Title: Spectral gap implies rapid mixing for commuting Hamiltonians

Speakers: Angela Capel Cuevas

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Abstract: Quantum systems typically reach thermal equilibrium rather quickly when coupled to an external thermal environment. The usual way of bounding the speed of this process is by estimating the spectral gap of the dissipative generator. However, the gap, by itself, does not always yield a reasonable estimate for the thermalization time in many-body systems: without further structure, a uniform lower bound on it only constraints the thermalization time to be polynomially growing with system size. In this talk, we will discuss that for all 2-local models with commuting Hamiltonians, the thermalization time that one can estimate from the gap is in fact much smaller than direct estimates suggest: at most logarithmic in the system size. This yields the so-called rapid mixing of dissipative dynamics. We will show this result by proving that a finite gap directly implies a lower bound on the modified logarithmic Sobolev inequality (MLSI) for the class of models we consider. The result is particularly relevant for 1D systems, for which we can prove rapid thermalization with a constant decay rate, giving a qualitative improvement over all previous results. It also applies to hypercubic lattices, graphs with exponential growth rate, and trees with sufficiently fast decaying correlations in the Gibbs state. This has consequences for the rate of thermalization towards Gibbs states, and also for their relevant Wasserstein distances and transportation cost inequalities.

Zoom link https://pitp.zoom.us/j/91315419731?pwd=TGpFTjlHWEJZVWZkdTh6bDFKMjhQZz09

Spectral gap implies rapid mixing for commuting Hamiltonians

Modified logarithmic Sobolev inequalities for quantum many-body systems

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Joint work with **A. Alhambra, J. Kochanowski and C. Rouzé**

Quantum Information Seminar, Perimeter Institute
15 October 2023

ACKNOWLEDGEMENTS

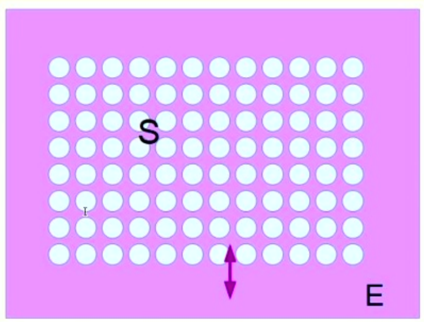
PERIMETER INSTITUTE

Simons Emmy Noether Fellowship Program

Motivation: Open quantum many-body systems

Open quantum many-body system.

No experiment can be executed at zero temperature or be completely shielded from noise.



- Finite lattice $\Lambda \subset\subset \mathbb{Z}^d$.
- Hilbert space associated to Λ is $\mathcal{H}_\Lambda = \bigotimes_{x\in\Lambda} \mathcal{H}_x$.
- Density matrices: $\mathcal{S}_\Lambda := \mathcal{S}(\mathcal{H}_\Lambda) = \{\rho_\Lambda \in \mathcal{B}_\Lambda \, : \, \rho_\Lambda \geq 0 \text{ and } \operatorname{tr}[\rho_\Lambda] = 1\}$.

- Dynamics of S is dissipative!
- The continuous-time evolution of a state on S is given by a q. Markov semigroup (**Markovian approximation**).

Quantum Markov semigroup / Dissipative quantum evolution

Quantum Markov semigroup

A **quantum Markov semigroup** is a 1-parameter continuous semigroup $\{\mathcal{T}_t\}_{t\geq 0}$ of completely positive, trace preserving (CPTP) maps (a.k.a. quantum channels) in $\mathcal{S}_\Lambda$.

Semigroup:

- $\mathcal{T}_t \circ \mathcal{T}_s = \mathcal{T}_{t+s}$.
- $\mathcal{T}_0 = \mathbb{1}$.

$$\frac{d}{dt}\mathcal{T}_t = \mathcal{T}_t \circ \mathcal{L}_\Lambda = \mathcal{L}_\Lambda \circ \mathcal{T}_t.$$

QMS generator

The infinitesimal generator $\mathcal{L}_\Lambda$ of the previous semigroup of quantum channels is usually called **Liouvillian**, or **Lindbladian**.

$$\mathcal{T}_t = e^{t\mathcal{L}_\Lambda} \Leftrightarrow \mathcal{L}_\Lambda = \frac{d}{dt}\mathcal{T}_t \mid_{t=0}.$$

For $\rho_\Lambda \in \mathcal{S}_\Lambda, \quad \mathcal{L}_\Lambda(\rho_\Lambda) = -i[H_\Lambda, \rho_\Lambda] + \sum_{k\in\Lambda} \widetilde{\mathcal{L}}_k(\rho_\Lambda)$ **GKLS equation.**

Mixing of dissipative quantum systems

Mixing $\Leftrightarrow$ **Convergence**

Primitive QMS

We assume that $\{\mathcal{T}_t\}_{t\geq 0}$ has a unique full-rank invariant state which we denote by σ_Λ.

Detailed balance condition

We also assume that the quantum Markov process studied is **reversible**, i.e., it satisfies the **detailed balance condition** w.r.t. $\sigma \equiv \sigma_\Lambda$:

$$\langle f, \mathcal{L}^*_\Lambda(g) \rangle_\sigma = \langle \mathcal{L}^*_\Lambda(f), g \rangle_\sigma ,$$

for every $f, g \in \mathcal{B}_\Lambda$ and Hermitian, where

$$\langle f, g \rangle_\sigma = \operatorname{tr}\left[f \, \sigma^{1/2} \, g \, \sigma^{1/2} \right] .$$

Notation: $\rho_t := \mathcal{T}_t(\rho)$.

$$\rho_\Lambda \xrightarrow{t} \rho_t := \mathcal{T}_t(\rho_\Lambda) = e^{t\mathcal{L}_\Lambda}(\rho_\Lambda) \xrightarrow{t\to\infty} \sigma_\Lambda$$

Mixing time

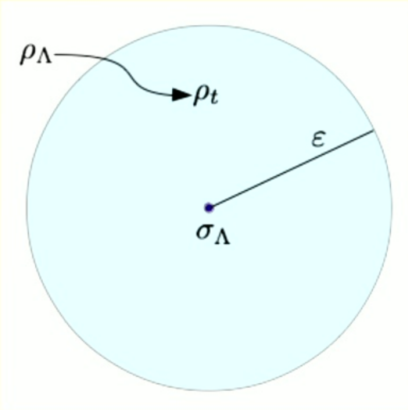
- Under the previous conditions, there is always convergence to σ_Λ.
- How fast does convergence happen?

Note $\mathcal{T}_\infty(\rho_\Lambda) := \sigma_\Lambda$ for every ρ_Λ.

