Title: Quantum Steenrod operations of symplectic resolutions

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Abstract: We study the quantum connection in positive characteristic for conical symplectic resolutions. We conjecture the equivalence of the p-curvature of such connections with (equivariant generalizations of) quantum Steenrod operations of Fukaya and Wilkins, which are endomorphisms of mod p quantum cohomology deforming the Steenrod operations. The conjecture is verified in a wide range of examples, including the Springer resolution, thereby providing a geometric interpretation of the p-curvature and a full computation of quantum Steenrod operations. The key ingredients are a new compatibility relation between the quantum Steenrod operations and the shift operators, and structural results for the mod p quantum connection recently obtained by Etingof--Varchenko.

Zoom link: https://pitp.zoom.us/j/91010341249?pwd=QXJuMlJrWWd0dHpPdUpDUGVqVmYvZz09

Quantum Steenrod Operations of Symplectic Resolutions

Goal: Compute symplectic geometric / Floer theoretic invts
for (conical) symplectic resolutions

$\hookrightarrow X/\mathbb{C}$ smooth, $\omega_{\mathbb{C}}$ holo. symp. $X \to \mathrm{Spec}(\mathcal{O}_X)$ is resolution

e.g. T^*(flag varieties), quiver varieties

Today, I focus on

<u>quantum connection</u> <u>in mod p</u>

and

<u>Quantum Steenrod Operations</u>

for conical symplectic resolutions.

$a \in H^2(X;\mathbb{F}_p)$

$\nabla_a : QH^*(X)[[t]] \circlearrowleft$

$b \in H^*(X;\mathbb{F}_p)$

$\Sigma_b : QH^*(X)[[t,\theta]] \circlearrowleft$

Quantum connection in mod p

(X,ω) Kähler manifold $\boxed{c_1 \geq 0}$ semipositive

$\Lambda = \mathbb{F}_p[[q^A : A \in H_2(X;\mathbb{Z})]]$ (Novikov ring)

$\rightsquigarrow QH^*(X) = (H^*(X;\Lambda), *)$ defined: $a * b = \sum_{\substack{c \\ A \in H_2}} \#\{\ \} \, c \, q^A$

$*|_{q=0} = \cup$

$a \in H^2(X) \rightsquigarrow \partial_a : \Lambda \longrightarrow \Lambda$, $\partial_a q^A = (a \cdot A) q^A$

$\Rightarrow$ quantum connection $\nabla_a = t\,\partial_a + \underline{a *}$ (quantum connection)

formal param $|t| = 2$

n in mod p

→ semipositive

anifold $\boxed{(G \geq 0)}$

$H_2(X;\mathbb{Z})]]$ (Novikov ring)

$(H^*(X;\Lambda), *)$ defined: $a * b = \sum_{\substack{c \\ A \in H_2}} \#\{ \ \}\, c\, q^A$

$*|_{q=0} = \smile$

$\Lambda \longrightarrow \Lambda \qquad \partial_a q^A = (a \cdot A)\, q^A$

formal param $|t| = 2$

$\nabla_a = t\,\partial_a + \underline{a*}$ (quantum connection) : $QH^*(X) \longrightarrow QH^{*+2}(X)[[t]]$

