Title: Factorization homology in quantum topology

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Series: Mathematical Physics

Date: October 12, 2023 - 1:30 PM

URL: https://pirsa.org/23100092

Abstract: Abstract TBA

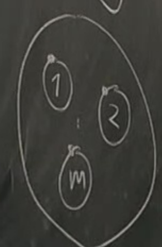
Zoom link: https://pitp.zoom.us/j/91816652867?pwd=bnQzYzA0SXBuL3BTTmNUMFBMcEFWdz09

Factorization homology in quantum topology

with A. Brochier, L. Müller

1. Factorization homology

Homology theory for E_n-algebras in a higher sym. mon. cat. $\mathcal{S}$



$$fE_n(m) = \left\{ \begin{array}{l} \text{embeddings } (D^n)^{\sqcup m} \hookrightarrow D^n \\ \text{composed of translations, rescalings \& rotations} \end{array} \right\}$$

Factorization homology in quantum topology

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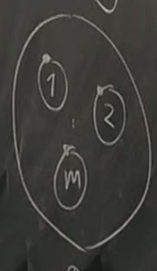
① Factorization homology

Homology theory for fE_n-algebras $\mathcal{A}$ in a higher sym. mon. cat. $\mathcal{S}$

$\mathcal{A} \in \mathcal{S}$



$$\longmapsto \mathcal{A}^{\boxtimes m} \longrightarrow \mathcal{A}$$

$$\cap$$

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...topology

... sym. mon. cat. $\mathcal{S}$

... D^n

...scalings & rotations

Factorization homology on an oriented n-dim. oriented manifold...

$$\int_\Sigma \mathcal{A} := \underset{\substack{m \geq 0 \\ \varphi : (D^n)^{\sqcup m} \hookrightarrow \Sigma}}{\mathrm{hocolim}} \mathcal{A}^{\boxtimes m} \in \mathcal{S}$$

[Beilinson-Drinfeld 04, Lurie 09, Ayala-Francis 15, ...]

Example. $\mathcal{S} = Ch_k$, $\mathcal{A}$ just a k-algebra, $n=1$, $\Sigma = S^1$

Then $\int_{S^1} \mathcal{A} \simeq$ Hochschild chains on $\mathcal{A} \in Ch_k$

(2) Fact...

[Ben-Zvi...

cat. $\mathcal{S}$

otations

Factorization homology on an oriented n-dim. oriented manifold Σ

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② Factorizati

[Ben-Zvi–Broch

...m. oriented manifold Σ with coefficients in $\mathcal{A}$

② Factorization homology of balanced braided categories

[Ben-Zvi–Brochier–Jordan 18]

We fix $n=2$, $\mathcal{S} = \mathrm{Rex} = \left\{ \begin{array}{l} \text{fin. cocomplete lin. cat.} \\ \text{right exact functors} \\ \text{linear nat. isos} \end{array} \right\}$ + Deligne product $\boxtimes$

$\Sigma = S^1$

A fE_2-alg $\mathcal{A}$ in Rex is a balanced braided category [Salvatore-Wahl 03].

It has a

- monoidal product $\otimes : \mathcal{A} \boxtimes \mathcal{A} \longrightarrow \mathcal{A}$ with unit $I \in \mathcal{A}$
- braiding $c_{X,Y} : X \otimes Y \xrightarrow{\sim} Y \otimes X$ + YBE

...manifold Σ with coefficients in $\mathcal{A}$

② Factorization homology of balanced braided categories

[Ben-Zvi – Brochier – Jordan 18]

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- braiding $c_{X,Y} : X \otimes Y \xrightarrow{\sim} Y \otimes X$ + YBE
- balancing $\theta_X : X \xrightarrow{\sim} X$ with $\theta_I = \mathrm{id}_I$

and $\theta_{X \otimes Y} = c_{Y,X}\, c_{X,Y} (\theta_X \otimes \theta_Y)$.