Mixing time

We define the **mixing time** of $\{\mathcal{T}_t\}$ by

$$t_{\text{mix}}(\varepsilon) = \min\left\{ t > 0 : \sup_{\rho_\Lambda \in \mathcal{S}_\Lambda} \| \mathcal{T}_t(\rho_\Lambda) - \mathcal{T}_\infty(\rho_\Lambda) \|_1 \leq \varepsilon \right\}.$$



Rapid mixing

Mixing time

We define the **mixing time** of $\{\mathcal{T}_t\}$ by

$$t_{\text{mix}}(\varepsilon) = \min\left\{ t > 0 : \sup_{\rho_\Lambda \in \mathcal{S}_\Lambda} \|\rho_t - \sigma_\Lambda\|_1 \leq \varepsilon \right\}.$$

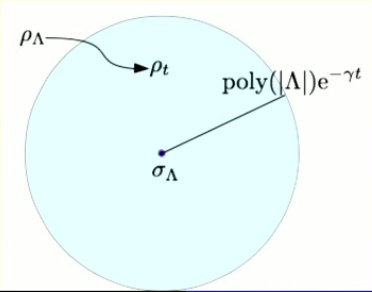
Recall: $\rho_t := \mathcal{T}_t(\rho_\Lambda)\,,\; \sigma_\Lambda := \mathcal{T}_\infty(\rho_\Lambda)$.

Rapid mixing

We say that $\mathcal{L}_\Lambda$ satisfies **rapid mixing** if

$$\sup_{\rho_\Lambda \in \mathcal{S}_\Lambda} \|\rho_t - \sigma_\Lambda\|_1 \leq \text{poly}(|\Lambda|)e^{-\gamma t}.$$

$t_{\text{mix}}(\varepsilon) \sim \text{poly}\log(|\Lambda|)$.



Applications to quantum information/quantum computing

What are the implications of rapid mixing?

Rapid mixing

$$\sup_{\rho \in \mathcal{S}(\mathcal{H}_\Lambda)} \|T_t(\rho) - \sigma\|_1 \leq \text{poly}(|\Lambda|) e^{-\gamma t}$$

Mixing time: $\tau(\epsilon) = \mathcal{O}(\text{polylog}(|\Lambda|))$

"Negative" point of view:

- Quantum properties that hold in the ground state but not in the Gibbs state are **suppressed too fast** for them to be of any reasonable use.

"Positive" point of view:

- Thermal states with short mixing time can be **constructed efficiently** with a quantum device that simulates the effect of the thermal bath.
- This has important implications as a self-studying open problem as well as in optimization problems via simulated annealing type algorithms.

APPLICATIONS TO QUANTUM INFORMATION/QUANTUM COMPUTING

If **rapid mixing**, no **error correction:**

Rapid mixing

$$\sup_{\rho \in \mathcal{S}(\mathcal{H}_\Lambda)} \|T_t(\rho) - \sigma\|_1 \leq \text{poly}(|\Lambda|) e^{-\gamma t}$$

Mixing time: $\tau(\varepsilon) = \mathcal{O}(\text{polylog}(|\Lambda|))$

Easy $t_{\text{mix}} \sim \log(n)$ — $t_{\text{mix}} \sim \text{poly}(n)$ — $t_{\text{mix}} \sim \exp(n)$ **Hard**

Efficient prediction
Speed-up for SDP solvers
Error correction
Topological order
Self-correction
Quantum memories

Main applications or consequences:

- Robust and efficient **preparation of topologically ordered phases** of matter via dissipation.
- Design of more efficient **quantum error-correcting codes** optimized for correlated Markovian noise models.
- **Stability** against local perturbations (Cubitt, Lucia, Michalakis, Pérez-García '15)
- **Area law** for mutual information (Brandao, Cubitt, Lucia, Michalakis, Pérez-García '15)
- Gaussian **concentration inequalities** (Lipschitz observables) (C., Rouzé, S. Franca '20)
- Finite blocklength refinement of **quantum Stein lemma** (C., Rouzé, Stilck Franca '20)
- **Quantum annealers:** Output an energy closed to that of the fixed point after short time (C., Rouzé, Stilck Franca '20)
- **Preparation Gibbs states:** Existence of local quantum circuits with logarithmic depth to prepare the Gibbs state (C., Rouzé, Stilck Franca '20)
- Establish the absence of **dissipative phase transitions** (Bardet, C., Gao, Lucia, Pérez-García, Rouzé '21)
- Examples of interacting **SPT phases** with decoherence time growing logarithmically with the system size for thermal noise (Bardet, C., Gao, Lucia, Pérez-García, Rouzé '21)

And many more...

Modified logarithmic Sobolev inequality (MLSI)

Recall: $\rho_t := \mathcal{T}_t(\rho)$.

Master equation:

$$\partial_t \rho_t = \mathcal{L}_\Lambda(\rho_t).$$

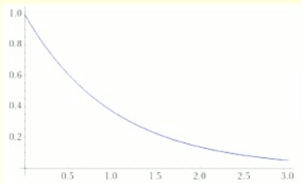
Relative entropy of ρ_t and σ_Λ:

$$D(\rho_t || \sigma_\Lambda) = \operatorname{tr}[\rho_t(\log \rho_t - \log \sigma_\Lambda)].$$

Differentiating:

$$\partial_t D(\rho_t || \sigma_\Lambda) = \operatorname{tr}[\mathcal{L}_\Lambda(\rho_t)(\log \rho_t - \log \sigma_\Lambda)].$$

Lower bound for the derivative of $D(\rho_t || \sigma_\Lambda)$ in terms of itself:



$$2\alpha D(\rho_t || \sigma_\Lambda) \leq -\operatorname{tr}[\mathcal{L}_\Lambda(\rho_t)(\log \rho_t - \log \sigma_\Lambda)].$$

Modified logarithmic Sobolev inequality

Modified logarithmic Sobolev inequality

MLSI constant

The **MLSI constant** of $\mathcal{L}_\Lambda$ is defined as:

$$\alpha(\mathcal{L}_\Lambda) := \inf_{\rho_\Lambda \in \mathcal{S}_\Lambda} \frac{-\operatorname{tr}[\mathcal{L}_\Lambda(\rho_\Lambda)(\log \rho_\Lambda - \log \sigma_\Lambda)]}{2D(\rho_\Lambda || \sigma_\Lambda)}$$

If $\liminf_{\Lambda \nearrow \mathbb{Z}^d} \alpha(\mathcal{L}_\Lambda) > 0$:

$$D(\rho_t || \sigma_\Lambda) \leq D(\rho_\Lambda || \sigma_\Lambda) e^{-2\,\alpha(\mathcal{L}_\Lambda)\,t},$$

and **Pinsker's inequality** $\left(\frac{1}{2} \| \rho - \sigma \|_1^2 \leq D(\rho \| \sigma) \text{ for } \|A\|_1 := \operatorname{tr}[|A|] \right)$

$$\|\rho_t - \sigma_\Lambda\|_1 \leq \sqrt{2D(\rho_\Lambda || \sigma_\Lambda)}\, e^{-\alpha(\mathcal{L}_\Lambda)\,t} \leq \sqrt{2\log(1/\sigma_{\min})}\, e^{-\alpha(\mathcal{L}_\Lambda)\,t}.$$

For thermal states $\sigma_\Lambda = \mathrm{e}^{-\beta H} / \operatorname{tr}[\mathrm{e}^{-\beta H}]$,
$\sigma_{\min} \sim 1/\exp(|\Lambda|)$.

