Title: Oper and Integrable Systems

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Series: Mathematical Physics

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Abstract: I will introduce (q-)opers on a projective line in the presence of twists and singularities and will discuss the space of such opers. We will see how Bethe Ansatz equations for quantum spin chains and energy level equations of classical soluble models of Calogero-Ruijsenaars type naturally appear from the oper construction. Both can also be described in terms of so-called QQ-systems, which have their origins in algebra and representation theory. Our construction is universal and works for any simple, simply-connected complex Lie group G.

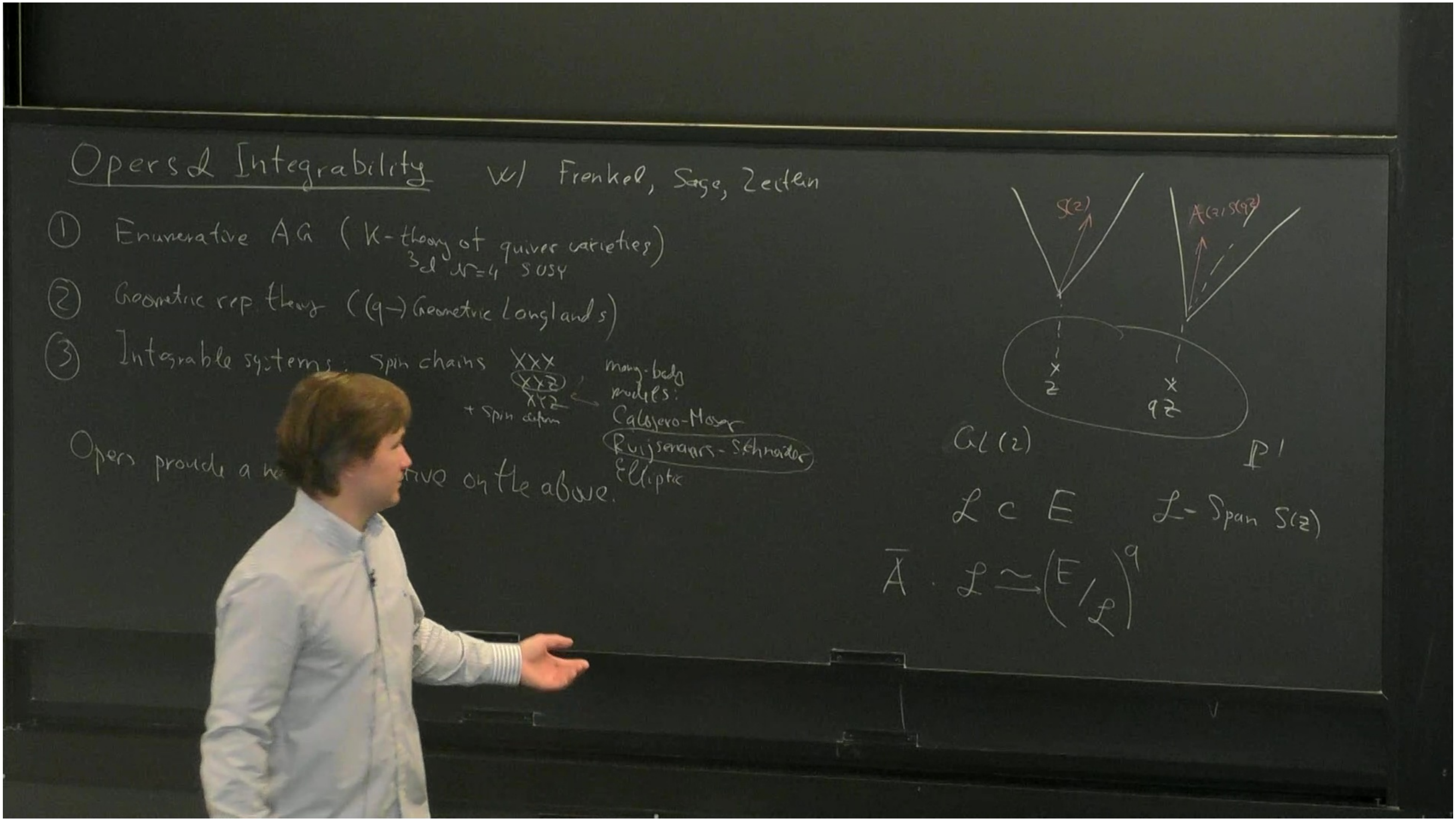
Zoom link https://pitp.zoom.us/j/94393767928?pwd=Qkg2eG5CME5sQjNEV2lDVVNyMktDQT09

