 0$: e condensed boundary

(2) $b| = 0$: m condensed boundary

- Two Lagrangian subgroups: (1) generated by $(1,0)$, (2) generated by $(0,1)$

Example: 1+1d Symmetry ⇔ 2+1d TQFT

[Cordova,Zhang]...

- Symmetry: ``Ising symmetry'' (e.g. in 1+1d critical Ising model)

$$(\eta, D): \qquad \eta^2 = 1, \qquad D \times D = 1 + \eta$$

D: Kramers-Wannier duality. η: Z_2 spin-flipping symmetry

- Bulk TQFT Z_2 gauge theory: gauging Z_2 spin-flipping symmetry
- What is D in the bulk? Electromagnetic duality: exchange electric/magnetic charge

$$(q_e, q_m) \leftrightarrow (q_m, q_e)$$

[Freed,Teleman]

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Example: 1+1d Symmetry ⇔ 2+1d TQFT

- Dynamics question: can there be symmetric gapped phase?
- Method: is there gapped boundary of TQFT preserving em duality?
- What are the gapped boundaries?

2+1d TQFT Z_2 gauge theory has two types of boundaries: condensation of electric charge or magnetic charge

- No gapped boundaries of TQFT preserve the em duality symmetry

i.e. no Lagrangian subgroup invariant under the em duality symmetry

- Therefore, no symmetric gapped phase for Ising symmetry in 1+1d

Classification of TQFTs in 4+1d

[Johnson-Freyd,Yu]...

- Particle, loop, membrane excitations (line, surface and volume operators). Particles braid with membrane, do not braid with loop
- 4+1d: particles can only be bosons or fermions
- Suppose there are fermion particles, it can be obtained from a spin TQFT with local fermions by gauging the fermion parity (bosonization)

⇒ Sufficient to classify bosonic TQFTs with boson particles and spin TQFTs

- In spin TQFTs we can condense boson= fermion+ local fermion
- In both cases, we can condense all particles ⇒ leaving only loop excitations (membranes are confined: braid with particles)

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Classification of TQFTs in 4+1d with only loops

[Johnson-Freyd, Yu]

- What are 4+1d TQFT with only loop excitations?
- <u>Claim</u>: the loops must obey Abelian fusion rule
- Proof: suppose the loops s, s' obey non-Abelian fusion

$$s \times s' = \sum s_i$$

Shrinking the loops give trivial particle $1 \times 1 = \sum 1$, consistent only if there is one s_i in fusion outcome: group-like Abelian fusion rule

- Abelian loops are described by two-form Abelian gauge theory

$$L = \sum_{I,J} \frac{K_{IJ}}{4\pi} \int B^I dB^J$$

4+1d Z_N two-form gauge theory

- BF theory action: $\frac{N}{2\pi}\int b_e d b_m$, 2-forms b_e, b_m
- What are the observables? Bosonic loop excitations (surface operators)

$\mathrm{Z}_N \times \mathrm{Z}_N$ loops (q_e, q_m) : electric loop $\mathrm{e}^{iq_e\int b_e}$, magnetic loop $\mathrm{e}^{iq_m\int \mathrm{b_m}}$

- Statistics: $(1,0)$ and $(0,1)$ have mutual braiding $e^{2\pi i/N}$
- What are topological boundary conditions?
- Lagrangian subgroup: condensing the loops give trivial theory

$$\frac{1}{2\pi} b'_e d b'_m, \qquad \begin{pmatrix} b'_e \\ b'_m \end{pmatrix} = M \begin{pmatrix} b_e \\ b_m \end{pmatrix}, \qquad M \in GL(2,Z)$$

$N = \det M$; Gapped boundaries of Z_N $\Leftrightarrow$ M: $\det M = N$

Condensed operators: $e^{i\int b'_e} = 1, e^{i\int b'_m} = 1$ i.e. Lagrangian $\Lambda = \mathrm{Img}\, M$

Example: 3+1d Symmetry ⇔ 4+1d TQFT

- Kramers-Wannier like duality in 3+1d theory with anomaly-free Z_N one-form symmetry

$$D_{(N)} \times \overline{D}_{(N)} = \sum_{\text{surface } S} \eta(S), \qquad \eta^N = 1$$