Rapid mixing
$\|\rho_t - \sigma_\Lambda\|_1 \leq \text{poly}(|\Lambda|) e^{-\gamma t}$

MLSI $\Rightarrow$ Rapid mixing.

Using the spectral gap (Kastoryano-Temme '13):

$$\|\rho_t - \sigma_\Lambda\|_1 \leq \sqrt{1/\sigma_{\min}}\, e^{-\lambda(\mathcal{L}_\Lambda^*)\,t}.$$

QUANTUM SPIN SYSTEMS

Modified Logarithmic Sobolev Inequality (MLSI)

$$D(T_t(\rho)\|\sigma) \leq D(\rho\|\sigma)\, e^{-2\alpha(\mathcal{L})t}$$

Relative entropy: $D(\rho\|\sigma) := \text{tr}[\rho(\log\rho - \log\sigma)]$

Kastoryano-Temme, '13

Rapid mixing

$$\sup_{\rho\in\mathcal{S}(\mathcal{H}_\Lambda)} \|T_t(\rho)-\sigma\|_1 \leq \text{poly}(|\Lambda|)e^{-\gamma t}$$

Mixing time: $\tau(\varepsilon) = \mathcal{O}(\text{polylog}(|\Lambda|))$

Kastoryano-Temme, '13

Poincaré Inequality

$$\text{Var}_\sigma(T_t(\rho)) \leq \text{Var}_\sigma(\rho)\, e^{-2\alpha(\lambda L)t}$$

Variance: $\text{Var}_\sigma(\rho) := \text{tr}[(\rho^{1/2}\sigma)^2] - \text{tr}[\rho\sigma]^2$

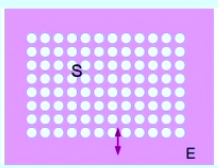
"Slower" mixing

$$\sup_{\rho\in\mathcal{S}(\mathcal{H}_\Lambda)} \|T_t(\rho)-\sigma\|_1 \leq \exp(\sqrt{|\Lambda|})e^{-\delta t}$$

Mixing time: $\tau(\varepsilon) = \mathcal{O}(\sqrt{|\Lambda|})$

$$e^{t\mathcal{L}}(\rho) \xrightarrow{t\to\infty} \sigma$$

$\sigma = \frac{e^{-\beta H}}{\text{tr}[e^{-\beta H}]}$ Gibbs state of a local, commuting Hamiltonian



Notation: $\Lambda \subset\subset \mathbb{Z}^d$ lattice
$\{T_t\}_{t\geq 0}$ Quantum Markov semigroup
$\mathcal{L}$ Inf. generator (Lindbladian)

Mixing time of the semigroup $\{T_t\}_{t\geq 0}$

$$\tau(\varepsilon) = \min\left\{ t>0 : \sup_{\rho\in\mathcal{S}(\mathcal{H}_\Lambda)} \|T_t(\rho)-\sigma\|_1 \leq \varepsilon \right\}$$

Quantum spin systems

Thermalization

MLSI

$D(T_t(\rho)\|\sigma) \leq D(\rho\|\sigma)\, e^{-2\alpha(\mathcal{L})t}$

Rel. entropy: $D(\rho\|\sigma) := \mathrm{tr}[\rho(\log\rho - \log\sigma)]$

KT'13

Rapid mixing

$\sup_{\rho \in \mathcal{S}(\mathcal{H}_\Lambda)} \|T_t(\rho) - \sigma\|_1 \leq \mathrm{poly}(|\Lambda|) e^{-\gamma t}$

Mixing time: $\tau(\varepsilon) = \mathcal{O}(\mathrm{polylog}(|\Lambda|))$

KT'13

$$e^{t\mathcal{L}}(\rho) \overset{t\to\infty}{\longrightarrow} \sigma$$

Poincaré Inequality

$\mathrm{Var}_\sigma(T_t(\rho)) \leq \mathrm{Var}_\sigma(\rho)\, e^{-2\alpha(\lambda L)t}$

Variance: $\mathrm{Var}_\sigma(\rho) := \mathrm{tr}[(\rho^{1/2}\sigma)^2] - \mathrm{tr}[\rho\sigma]^2$

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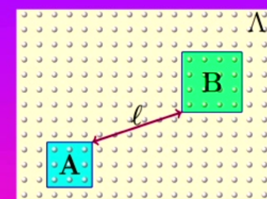
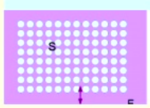
Decay of correlations

?



$\sigma = \frac{e^{-\beta H}}{\mathrm{tr}[e^{-\beta H}]}$ Gibbs state of a local, commuting Hamiltonian



Mixing time of $\{T_t\}_{t\geq 0}$

$\tau(\varepsilon) = \min\left\{ t > 0 : \sup_{\rho \in \mathcal{S}(\mathcal{H}_\Lambda)} \|T_t(\rho) - \sigma\|_1 \leq \varepsilon \right\}$

Notions: $I_\sigma(A:B) := D(\sigma_{AB}\|\sigma_A \otimes \sigma_B)$

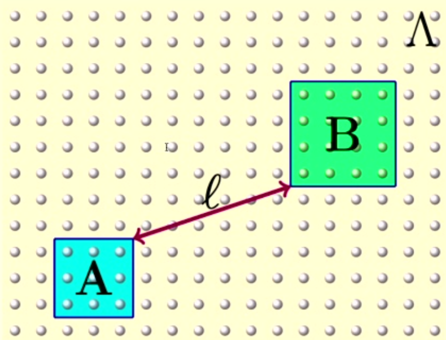
$\mathrm{Cov}_\sigma(A:B) := \sup_{\|O_A\|,\|O_B\|\leq 1} |\mathrm{tr}[(\sigma_{AB} - \sigma_A \otimes \sigma_B) O_A \otimes O_B]|$

Decay of correlations on Gibbs state

Motivation

Describe the **correlation properties** of **Gibbs states** of local Hamiltonians.

- **Hamiltonian:** $H_\Lambda = H_A + H_B + H_{(A\cup B)^c} + H_{\partial A} + H_{\partial B}$,
- **Gibbs state:** $\sigma_\Lambda(\beta) = \mathrm{e}^{-\beta H_\Lambda} / \operatorname{Tr}[\mathrm{e}^{-\beta H_\Lambda}]$.



$$\ell := \operatorname{dist}(A, B)$$

Questions:

For non-commuting Hamiltonians:

$$\mathrm{e}^{-\beta H_{A\cup B}} \approx \mathrm{e}^{-\beta H_A}\, \mathrm{e}^{-\beta H_B} \ ?$$

$$\operatorname{tr}_{A^c}[\sigma_\Lambda] \otimes \operatorname{tr}_{B^c}[\sigma_\Lambda] := (\sigma_\Lambda)_A \otimes (\sigma_\Lambda)_B \approx \operatorname{tr}_{(A\cup B)^c}[\sigma_\Lambda] := (\sigma_\Lambda)_{A\cup B} \ ?$$

Decay of correlations on Gibbs state

3 different forms of **decay of correlations**.