Opers & Integrability   w/ Frenkel, Sage, Zeitlin

(1) Enumerative AG (K-theory of quiver varieties)
3d $\mathcal{N}=4$ SUSY

(2) Geometric rep theory ((q-) Geometric Langlands)

(3) Integrable systems : Spin chains XXX
XXZ
XYZ $\longleftrightarrow$ many-body models:
+ spin deform
Calogero-Moser
Ruijsenaars-Schneider
Elliptic



Opers & Integrability w/ Frenkel, Sage, Zeitlin
① Enumerative AG (K-theory of quiver varieties)
3d N=4 SUSY
② Geometric rep. theory ((q-)Geometric Langlands)
③ Integrable systems: Spin chains XXX
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many-body models:
Calogero-Moser
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Elliptic
Opers provide a
on the above.
S(z)
A(z)S(qz)
z
qz
GL(2)
P^1
L ⊂ E
L - Span S(z)
Ā · L ≃ (E/L)^q

Opers provide a new perspective on the above. Elliptic

$\mathcal{L} \subset E$ $\quad$ $\mathcal{L}$ - Span $S(z)$

$\bar{A} : \mathcal{L} \xrightarrow{\sim} \left(E/\mathcal{L}\right)^q$

1 (G,q)-opers $\quad$ G - simple, simply-connected Lie group $/\mathbb{C}$

$G = GL(r+1)$

Let E be holomorphic vector bundle of rank $r+1$ over $\mathbb{P}^1$

$M_q : \mathbb{P}^1 \to \mathbb{P}^1$

$z \mapsto qz$, $q \in \mathbb{C}^\times$, $\quad E^q$ - pullback under M_q

Def: A meromorphic $(GL(r+1), q)$-oper on $\mathbb{P}^1$ is $(A, E, \mathcal{L}_\bullet)$

where $\mathcal{L}^\bullet = \mathcal{L}_{r+1} \subset \mathcal{L}_r \subset \dots \subset \mathcal{L}_2 \subset \mathcal{L}_1 = E$

complete flag of subbundles

$A \in \mathrm{Hom}_{\mathcal{O}_U}(E, E^q)$

U open Zariski-dense subset of $\mathbb{P}^1$

Satisfying:

i) $A\,\mathcal{L}_i \subset \mathcal{L}_{i-1}$

ii) Restriction of A to $V = U \cap M_q^{-1}(U)$

$\bar{A}_i : \mathcal{L}_i/\mathcal{L}_{i+1} \xrightarrow{\sim} \left(\mathcal{L}_{i-1}/\mathcal{L}_i\right)^q$

are isomorphisms on V

$\det A = 1 \Rightarrow (SL(r+1), q)$-oper

q-gauge transformations

$g(z) \in SL(r+1)(z)$ - change of trivialisation

$$A(z) \longmapsto g(qz)\,A(z)\,g(z)^{-1}$$

Let $\mathcal{L}_{r+1} = \operatorname{Span} s(z)$

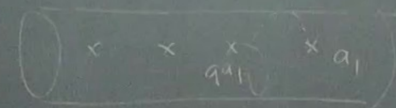
$$W_i(s)(z) = s(z) \wedge A(z)s(qz) \wedge A(z)A(qz)\,s(q^2 z) \ldots\ldots A(z)\ldots A(q^{i-2}z)\,s(q^{i-1}z) \quad \Lambda^i \mathcal{L}_{r+1}$$

ii) is equivalent to $W_i(s)(z) \neq 0$

<u>Def</u>: An $(SL(r+1), q)$-oper with reg sing at $a_1 \ldots a_L$ and weights $k_1 \ldots k_L$ is:

$$W_i(s)(z) = \Lambda(z) = \prod_{i=1}^{L} \prod_{\ell=0}^{k_i-1} (z - q^{\ell} a_i)$$

<u>Def</u>: Z-twisted q-oper.