Quantum connection is easy to define, but rich

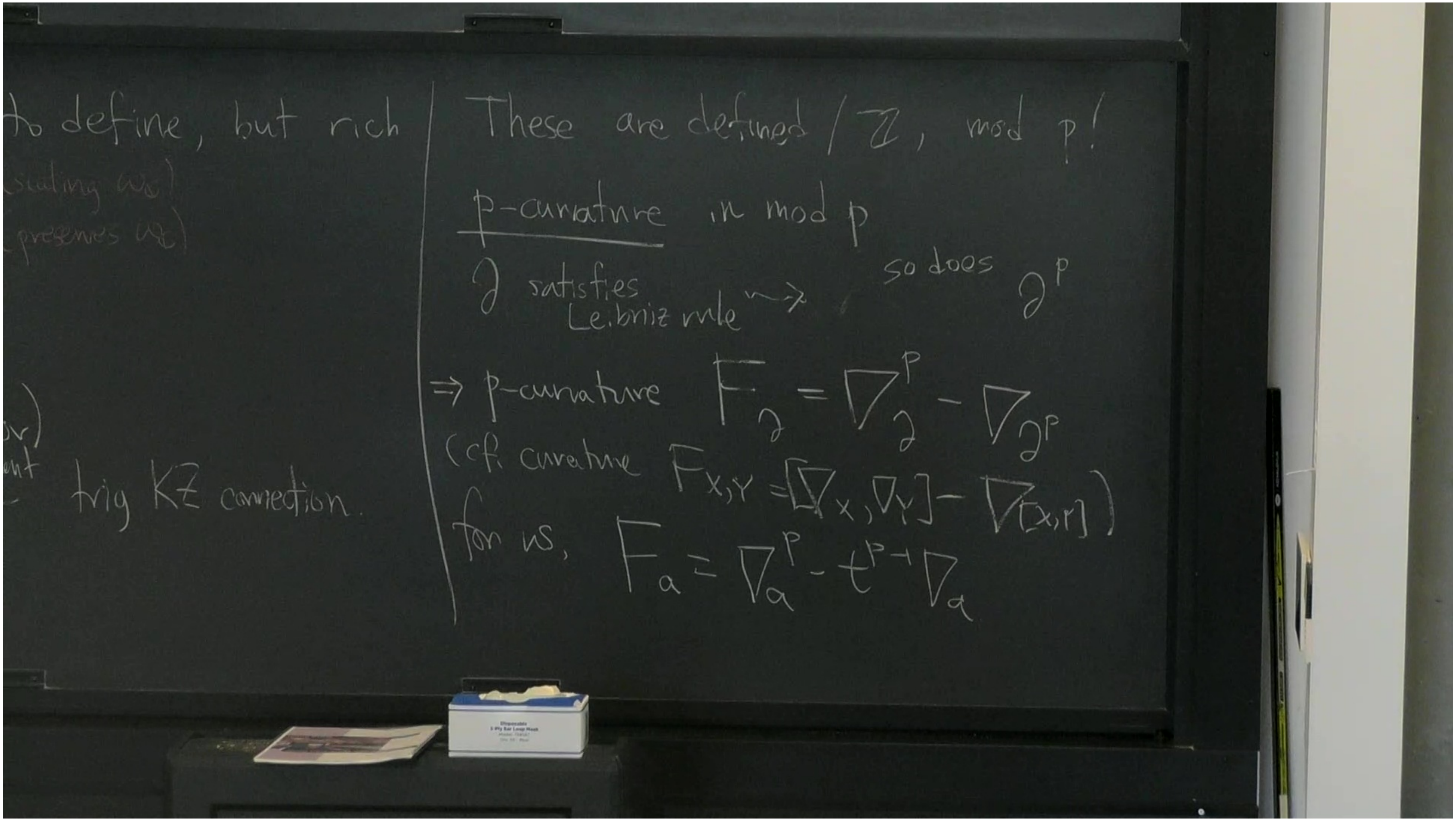
$X =$ conical symplectic resolutions $\circlearrowleft$ S^1-action (scaling $\omega_{\mathbb{C}}$)
T-action (preserves $\omega_{\mathbb{C}}$)

$\rightsquigarrow QH^*_{eq}$, ∇^{eq} ← essential, ...

Thm (Braverman-Maulik-Okounkov)

∇^{eq} for $X = T^*G/B \underset{\text{gauge equivalent}}{\sim}$ trig KZ connection.

[t]]

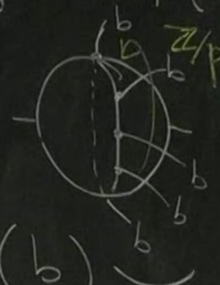

to define, but rich
(...uting ω_0)
(preserves ω_0)
trig KZ connection.
These are defined / $\mathbb{Z}$, mod p!
p-curvature in mod p
∂ satisfies Leibniz rule $\rightsquigarrow$ so does ∂^p
$\Rightarrow$ p-curvature $F_\partial = \nabla_\partial^p - \nabla_{\partial^p}$
(cf. curvature $F_{X,Y} = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$)
for us, $F_a = \nabla_a^p - t^{p-1}\nabla_a$

(quantum connection)

Quantum Steenrod Operations [Fukaya '97, Wilkins '18]

$b \in H^*(X;\mathbb{F}_p) \rightsquigarrow \Sigma_b : QH^*(X;\mathbb{F}_p) \longrightarrow (QH^*(X)[[t,\theta]])^{*+p|b|}$ $|t|=2$, $|\theta|=1$

$\cong QH^*(X) \otimes H^*(B\mathbb{Z}/p;\mathbb{F}_p)$ $(p>2)$

defined from counts of  $\longrightarrow X$

Properties) (i) $\Sigma_b|_{q=0} = St(b)$

(ii) $\Sigma_b|_{(t,\theta)=0} = \underbrace{b * b * \cdots * b}_{p \text{ times}}$