- $D_{(N)}$: gauge Z_N one-form symmetry on half space, a topological defect for theory self dual under gauging
- Bulk TQFT: Z_N two-form gauge theory by gauging one-form symmetry
- What is $D_{(N)}$ in bulk? electromagnetic duality $(q_e, q_m) \leftrightarrow (q_m, -q_e)$

Boundary global 1-form symmetry: e condensed
Boundary gauge 1-form symmetry: m condensed

Electromagnetic duality

Example: 3+1d Symmetry ⇔ 4+1d TQFT

- Dynamics question: can there be symmetric gapped phase?
- Method: is there gapped boundary M of TQFT preserving em duality?
- What are the symmetric gapped boundaries?

em duality: $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = iY$, Symmetric: $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^{-1} M \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = M$

Symmetric gapped boundary: $M = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} = \alpha I + \beta iY \in GL(2, \mathrm{Z}_N)$

$N = \det M = \alpha^2 + \beta^2$, Lagrangian $\Lambda = \mathrm{Img}\, M$

- Symmetry $D_{(N)}$ never realized in symmetric gapped phase if $N \neq$ sum of two integer-square $N \neq 2, 4, 5, 8, 9, 10, 13, 16, 17, 18, 20, 25, 26, 29, 32, 34, 36, 37 \ldots$

Alternative argument with same result: [Apte, Cordova, Lam]

Example: 3+1d Symmetry ⇔ 4+1d TQFT

- Generalization: bulk TQFT has huge symmetry by permuting loop excitation, not just electromagnetic duality
- Symmetry ST^n: $\begin{pmatrix} 0 & 1 \\ -1 & n \end{pmatrix}$; non-invertible symmetry in 3+1d $\mathrm{D}(ST^n)$
- Symmetric gapped boundaries M: $\begin{pmatrix} 0 & 1 \\ -1 & n \end{pmatrix}^{-1} M \begin{pmatrix} 0 & 1 \\ -1 & n \end{pmatrix} = M$

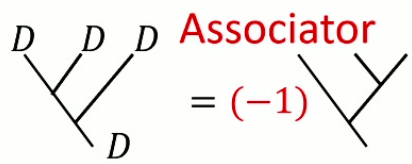
Given by $\mathrm{M} = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha + n\beta \end{pmatrix} \Rightarrow N = \det M = \alpha^2 + \beta^2 + n\alpha\beta$

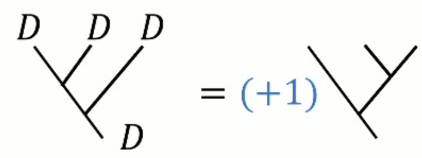
- Symmetry $\mathrm{D}(ST^n)$ never realized in symmetric gapped phase if $N \neq \alpha^2 + \beta^2 + n\alpha\beta$ for two integers α, β

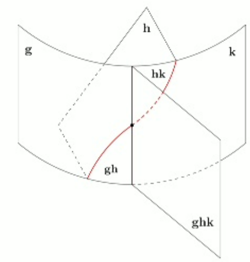
$n = 1$: $N \neq$ 3, 4, 7, 9, 12, 13, 16, 19, 21, 25, 27, 28, 31, 36, 37, 39, 43, 48, 49, 52, 57, 61, 63, 64, 67, 73, 75, 76, 79, 84, 91, 93, 97, 108, ...

Non-Associativity: “Frobenius-Schur indicator”

- Data on the symmetry: fusion rule and associator
- Example: 1+1d two symmetries with the same fusion $D \times D = 1 + \eta$

“Not associative”: 

“Associative”: 



- Symmetry generator is on the boundary of bulk invertible symmetry $G = \mathrm{Z}_2$, associators are classified by G-SPT phases in the bulk

$H^3(\mathrm{Z}_2, U(1)) = \mathrm{Z}_2$ two symmetries with different associators

- Associators for fusion involving only D: “Frobenius-Schur indicator”

Physics review by [Kitaev]

- How to describe associators using bulk TQFT? Stack G-SPTs in the bulk

Associator and Symmetric Gapped Phases

- Suppose symmetries have nontrivial associator, is that an obstruction to symmetric gapped phase?
- Asking if the stacked G-SPT phases have symmetric gapped boundaries
- For associator described by "group-cohomology SPTs"

$H^{d+1}(G, U(1))$ for bulk dimension d+1 [Dijkgraaf,Witten],[Chen,Gu,Liu,Wen]...