$\exists\, g(z) \in SL(r+1)(z)$ s.t.

$$g(qz)A(z)g(z)^{-1} = Z \subset H \subset H(z) \in SL(r+1)(z)$$

Let $\mathcal{L}_{r+1} = \mathrm{Span}\, s(z)$

$$\mathcal{W}_i(s)(z) = \Big(s(z) \wedge A(z)s(qz) \wedge A(z)A(qz)\,s(q^2z) \ldots\ldots A(z)\ldots A(q^{i-2}z)\,s(q^{i-1}z) \Big) \Big|_{\Lambda^i \mathcal{L}_{r+1}}$$

ii) is equivalent to $\mathcal{W}_i(s)(z) \neq 0$

$$A(z) = \begin{pmatrix} \Lambda_1 & & & & & & \\ & \Lambda_2 & & & & 0 & \\ & & \Lambda_3 & & & & \\ & & & * & & & \\ & * & & & * & & \\ & & & & & \Lambda_r & \\ & & & & & & * \end{pmatrix}$$

Def. Miura $(SL(r+1), q)$-oper is $(E, A, \mathcal{L}^\circ, \hat{\mathcal{L}}^\circ)$,

$\hat{\mathcal{L}}^\circ$ is preserved by A

$\mathcal{F}_G, A, \mathcal{F}_{B_+}, \mathcal{F}_{B_-}$

Pick basis for $\hat{\mathcal{L}}^\circ$. $e_1, e_2, \ldots e_{r+1}$

Expect $\mathcal{L}^\circ$, $\hat{\mathcal{L}}^\circ$ to be in a generic relative position, a[...]

have additional singularities when they align.

$ab^{-1} = 1 \qquad \mathcal{F}_{B_+,x} = aB_+,\ \mathcal{F}_{B_-,x} = b[\ldots]$

$$\mathcal{D}_i(s)(z) = e_1 \wedge \ldots \wedge e_{r+1-i} \wedge s(z) \wedge A(z)s(qz) \wedge \ldots \wedge A(z)\ldots A(q^{i-2}z)\,s(q^{i-1}z)\Big) = \mathcal{W}_i(z) [\ldots]$$

$B_- \backslash G / B_+ = [\ldots]$

have additional singularities when they align:

$a \cdot b^{-1} = 1 \qquad \mathcal{F}_{B_+,x} = a B_+, \ \mathcal{F}_{B_-,x} = b \cdot B_-$

$$D_i(s)(z) = e_1 \wedge \dots \wedge e_{r+1-i} \wedge s(z) \wedge A(z)s(qz) \wedge \dots \wedge A(z)\dots A(q^{i-2}z)\, s(q^{i-1}z)\big) = \mathcal{W}_i(z) \cdot V_i(z)$$

$$B_- \backslash G / B_+ = W / G$$

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Def: Z-twisted q-oper:

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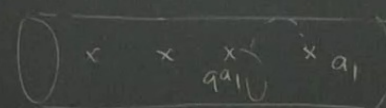
$$g(qz)A(z)g(z)^{-1} = Z \subset H \subset H(z) \in SL(r+1)(z)$$



Let $\mathcal{L}_{r+1} = \operatorname{Span} s(z)$

$$\mathcal{W}_i(s)(z) = s(z) \wedge A(z)s(qz) \wedge A(z)A(qz)\, s(q^2 z) \dots\dots A(z)\dots A(q^{i-2}z)\, s(q^{i-1}z)\big) \quad \Big| \ \Lambda^i \mathcal{L}_{r+1}$$

ii) is equivalent to $\mathcal{W}_i(s)(z) \neq 0$

$z \mapsto qz$, $q \in \mathbb{C}$,

Def: A meromorphic $(GL(r+1), q)$-oper on $\mathbb{P}^1$ is $(A, E, \mathcal{L}_\bullet)$,

where $\mathcal{L}^\bullet = \mathcal{L}_{r+1} \subset \mathcal{L}_r \subset \dots \subset \mathcal{L}_2 \subset \mathcal{L}_1 = E$

complete flag of subbundles

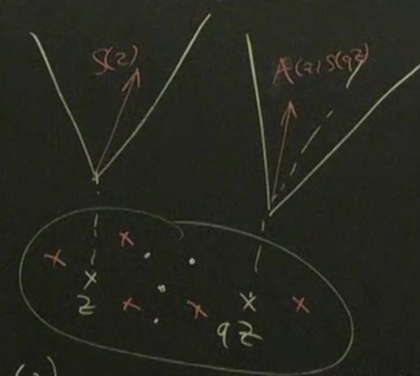
ii) Restriction of A to $V = U \cap M_q^{-1}(U)$

$$\bar{A}_i : \mathcal{L}_i / \mathcal{L}_{i+1} \xrightarrow{\sim} \left(\mathcal{L}_{i-1} / \mathcal{L}_i \right)^q$$