Operator correlation

$$\text{Cov}_\sigma(A:B) := \sup_{\|O_A\|=\|O_B\|=1} |\text{tr}[O_A \otimes O_B(\sigma_{AB} - \sigma_A \otimes \sigma_B)]|$$

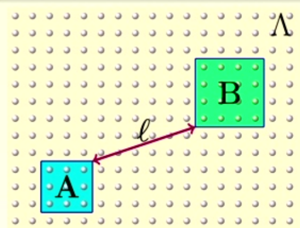
Mutual information

$$I_\sigma(A:B) := D(\sigma_{AB}||\sigma_A \otimes \sigma_B)$$

for $D(\rho\|\sigma) = \text{Tr}[\rho(\log\rho - \log\sigma)]$

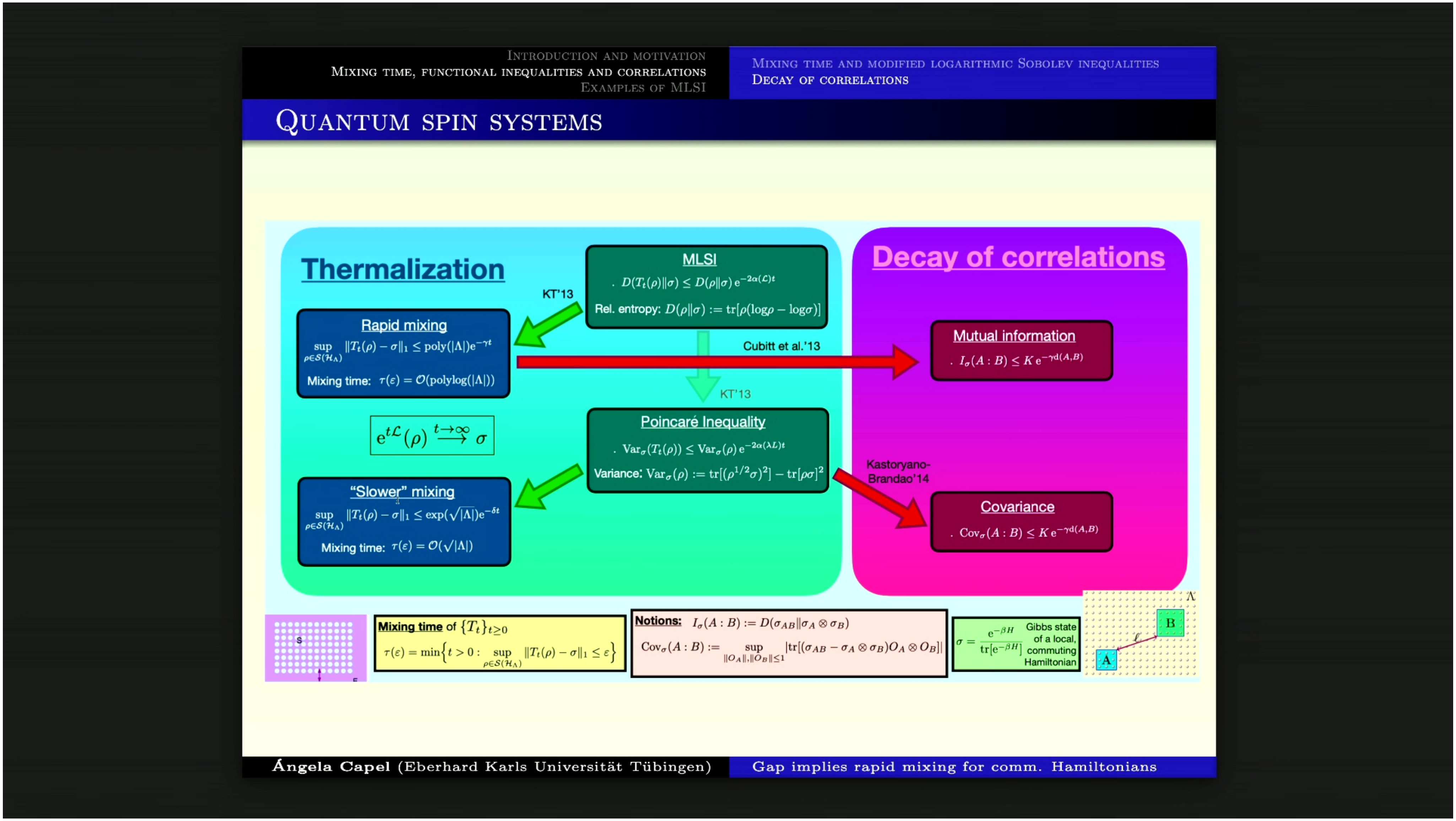
Mixing condition

$$\|h(\sigma_{AB})\|_\infty = \left\| \sigma_A^{-1/2} \otimes \sigma_B^{-1/2} \sigma_{AB} \sigma_A^{-1/2} \otimes \sigma_B^{-1/2} - \mathbb{1}_{AB} \right\|_\infty$$



Relation:

$$\frac{1}{2}\text{Cov}_\sigma(A:B)^2 \leq I_\sigma(A:B)$$
$$\leq \left\| \sigma_A^{-1/2} \otimes \sigma_B^{-1/2} \sigma_{AB} \sigma_A^{-1/2} \otimes \sigma_B^{-1/2} - \mathbb{1}_{AB} \right\|_\infty .$$