(cf. classical Steenrod operations $St : H^*(X;\mathbb{F}_p) \longrightarrow H^*(X;\mathbb{F}_p)[[t,\theta]]$)

Σ_b has applications in Ham. dynamics, mod p mirror symmetry

But, it's hard to compute

Main Result Thm. (L., in preparation)

$X =$ conical symplectic resolutions (X^T discrete)

Fix $b \in H^2(X)$, then $F_b - \Sigma_b =$ nilpotent

Moreover, if $b*$ has distinct eigenvalues, $F_b - \Sigma_b = 0$ (i.e. p-curvature is qu s applications in Ham. dynamics, mod p mirror symmetry

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if $b*$ has distinct eigenvalues, $F_b - \Sigma_b = 0$ (i.e. p-curvature is quantum Steenr

(iii) $\Sigma_b \circ \Sigma_{b'} = \Sigma_{b * b'}$ primes

Methods of proof: shift operators

ho

For every cocharacter $\sigma : S^1 \longrightarrow (T \times S^1) \circlearrowright_z X$

$$S_\sigma : QH^*(X) \longrightarrow QH^*(X)[[t]]$$

defined from counts of holomorphic sections of

$$\begin{tikzcd} X \arrow[r, hook] & E_\sigma = D^2 \times X \cup_\sigma D^2 \times X \arrow[d] \\ & \mathbb{P}^1 \arrow[u, bend right, "s"'] \end{tikzcd}$$

Key property: $S_\sigma f(z,t) = f(z+\sigma t, t)\, S_\sigma$

Thm (Okounkov-Pandharipande) $\quad \nabla_a\big|_{z \mapsto z+\sigma t} \circ S_\sigma = S_\sigma \circ \nabla_a$

(compatibility)

Thm (Seidel-Wilkins '22) $\quad \nabla_a \circ \Sigma_b = \Sigma_b \circ \nabla_a$

Thm (Okounkov-Pandharipande)

$$\nabla_a\big|_{z\mapsto z+\sigma t} \circ S_\sigma = S_\sigma \circ \nabla_a$$

(compatibility)

Thm (Seidel-Wilkins '22)

$$\nabla_a \circ \Sigma_b = \Sigma_b \circ \nabla_a$$

(covariant constancy)

partially determine
Σ_b in <u>low degr</u>

Thm. (L.)

$$\Sigma_b\big|_{z\mapsto z+\sigma t} \circ S_\sigma = S_\sigma \circ \Sigma_b$$

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(compatibility)

Thm (Seidel–Wilkins '22) $\nabla_a \circ \Sigma_b = \Sigma_b \circ \nabla_a$ → partic

Σ

(covariant constancy)

Thm. (L) $\Sigma_b \big|_{z \mapsto z+\sigma t} \circ S_\sigma = S_\sigma \circ \Sigma_b$ (*)

$$\underset{\|}{b^p - t^{p-1} b}$$

(i) $F_b, \Sigma_b \big|_{q=0} = St(b) \smile$, (ii) $F_b, \Sigma_b \big|_{(t,\theta)=0} = \underbrace{b * \cdots * b}_{p \text{ times}} *$, (*)

kov-Pandharipande) $\nabla_a\big|_{z\mapsto z+\sigma t} \circ S_\sigma = S_\sigma \circ \nabla_a$

(compatibility)

l-Wilkins '22) $\nabla_a \circ \Sigma_b = \Sigma_b \circ \nabla_a$ → partially determine Σ_b in low degrees

(covariant constancy)

$\underline{b^p - t^{p-1} b}$ $\Sigma_b\big|_{z\mapsto z+\sigma t} \circ S_\sigma = S_\sigma \circ \Sigma_b$ (*)

$\Sigma_b\big|_{q=0} = St(b) \smile$, (ii) F_b, $\Sigma_b\big|_{(t,\theta)=0} = \underbrace{b * \cdots * b}_{p \text{ times}} *$, (*) $\Rightarrow$ main result.

Thm (Etingof–Varchenko)

for ∇^{eq} in mod p,

$(b \in H^2)$ if $b*$ has distinct eigenvalues

$\Rightarrow$ so does F_b

known for many symplectic resolutions

- T^*G/B, T^*G/P
- quiver varieties type A
- $\mathrm{Hilb}^n(\mathbb{C}^2)$

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Conj. $F_b = \Sigma_b$ on $b \in H^2(X)$ agree always!

(quadrics in $\mathbb{CP}^5$)