are isomorphisms on V

$\mathcal{D}_i(s)(z) = \det$ $\underbrace{\begin{vmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \;\; s \;\; As \end{vmatrix}}_{i}$

twisted q-Wronskian

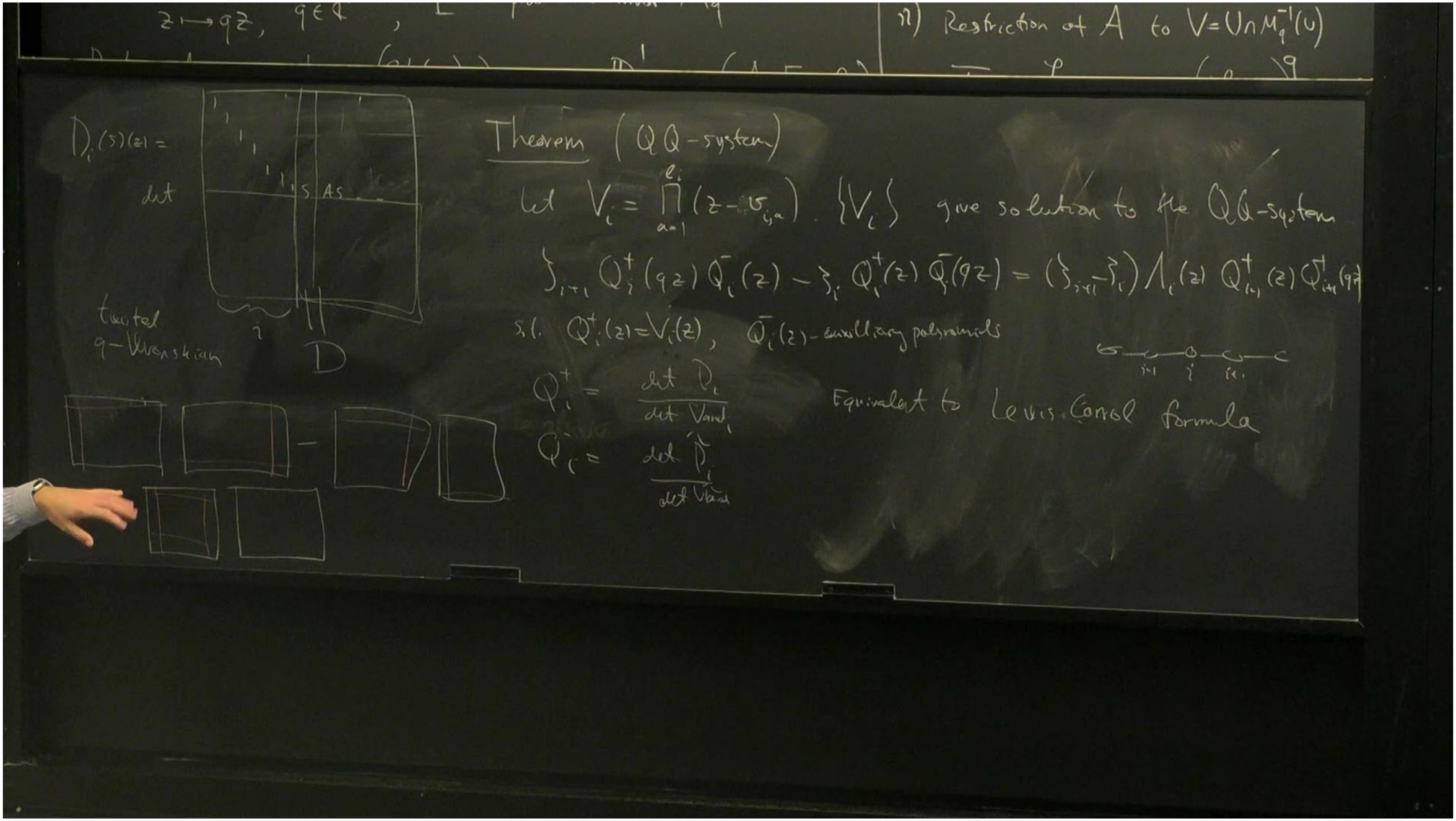


$GL(2)$

$\mathbb{P}^1 \setminus \{0, \infty\}$

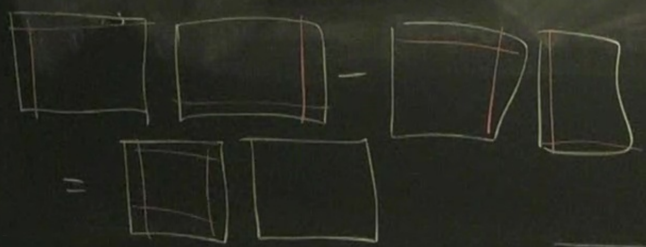
$\mathcal{L} \subset E$ $\quad$ $\mathcal{L}$ - Span $s(z)$