G-SPTs for finite G always have a symmetric gapped boundary ⇒ no obstruction [Wang,Wen Witten]

- For certain "beyond group cohomology" SPTs there is no gapped boundary
- Obstructions to Symmetric Gapped Phases: (1) find symmetric Lagrangian (2) "beyond group cohomology" associator

"Symmetry-Enforced Gaplessness": [Wang,Senthil],[Cordova,Ohmori]

Associator enforced gaplessness

- Partition function of unitary 3+1d TQFTs on K3 is nonzero [Cordova,Ohmori]

$$Z[K3] \neq 0$$

- Associator that produces nontrivial phase on K3: cannot be realized in 3+1d TQFTs
- Example: associator for beyond group cohomology 4+1d SPT

$$\int A(p_1/3)$$

- On K3, $p_1 = -48$, partition function transforms under $A \to A + d\phi$

$$Z[K3] = Z[K3]e^{-16i\phi}$$

- For $16\phi \neq 2\pi$ integer: $Z[K3] = 0$ cannot realize by gapped phase
- (Remark: similar argument applies for Witten anomaly on S^4)

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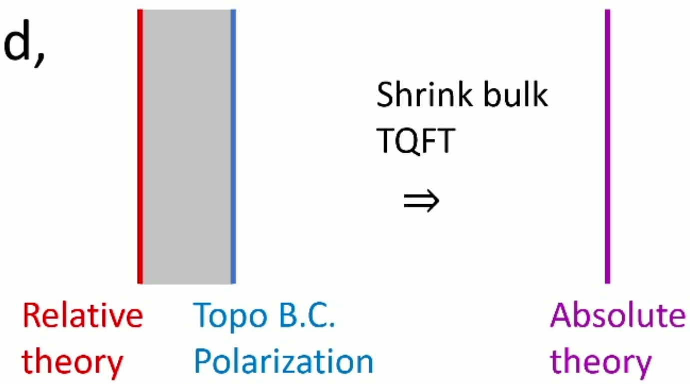
Get absolute theories by fixing polarization