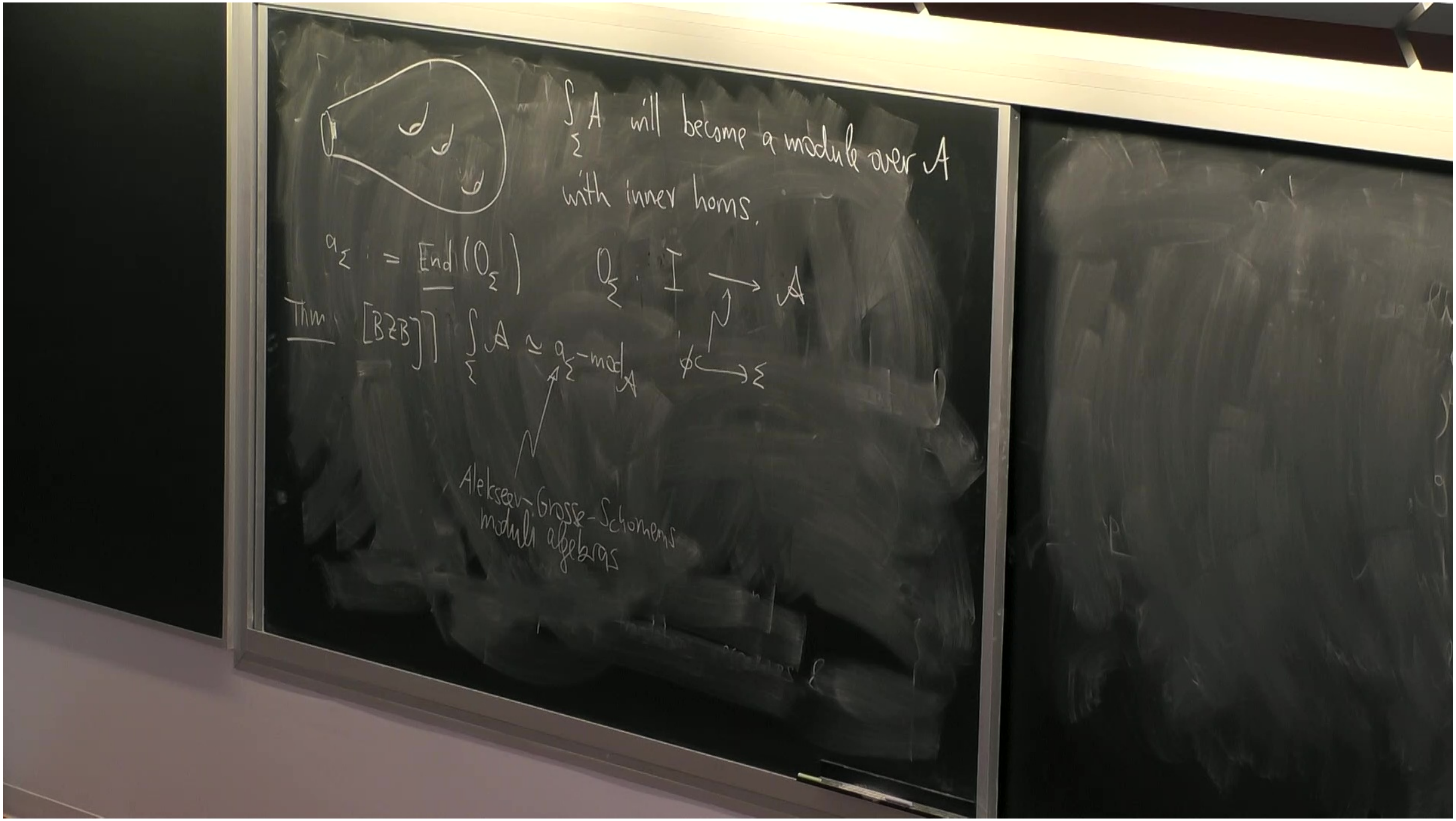
We say that $\mathcal{A}$ is finite if $\mathcal{A} \simeq B\text{-mod}$, B f.d. alg.

rigid if every object $X \in \mathcal{A}$ has a dual $X^\vee$ (evaluation $X^\vee \otimes X \to 1$, coev $1 \to X \otimes X^\vee$ + zigzag id.)

A finite ribbon cat. is a finite rigid bal. br. cat. with simple unit and $\theta_{X^\vee} = \theta_X^\vee$.

