Title: Q-functions in spin chains. QSC for spin chains

Speakers: Paul Ryan

Collection: Quantum Spectral Curve and Three Point Functions mini-course

Date: October 16, 2023 - 2:00 PM

URL: https://pirsa.org/23100083

* QSC for $su(2)$, $sl(2)$ spin chains

- Modern formulation of spectral problem

QSC = QQ system

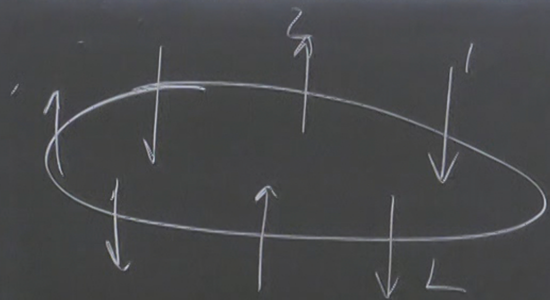
Complex Analysis $\quad Q_A(u)$

index

- Riemann surface (model dependent)

- Functional Rels. (symmetry)

su(2)-invariant Heisenberg spin chain



L particles

$$H = \sum_{n=1}^{L} H_{n,n+1}$$

$$\downarrow$$

$$\frac{1}{4}\left(\sigma^x_n \otimes \sigma^x_{n+1} + \sigma^y_n \otimes \sigma^y_{n+1} + \sigma^z_n \otimes \sigma^z_{n+1} - 1 \otimes 1\right)$$

Periodicity $L+1 := 1$

* Global su(2) symmetry

$$[H, \vec{S}] = 0$$

Eigenspaces of H $\Rightarrow$ irreps of su(2)

$$(\mathbb{C}^2)^{L} = \bigoplus_{\lambda} V_{\lambda}$$

λ ← Young diagram

$$\square \otimes \square = \square\!\square \oplus \begin{smallmatrix}\square \\ \square\end{smallmatrix}$$

$V_\lambda, \; S^+|\Psi_\lambda\rangle = 0$

$= E_\lambda |\Psi_\lambda\rangle$

Vacuum $|0\rangle = |\underbrace{\uparrow\uparrow\ldots\uparrow}_{L}\rangle$

$= \frac{L}{2}|0\rangle$

Flip spins

$$|\Psi\rangle \sim |\underbrace{\downarrow\downarrow\ldots\downarrow}_{S\text{-magnon state}}\uparrow\ldots\uparrow\rangle$$

$$S^z|\Psi\rangle = \left(\frac{L}{2}-S\right)|\Psi\rangle$$

SU(2) Q-system

$$Q_1(u),\ Q_2(u),\ Q_{12}(u)$$

$$f^{\pm} := f\left(u \pm \tfrac{i}{2}\right)$$

$$Q_{12} = Q_1^+ Q_2^- - Q_2^+ Q_1^-$$

Ana

2-3:30

HW due

Analytic properties $su(2)$ spin chain

- Boundary Condition $Q_{12}(u) = u^L$

- $Q_1(u), Q_2(u)$ = polynomial

* Completeness: 1 $\Leftrightarrow$ 1 correspondence solutions $\Leftrightarrow$ H eigenspaces

S-magnon state

Bethe roots

$$Q_1 \sim A_1 u^S = A_1 \prod_{k=1}^{S} (u - u_k), \quad E = -\frac{i}{2} \partial_u \log \frac{Q_1(u+i/2)}{Q_1(u-i/2)} \Big|_{u=0}$$

$$Q_2 \sim A_2 u^{L-S+1}$$

* Global su(2) symmetry

$$[H, \vec{S}] = 0$$

Eigenspaces of $H \to$ irreps of su(2)

$$(\mathbb{C}^2)^L = \bigoplus_\lambda V_\lambda$$

$$\square \otimes \square = \square\square \oplus \begin{array}{|c|}\hline \\ \hline \\ \hline \end{array}$$