Introduction and motivation
Mixing time, functional inequalities and correlations
Examples of MLSI
Mixing time and modified logarithmic Sobolev inequalities
Decay of correlations
Quantum spin systems
Thermalization
MLSI
$D(T_t(\rho)\|\sigma) \leq D(\rho\|\sigma)\, e^{-2\alpha(\mathcal{L})t}$
Rel. entropy: $D(\rho\|\sigma) := \mathrm{tr}[\rho(\log\rho - \log\sigma)]$
KT'13
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$\sup_{\rho\in\mathcal{S}(\mathcal{H}_\Lambda)} \|T_t(\rho)-\sigma\|_1 \leq \mathrm{poly}(|\Lambda|)e^{-\gamma t}$
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Cubitt et al.'13
KT'13
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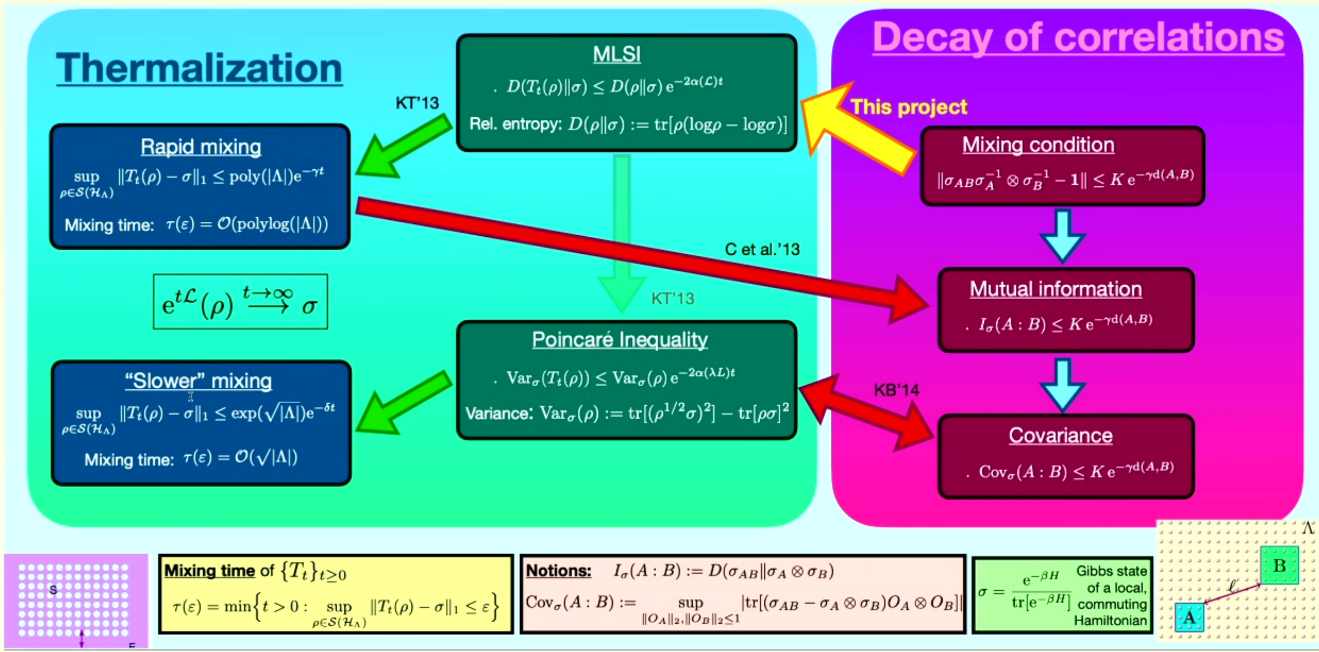
B
A
Ángela Capel (Eberhard Karls Universität Tübingen)
Gap implies rapid mixing for comm. Hamiltonians

Quantum spin systems



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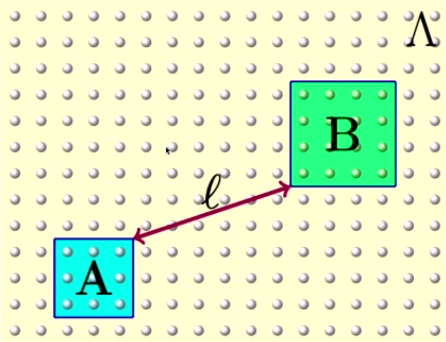

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Decay of correlations on Gibbs state

3 different forms of **decay of correlations**.

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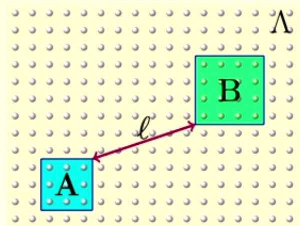
$$I_\sigma(A:B) := D(\sigma_{AB}||\sigma_A \otimes \sigma_B)$$

for $D(\rho\|\sigma) = \mathrm{Tr}[\rho(\log\rho - \log\sigma)]$

Mixing condition

$$\|h(\sigma_{AB})\|_\infty = \left\|\sigma_A^{-1/2} \otimes \sigma_B^{-1/2}\sigma_{AB}\sigma_A^{-1/2} \otimes \sigma_B^{-1/2} - \mathbb{1}_{AB}\right\|_\infty$$



Relation:

$$\frac{1}{2}\mathrm{Cov}_\sigma(A:B)^2 \leq I_\sigma(A:B)$$
$$\leq \left\|\sigma_A^{-1/2} \otimes \sigma_B^{-1/2}\sigma_{AB}\sigma_A^{-1/2} \otimes \sigma_B^{-1/2} - \mathbb{1}_{AB}\right\|_\infty .$$

Setting and questions

Given:

- H_Λ local (commuting) Hamiltonian $\mapsto$ $\sigma_\Lambda := \frac{e^{-\beta H_\Lambda}}{\text{tr}[e^{-\beta H_\Lambda}]}$ Gibbs state .
- $\mathcal{L}_\Lambda$ local Lindbladian with unique stationary state σ_Λ ($\mathcal{L}_\Lambda(\sigma_\Lambda) = 0$).

Questions:

- Does $\mathcal{L}_\Lambda$ have a positive, constant (or poly log) MLSI?
- How do correlations decay in σ_Λ between spatially separated regions?

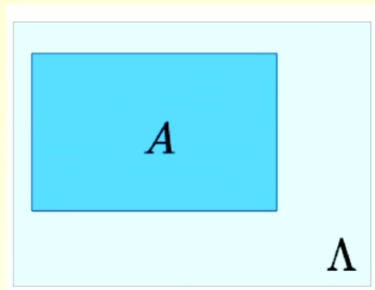
Objective

MLSI constant

$$\alpha(\mathcal{L}_\Lambda) := \inf_{\rho_\Lambda \in \mathcal{S}_\Lambda} \frac{-\operatorname{tr}[\mathcal{L}_\Lambda(\rho_\Lambda)(\log \rho_\Lambda - \log \sigma_\Lambda)]}{2D(\rho_\Lambda || \sigma_\Lambda)}$$

What do we want to prove?

$$\liminf_{\Lambda \nearrow \mathbb{Z}^d} \alpha(\mathcal{L}_\Lambda) \geq \Psi(|\Lambda|) > 0 \qquad (\text{or } = 0 \text{ very "slowly", like } \Omega\left(\frac{1}{\text{poly } \log(|\Lambda|)}\right))$$



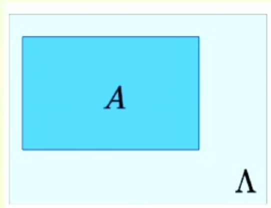
Can we prove something like

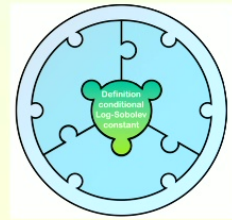
$$\alpha(\mathcal{L}_\Lambda) \geq \Psi(|A|)\ \alpha(\mathcal{L}_A) > 0 \ ?$$

No, but we can prove

$$\alpha(\mathcal{L}_\Lambda) \geq \Psi(|A|)\ \alpha_\Lambda(\mathcal{L}_A) > 0 \ .$$