A finite ribbon cat. $\mathcal{A}$ is modular if

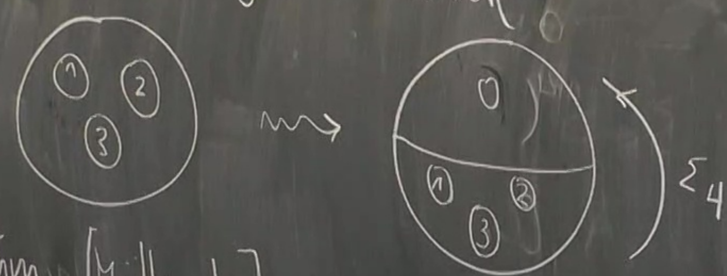
$$c_{Y,X}\, c_{X,Y} = \mathrm{id}_{X \otimes Y} \ \forall Y \Rightarrow X \cong I \oplus \cdots \oplus I$$


∫_Σ A will become a module over A
with inner homs.
a_Σ = End(O_Σ)
O_Σ : I → A
φ ↪ Σ
Thm [BZBJ] ∫_Σ A ≃ a_Σ-mod_A
Alekseev-Grosse-Schomerus
moduli algebras

over $\mathcal{A}$

③ Cyclic fE_2-algebras

fE_2 has a cyclic structure



Thm: [Müller-W] The cyclic structure for a fE_2-algebra amounts to a ribbon Grothendieck-Verdier duality, i.e. an equivalence $D: \mathcal{A} \to \mathcal{A}^{op}$ making $\mathcal{A}(X \otimes Y, k)$ representable: $\mathcal{A}(X \otimes Y, k) \cong \mathcal{A}(Y, DX)$

$k = D1$

plus $D\theta_X = \theta_{DX}$.

manifold Σ wit

② Factorization

[Ben-Zvi – Brochie

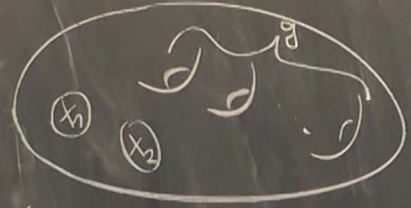
We fix $n = 2$,

A fE_2-alg $\mathcal{A}$ in

It has a

- monoidal
- braiding
- balancing

Thm [MW] This extends uniquely to 3d oriented handlebodies



$X_1, X_2 \in \mathcal{A}$

$\xmapsto{\text{angular functor}} \mathcal{A}(X_1 \otimes \cdots \otimes X_n \otimes \mathbb{A}^{\otimes g}, K)^*$

space of conformal blocks for $\mathcal{A}$

$$\mathbb{A} = \int_X X \otimes DX$$

cks

(4) Classification of modular functors

modular functor = consistent system of mapping class grp rep.

Any modular functor can be evaluated on genus zero surfaces and produces a cyclic framed E_2-algebra.

Thm. [Brochier-W.] The space of extensions for a cyclic fr. E_2-alg. to a modular functor is contractible or empty.

$\mathcal{A}$: cyclic fr. E_2-alg.

H handlebody, $\partial H = \Sigma$

define

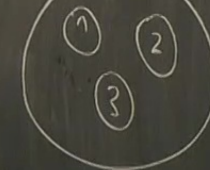
$$\Phi_{\mathcal{A}}(H) : \int_{\Sigma} \mathcal{A} \xrightarrow[\text{use the ansular functor}]{} \mathcal{A}^{\boxtimes \# \text{ bdy comp of } \Sigma}$$

We call $\mathcal{A}$ connected if $\Phi_{\mathcal{A}}(H) \cong \Phi_{\mathcal{A}}(H')$ for all H, H' with $\partial H = \partial H'$