Young diagram

$|\Psi_\lambda\rangle \in$

$H|\Psi_\lambda\rangle$

Usual	V

$H|0\rangle =$

$S^2|0\rangle =$

Bethe roots

$su(2)$

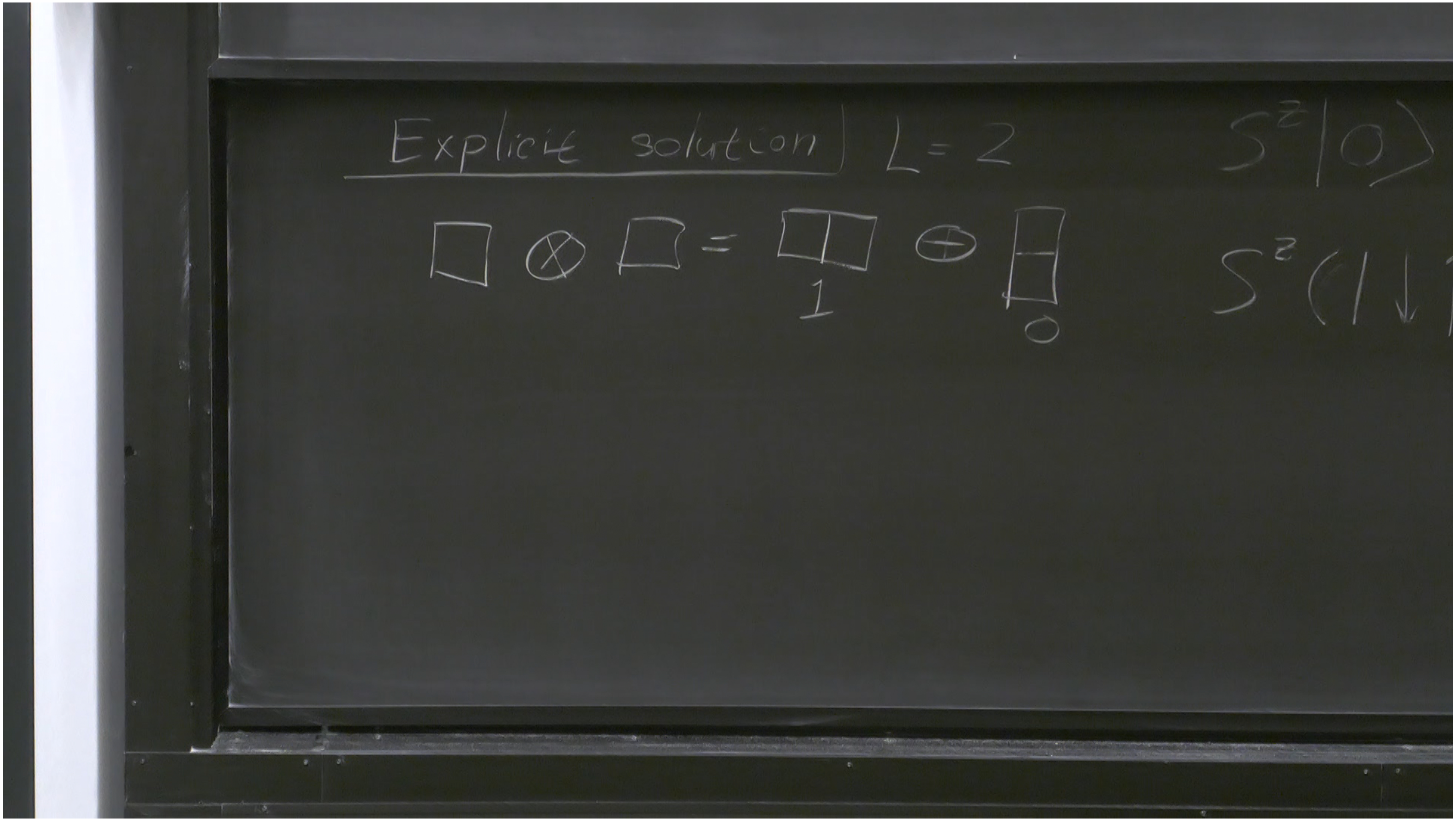
$|\Psi_\lambda\rangle \in V_\lambda , \; S^+ |\Psi_\lambda\rangle = 0$

$H|\Psi_\lambda\rangle = E_\lambda |\Psi_\lambda\rangle$

$\boxed{\text{Usual}}$ Vacuum $|0\rangle = |\underbrace{\uparrow\uparrow\ldots\uparrow}_{L}\rangle$

$H|0\rangle = 0$

$S^z|0\rangle = \frac{L}{2}|0\rangle$


Explicit solution
L = 2
1
0
S^z|0⟩

$$S^2|0\rangle = \frac{1}{2}|0\rangle = |0\rangle \sim \square\square$$

$$S^2\left(|\downarrow\uparrow\rangle - |\uparrow\downarrow\rangle\right) = 0 \sim \begin{smallmatrix}\square\\ \square\end{smallmatrix}$$

$$0\rangle = |0\rangle \sim \square\square \to S=0\,,\ Q_1(u)=A_1\,,\ Q_2\sim A_2u^3$$

$$|\uparrow\downarrow\rangle\rangle = 0 \sim \begin{smallmatrix}\square\\ \square\end{smallmatrix} \to S=1\,,\ Q_1(u)\sim A_1u\,,\ Q_2\sim A_2u^2$$

$$* \qquad Q_1(u) = A_1(u - u_1)$$

$$Q_2(u) = A_2 u^2 + au + b$$

Use symmetries $A_1 \to 1$. $Q_1 = u - u_1$.