$$\bar{A} : \mathcal{L} \xrightarrow{\sim} \left(E / \mathcal{L} \right)^q$$



$z \mapsto qz, \quad q \in \mathbb{C}$
ii) Restriction of A to $V = U \cap M_q^{-1}(U)$
$D_i(s)(z) = \det$
$S \quad AS$
twisted q-Wronskian
D
Theorem (QQ-system)
Let $V_i = \prod_{a=1}^{\ell_i}(z - v_{i,a})$. $\{V_i\}$ give solution to the QQ-system
$\xi_{i+1} Q_i^+(qz) Q_i^-(z) - \xi_i Q_i^+(z) Q_i^-(qz) = (\xi_{i+1} - \xi_i) \Lambda_i(z) Q_{i-1}^+(z) Q_{i+1}^+(qz)$
s.t. $Q_i^+(z) = V_i(z)$, $Q_i^-(z)$ - auxiliary polynomials
$i-1 \quad i \quad i+1$
$Q_i^+ = \dfrac{\det D_i}{\det \mathrm{Vand}_i}$
$Q_i^- = \dfrac{\det \hat{D}_i}{\det \hat{\mathrm{Vand}}_i}$
Equivalent to Lewis Carroll formula

q-Wronskian $\mathcal{D}$



$$Q_i^+ = \frac{\det \mathcal{D}_i}{\det \mathrm{Vand}_i}$$

$$Q_i^- = \frac{\det \hat{\mathcal{D}}_i}{\det \mathrm{Vand}_i}$$

Equivalent to Lewis-Carroll formula

$$Z = \mathrm{diag}(\zeta_1 \dots \zeta_{r+1})$$

(G,q)-oper

$G = N_- H N_+$

$g = n_- h n_+$

Principal minors:

V_i^+ - irrep of G w/ highest weight ω_i

$$h V^+_{\omega_i} = [h]^{\omega_i} V^+_{\omega_i}$$

$$\Delta_i : G \longrightarrow \mathbb{C}^\times$$

$$\Delta^{\omega_i}(g) = [h]^{\omega_i}, \quad i = 1..r$$

Generalized minors

$u, v \in W_G$

$$\Delta_{u\omega_i, v\omega_i}(g) = \Delta^{\omega_i}(u g v^{-1})$$

[Fomin-Zelevinsky]

$$\Delta \Delta - \Delta \Delta = \prod \Delta^{-a_{ij}}$$

q-Wronskian D

$$Q_i^+ = \frac{\det D_i}{\det \mathrm{Vand}_i}$$

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$-$

$=$

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Generalized minors

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V_i^+ - irrep of G w/ highest weight ω_i

$h V^+_{\omega_i} = [h]^{\omega_i} V^+_{\omega_i}$

$$\Delta_{u\omega_i, v\omega_i}(g) = \Delta^{\omega_i}(\bar{u}\, g\, \tilde{v}^{-1})$$

[Fomin–Zelevinski]

$$\Delta \Delta - \Delta \Delta = \prod \Delta^{-a_{ij}}$$

$v(z) \in B_+(z)$

$v(qz) A(z) v(z)^{-1} = Z$

$$\Delta_{v\omega_i, \omega_i}(v(z)) = Q^{\omega_i}(z)$$

$\exists\, g(z) \in SL(r+1)(z)$

Let $\mathcal{L}_{r+1} = \mathrm{Span}\, s(z)$

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$$\mathcal{W}_i(s)(z) = s(z) \wedge A(z)s(qz) \wedge A(z)A(qz)\, s(q^2z) \ldots\ldots A(z)\ldots A(q^{i-2}z)\, s(q^{i-1}z) \,\Big|\, \Lambda^i \mathcal{L}_{r+1}$$