Conditional MLSI constant





MLSI constant

The **MLSI constant** of $\mathcal{L}_\Lambda = \sum_{k \in \Lambda} \mathcal{L}_k$ is defined by

$$\alpha(\mathcal{L}_\Lambda) := \inf_{\rho_\Lambda \in \mathcal{S}_\Lambda} \frac{-\operatorname{tr}[\mathcal{L}_\Lambda(\rho_\Lambda)(\log \rho_\Lambda - \log \sigma_\Lambda)]}{2D(\rho_\Lambda || \sigma_\Lambda)}$$

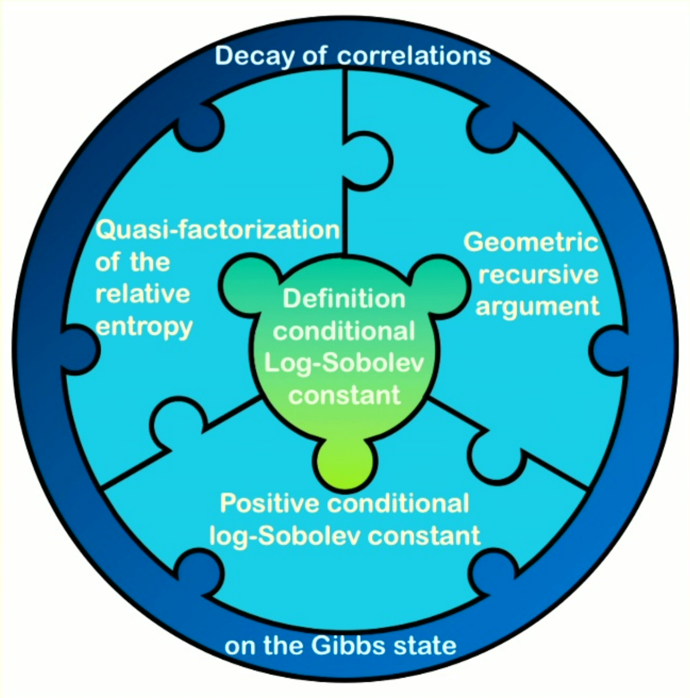
Conditional MLSI constant

The **conditional MLSI constant** of $\mathcal{L}_\Lambda$ on $A \subset \Lambda$ is defined by

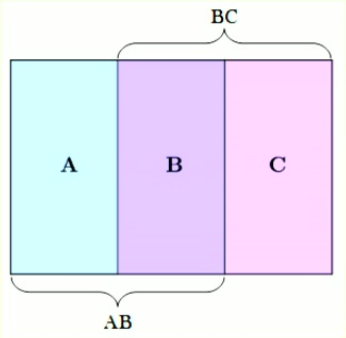
$$\alpha_\Lambda(\mathcal{L}_A) := \inf_{\rho_\Lambda \in \mathcal{S}_\Lambda} \frac{-\operatorname{tr}[\mathcal{L}_A(\rho_\Lambda)(\log \rho_\Lambda - \log \sigma_\Lambda)]}{2D_A(\rho_\Lambda || \sigma_\Lambda)}$$

Strategy

Used in (C.-Lucia-Pérez García '18) and (Bardet-C.-Lucia-Pérez García-Rouzé, '19).



Quasi-factorization of the relative entropy



Quasi-factorization of the relative entropy

Given $\Lambda = ABC$, it is an inequality of the form:

$$D(\rho_\Lambda \| \sigma_\Lambda) \leq \xi(\sigma_{ABC}) \left[D_{AB}(\rho_\Lambda \| \sigma_\Lambda) + D_{BC}(\rho_\Lambda \| \sigma_\Lambda) \right] ,$$

for $\rho_\Lambda, \sigma_\Lambda \in \mathcal{S}(\mathcal{H}_{ABC})$, where $\xi(\sigma_{ABC})$ depends only on σ_{ABC} and measures how far σ_{AC} is from $\sigma_A \otimes \sigma_C$.

How does the strategy work?

We want to prove:

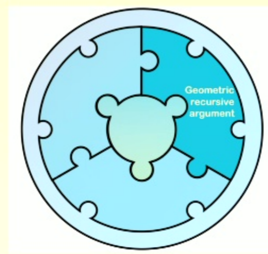
$\alpha(\mathcal{L}_\Lambda) := \inf_{\rho_\Lambda \in \mathcal{S}_\Lambda} \frac{-\operatorname{tr}[\mathcal{L}_\Lambda(\rho_\Lambda)(\log \rho_\Lambda - \log \sigma_\Lambda)]}{2D(\rho_\Lambda \| \sigma_\Lambda)}$

$$\boxed{\alpha(\mathcal{L}_\Lambda) \geq \Psi(|A|)\, \alpha_\Lambda(\mathcal{L}_A) > 0}$$

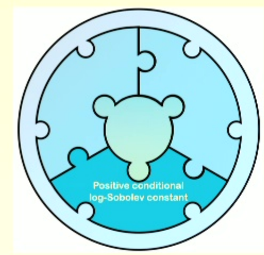
$\alpha_\Lambda(\mathcal{L}_A) := \inf_{\rho_\Lambda \in \mathcal{S}_\Lambda} \frac{-\operatorname{tr}[\mathcal{L}_A(\rho_\Lambda)(\log \rho_\Lambda - \log \sigma_\Lambda)]}{2D_A(\rho_\Lambda \| \sigma_\Lambda)}$

After choosing  and , we prove the following:

$D(\rho_\Lambda \| \sigma_\Lambda) \to D_A(\rho_\Lambda \| \sigma_\Lambda)$



$\Psi(|A|) > 0$



$\alpha_\Lambda(\mathcal{L}_A) > 0$

Example: Tensor product fixed point

(C.-Lucia-Pérez García '18)
(Beigi-Datta-Rouzé '18)

$$\mathcal{L}_\Lambda(\rho_\Lambda) = \sum_{x\in\Lambda} (\sigma_x \otimes \rho_{x^c} - \rho_\Lambda)$$ **heat-bath**