Thm [BW] modular functors $\simeq$ connected cyclic framed E_2-algebras

3 Cyclic fE_2

fE_2 has a cyclic



$\rightsquigarrow$

Thm: [Müller-W] The

to a ribbon Grothendi

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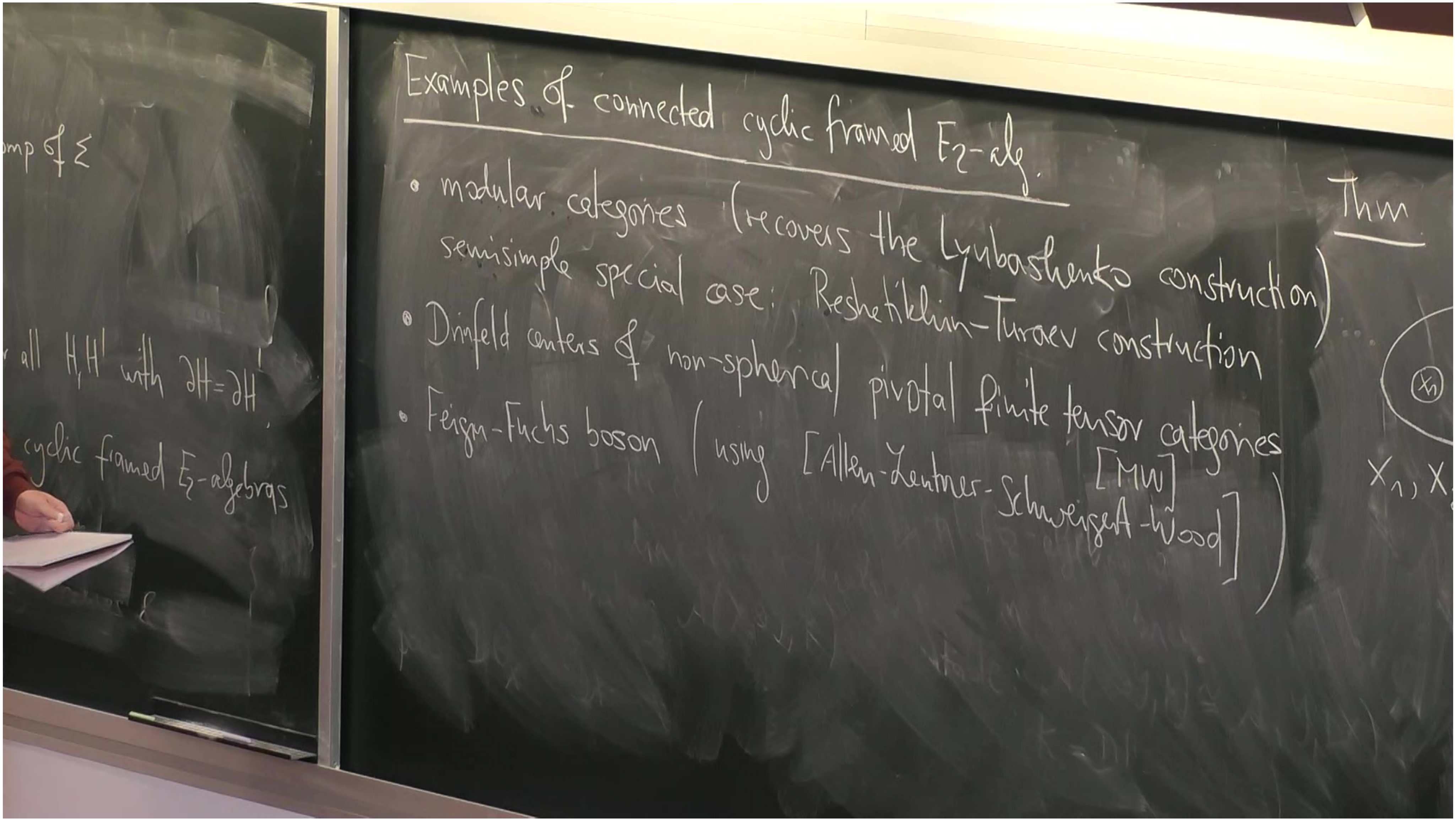
plus $D\Theta_X = \Theta$


Examples of connected cyclic framed E_2-alg.
• modular categories (recovers the Lyubashenko construction)
semisimple special case: Reshetikhin-Turaev construction
• Drinfeld centers of non-spherical pivotal finite tensor categories
• Feigin-Fuchs boson (using [Allen-Lentner-Schweigert-Wood] [MW])
Thm
X_1, X
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cyclic framed E_2-algebras

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$$\int_\Sigma \mathcal{A} \xrightarrow{\phi_{\mathcal{A}}(H) = M \otimes_{a_\Sigma} -} \mathcal{A}$$

$$\int_\Sigma \mathcal{A} = a_\Sigma\text{-mod}_{\mathcal{A}}$$

H, H' with $\partial H = \partial H'$

framed E_2-algebras

Thm [M

$X_1, X_2 \in$