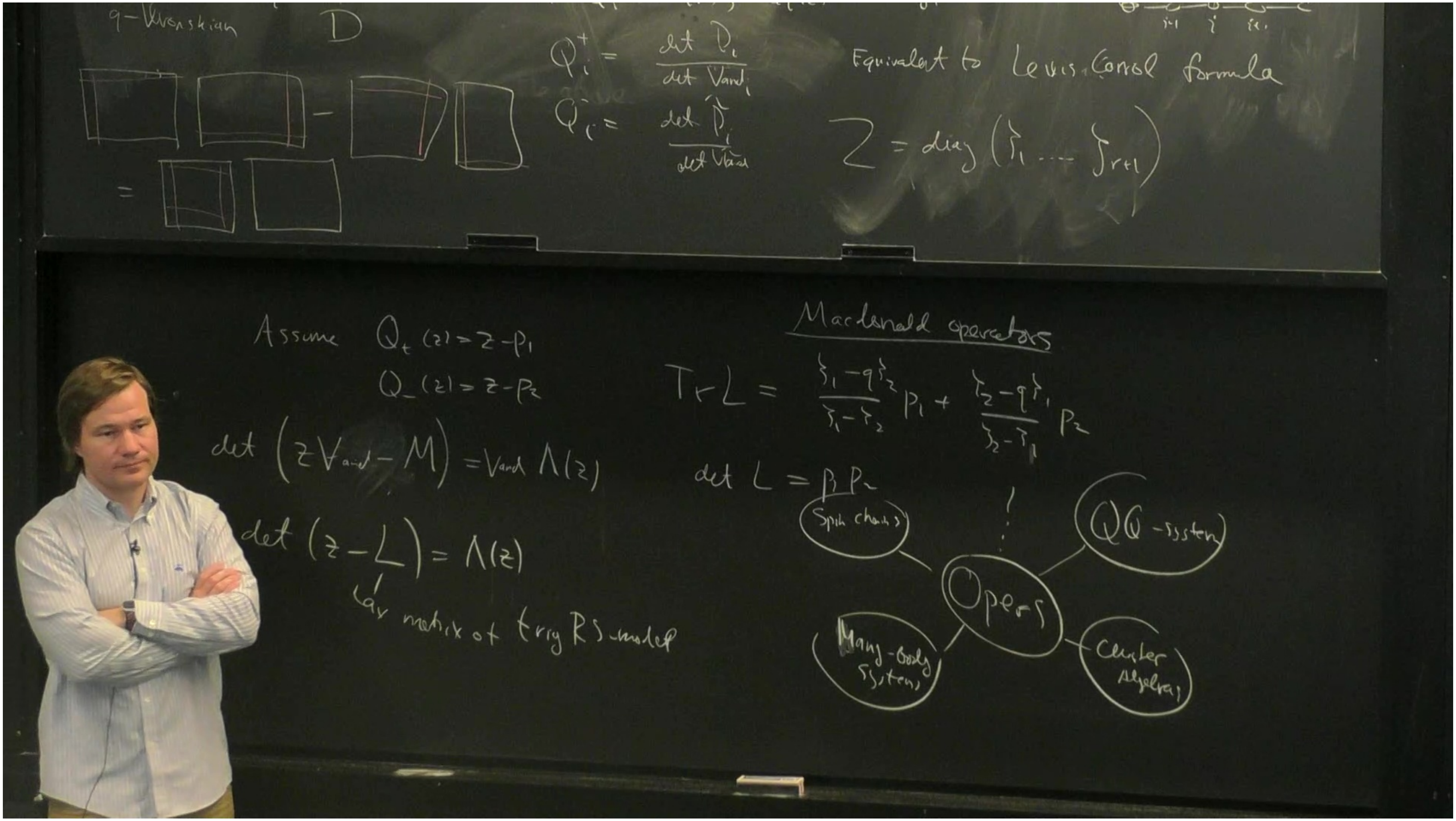
iv) is equivalent to $\mathcal{W}_i(s)(z) \neq 0$

$$s(z) = \begin{pmatrix} Q^{1,1} \\ Q^{2,1} \\ Q^{2,\ldots,1} \\ \vdots \\ Q^{w_0 \ldots} \end{pmatrix}$$

$(SL(2), q)$-oper $\qquad A(z) = Z = \begin{pmatrix} \zeta_1 & \\ & \zeta_2 \end{pmatrix}$

$$s(z) \wedge Z\, s(qz) = \Lambda(z)$$

$$\det \begin{pmatrix} Q_+(z) & \zeta_1 Q_+(qz) \\ Q_-(z) & \zeta_2 Q_-(qz) \end{pmatrix} = \Lambda(z)$$


q-Wronskian
Equivalent to Lewis Carrol formula
Macdonald operators
Assume
Spin chains
QQ-system
Opers
Many-body Systems
Cluster Algebras
Lax matrix of trig RS-model