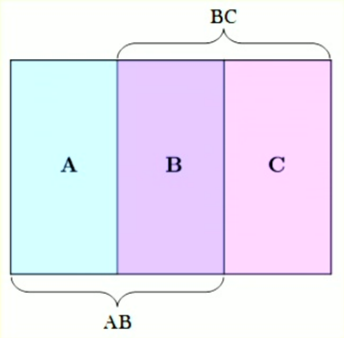
$$D_x(\rho_\Lambda \| \sigma_\Lambda) := D(\rho_\Lambda \| \sigma_\Lambda) - D(\rho_{x^c} \| \sigma_{x^c})$$

$$\sigma_\Lambda = \bigotimes_{x\in\Lambda} \sigma_x ,$$

$$\alpha_\Lambda(\mathcal{L}_x^*) := \inf_{\rho_\Lambda \in \mathcal{S}_\Lambda} \frac{-\operatorname{tr}[\mathcal{L}_x^*(\rho_\Lambda)(\log\rho_\Lambda - \log\sigma_\Lambda)]}{2 D_x(\rho_\Lambda \| \sigma_\Lambda)}$$

$$\begin{aligned}
D(\rho_\Lambda \| \sigma_\Lambda) &\leq \\
&\leq \sum_{x\in\Lambda} D_x(\rho_\Lambda \| \sigma_\Lambda) \\
&\leq \sum_{x\in\Lambda} \frac{-\operatorname{tr}[\mathcal{L}_x(\rho_\Lambda)(\log\rho_\Lambda - \log\sigma_\Lambda)]}{2\alpha_\Lambda(\mathcal{L}_x)} \\
&\leq \frac{1}{2\inf\limits_{x\in\Lambda} \alpha_\Lambda(\mathcal{L}_x)} \sum_{x\in\Lambda} -\operatorname{tr}[\mathcal{L}_x(\rho_\Lambda)(\log\rho_\Lambda - \log\sigma_\Lambda)] \\
&= \frac{1}{2\inf\limits_{x\in\Lambda} \alpha_\Lambda(\mathcal{L}_x)} \left(-\operatorname{tr}[\mathcal{L}_\Lambda(\rho_\Lambda)(\log\rho_\Lambda - \log\sigma_\Lambda)]\right) \\
&\leq \left(-\operatorname{tr}[\mathcal{L}_\Lambda(\rho_\Lambda)(\log\rho_\Lambda - \log\sigma_\Lambda)]\right).
\end{aligned}$$

Quasi-factorization of the relative entropy



Quasi-factorization of the relative entropy

Given $\Lambda = ABC$, it is an inequality of the form:

$$D(\rho_\Lambda \| \sigma_\Lambda) \leq \xi(\sigma_{ABC}) \left[D_{AB}(\rho_\Lambda \| \sigma_\Lambda) + D_{BC}(\rho_\Lambda \| \sigma_\Lambda) \right] ,$$

for $\rho_\Lambda, \sigma_\Lambda \in \mathcal{S}(\mathcal{H}_{ABC})$, where $\xi(\sigma_{ABC})$ depends only on σ_{ABC} and measures how far σ_{AC} is from $\sigma_A \otimes \sigma_C$.

Dynamics

Let $\sigma_\Lambda = \frac{e^{-\beta H_\Lambda}}{\operatorname{tr}\left[e^{-\beta H_\Lambda}\right]}$ be the Gibbs state of finite-range, commuting Hamiltonian.

Heat-bath generator

The **heat-bath generator** is defined as:

$$\mathcal{L}_\Lambda^H(\rho_\Lambda) := \sum_{x\in\Lambda} \left(\sigma_\Lambda^{1/2} \sigma_{x^c}^{-1/2} \rho_{x^c} \sigma_{x^c}^{-1/2} \sigma_\Lambda^{1/2} - \rho_\Lambda \right)$$

Davies generator

The **Davies generator** is given by:

$$\mathcal{L}_\Lambda^{D;*}(X) := i[H_\Lambda, X] + \sum_{x\in\Lambda} \widetilde{\mathcal{L}}_x^D(X) \, ,$$

where the $\mathcal{L}_x^D$ are defined in terms of the Fourier coefficients of the correlation functions in the bath and the ones of the system couplings.

Schmidt generator

The **Schmidt generator** (Bravyi-Vyalyi '05) can be written as:

$$\mathcal{L}_\Lambda^{S;*}(X) = \sum_{x\in\Lambda} \left(E_x^{S;*}(X) - X \right) ,$$

where the conditional expectations do not depend on system-bath couplings.

Previous results

Let us recall: For $\alpha(\mathcal{L}_\Lambda)$ a MLSI constant,

$$\|\rho_t - \sigma_\Lambda\|_1 \leq \sqrt{2\log(1/\sigma_{\min})}\, e^{-\alpha(\mathcal{L}_\Lambda)\,t}.$$

Using the spectral gap $\lambda(\mathcal{L}_\Lambda)$:

$$\|\rho_t - \sigma_\Lambda\|_1 \leq \sqrt{1/\sigma_{\min}}\, e^{-\lambda(\mathcal{L}_\Lambda)\,t}.$$

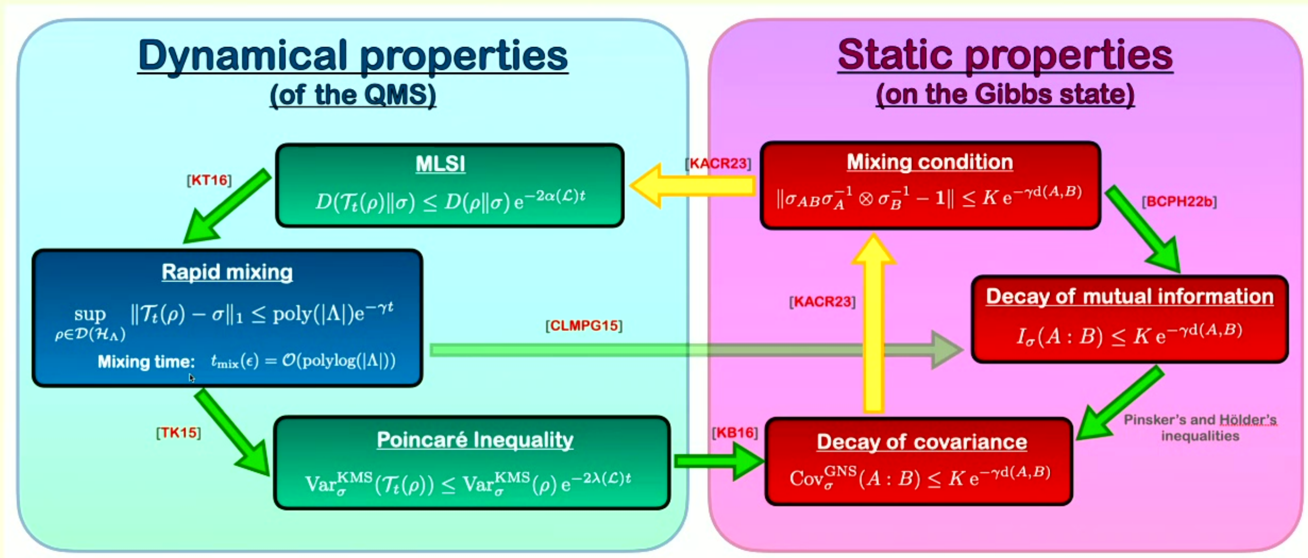
Spectral gap for Davies and heat-bath (Kastoryano-Brandao, '16)

Let $\mathcal{L}_\Lambda^{H,D}$ be the **heat-bath** or **Davies** generator in 1D. Then, $\mathcal{L}_\Lambda^{H,D}$ has a positive spectral gap, that is independent of the system size, for every temperature.