[Freed,Teleman],[Witten],[Eckhard,Kim,Schafer-Nameki,Willett],[Gukov,PH,Pei]…

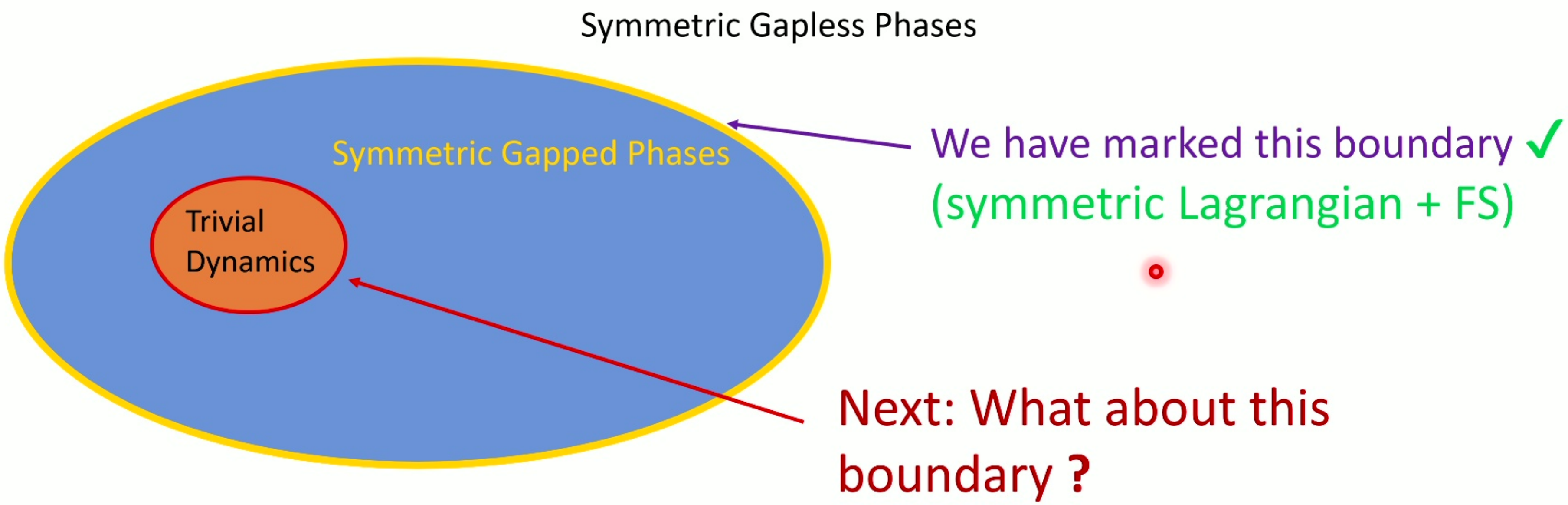
- A boundary is not a theory on its own: there is a bulk TQFT. Such theory is called a relative theory.
- To define an absolute theory without the bulk TQFT, we need to fix a polarization. Polarization may or may not exist depending on the bulk.
- Polarization: topological boundary conditions of bulk TQFT [Gukov,PH,Pei]
- If relative theory is symmetric and gapped,

Absolute theory is also symmetric gapped



This Talk So Far

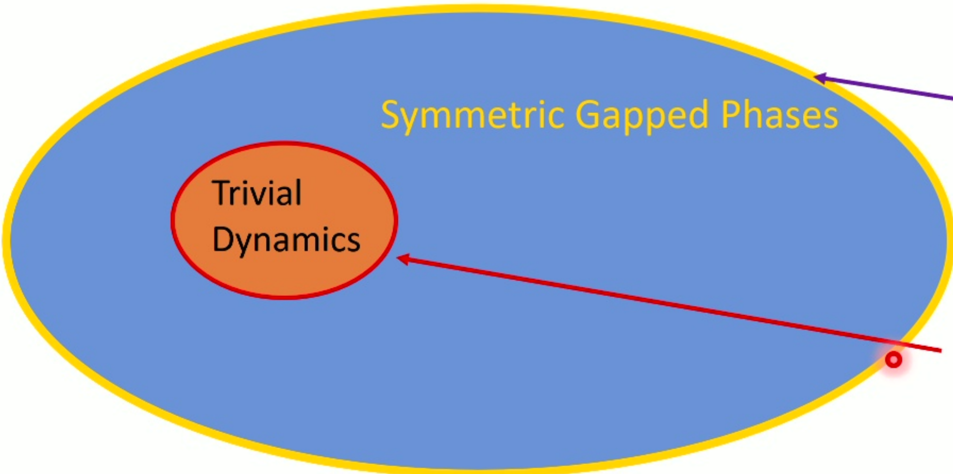
- Focus on unbroken symmetry



Symmetry Bootstrap from Bulk Viewpoint

- Focus on unbroken symmetry

Symmetric Gapless Phases



We have marked this boundary ✓
(symmetric Lagrangian + FS)

We have marked this boundary ✓
(& Polarization st $\Lambda \cap \Lambda_{\text{pol}} = 0$
+ FS match anomaly in defect TQFT)

Conclusion and Outlook: Power of TQFT

- Bulk TQFT is a powerful tool to study dynamical constraint imposed by symmetry in quantum systems
- We utilize bulk TQFT to study dynamical implications from novel and general non-invertible symmetries, which gives (1) simpler method reproduces known results in 1+1d and (2) give new results in 3+1d
- Important questions for TQFTs to understand dynamics of general quantum systems

(1) What are the gapped boundaries and topological defects

(2) What kind of anomalies can be realized in TQFT

Thank you