MLSI for heat-bath with tensor product fixed point (C.-Lucia-Pérez García, Beigi-Datta-Rouzé '18)

Let $\mathcal{L}_\Lambda^{H}$ be the **heat-bath** generator with tensor product fixed point. Then, it has a positive MLSI constant.

Main result


Dynamical properties
(of the QMS)
Static properties
(on the Gibbs state)
MLSI
$D(\mathcal{T}_t(\rho)\|\sigma) \leq D(\rho\|\sigma)\, e^{-2\alpha(\mathcal{L})t}$
[KACR23]
Mixing condition
$\|\sigma_{AB}\sigma_A^{-1}\otimes\sigma_B^{-1} - \mathbb{1}\| \leq K\, e^{-\gamma d(A,B)}$
[KT16]
[BCPH22b]
Rapid mixing
$\sup_{\rho\in\mathcal{D}(\mathcal{H}_\Lambda)} \|\mathcal{T}_t(\rho)-\sigma\|_1 \leq \text{poly}(|\Lambda|)e^{-\gamma t}$
Mixing time: $t_{\text{mix}}(\epsilon) = \mathcal{O}(\text{polylog}(|\Lambda|))$
[CLMPG15]
[KACR23]
Decay of mutual information
$I_\sigma(A:B) \leq K\, e^{-\gamma d(A,B)}$
[TK15]
Poincaré Inequality
$\text{Var}_\sigma^{\text{KMS}}(\mathcal{T}_t(\rho)) \leq \text{Var}_\sigma^{\text{KMS}}(\rho)\, e^{-2\lambda(\mathcal{L})t}$
[KB16]
Decay of covariance
$\text{Cov}_\sigma^{\text{GNS}}(A:B) \leq K\, e^{-\gamma d(A,B)}$
Pinsker's and Hölder's inequalities

MLSI for 2-colorable graphs

MLSI for 2-colorable graphs, (Alhambra-C.-Kochanowski-Rouzé, '23)

Let Λ be a 2-colorable graph and $\mathcal{L}_\Lambda^D$ be a **Davies** generator with unique fixed point σ_Λ given by the Gibbs state of a commuting, finite-range, 2-local Hamiltonian. If:

- i) The Lindbladian is **gapped**.
- ii) The Gibbs state satisfies **exponential decay of covariance**.

Then, $\mathcal{L}_\Lambda^D$ satisfies a **MLSI** with constant

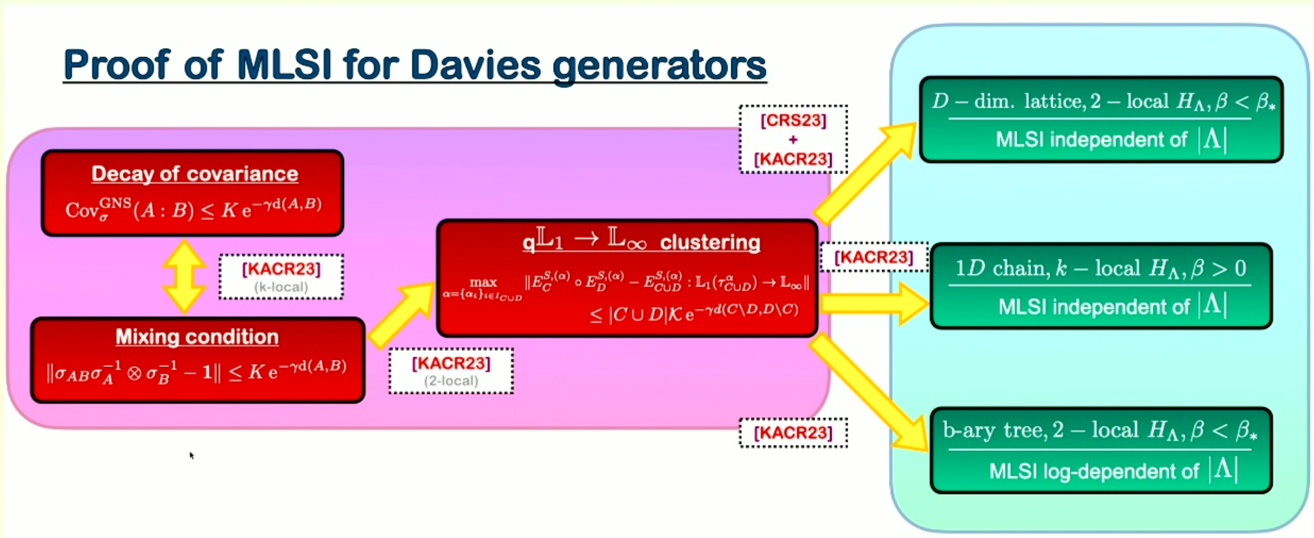
1) $\alpha(\mathcal{L}_\Lambda^D) = \Omega(1)_{|\Lambda| \to \infty}$, when Λ is a sub-exponential graph (e.g. hypercubic lattice), or
2) $\alpha(\mathcal{L}_\Lambda^D) = \Omega\left((\ln|\Lambda|)^{-1}\right)_{|\Lambda| \to \infty}$, if the correlation length of the fixed point is sufficiently small (e.g. b-ary trees).

Rapid mixing

- i) $\Lambda = \mathbb{Z}$ is 1-dimensional, H_Λ is k-local and $\beta > 0 \Rightarrow \mathcal{L}_\Lambda^D$ has a constant MLSI.
- ii) $\Lambda = \mathbb{Z}^D$ is D-dimensional, H_Λ is 2-local and $\beta < \beta_* \Rightarrow \mathcal{L}_\Lambda^D$ has a constant MLSI.
- iii) $\Lambda = \mathbb{T}_b$ is an inf. b-ary tree, H_Λ is 2-local and $\beta < \beta_* \Rightarrow \mathcal{L}_\Lambda^D$ has a log-size MLSI.

In all cases, $\mathcal{L}_\Lambda^D$ satisfies **rapid mixing**.

Ingredients of the proof



The last part uses "divide and conquer" arguments for the relative entropy.

$$+$$

Equivalence between dynamics:

$$D(\rho \| E_X^D(\rho)) \leq D(\rho \| E_X^S(\rho)) \leq D(\rho \| E_{X\partial}^D(\rho))$$

Open problems and lines of research

Open problems:

- Extend the chain of implications (in particular, decay of correlations $\Rightarrow$ MLSI) to k-local interactions.
- Extension to specific models.
  - 2D, quantum double models (positive spectral gap recently proven in (Lucia-Perez Garcia-Perez Hernandez, '21)).
- Extension to non-commuting Hamiltonians.
- Improve results of quasi-factorization for the relative entropy: More systems?
- New functional inequalities for different quantities, such as the Belavkin-Staszewski relative entropy:

$$D_{\mathrm{BS}}(\rho\|\sigma) = \operatorname{tr}\left[\rho \log\left(\rho^{1/2}\sigma^{-1}\rho^{1/2}\right)\right] .$$

Thank you for your attention!