$$Q_2 \to Q_2 - a\, Q_1.$$

$$\Rightarrow Q_2 = A_2 u^2 + b$$

$$u^L = Q_1(u+i/2)Q_2(u-i/2) - Q_2(u+i/2)Q_1(u-i/2)$$

$$A_2 = i, \quad u_1 = 0, \quad b = i/4 \Rightarrow E = -2.$$

$$u^L = A_1(u+i/2)^s(u-i/2)^c A_2 - A_1 A_2 (u-i/2)^s (u+i/2)^c$$

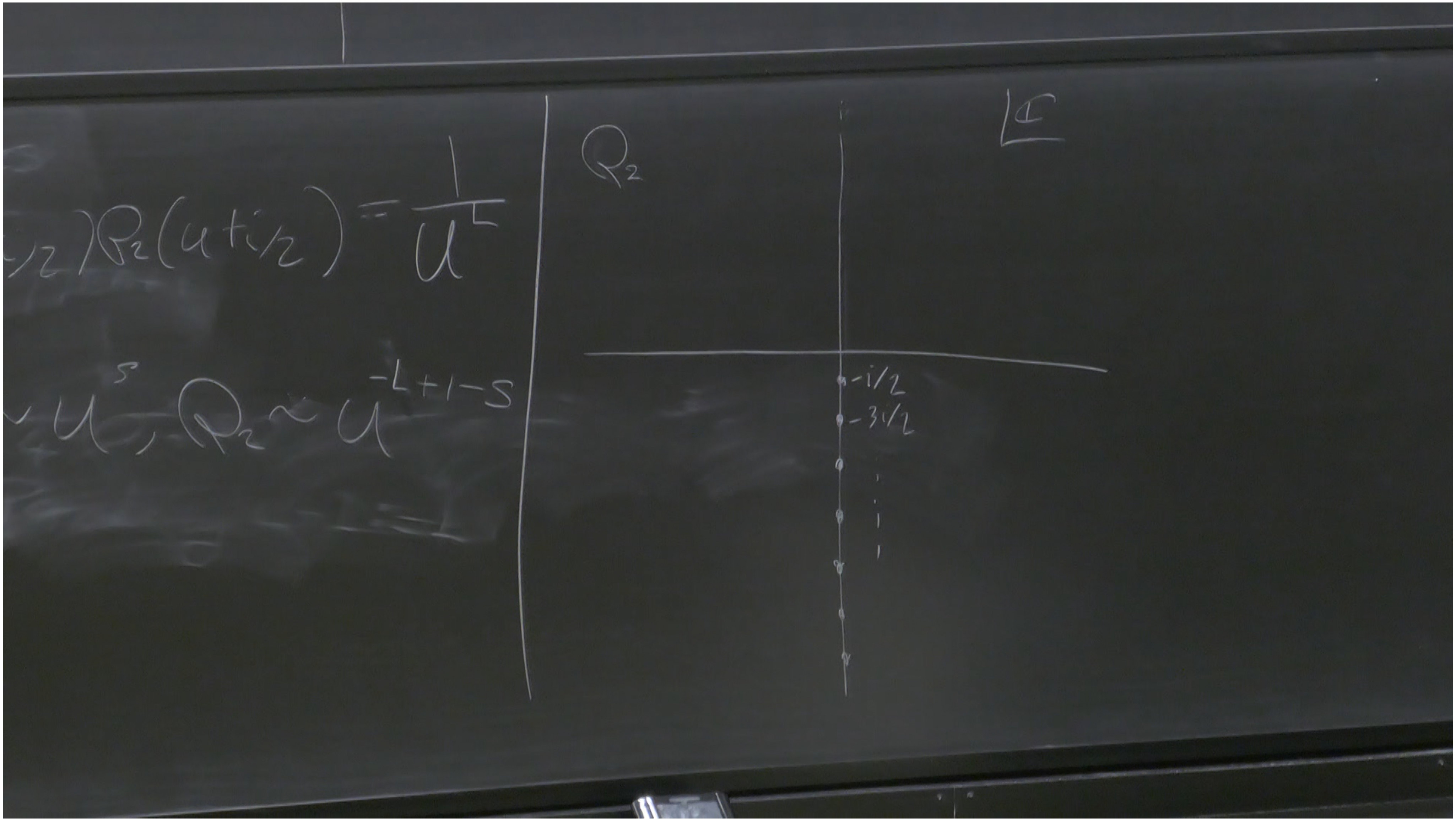
$$\sim A_1 A_2 (u^{s+c} - u^{s+c}) - u^{s+c-1}(\ldots)$$

$$= u^L \Rightarrow c = L+1-S$$

spin-1/2

$$Q_1(u+i/2)\,Q_2(u-i/2) - Q_1(u-i/2)\,Q_2(u+i/2) = \frac{1}{u^L}$$

* $Q_1(u)$ - polynomial, $Q_1 \sim u^s$, $Q_2 \sim u^{-L+1-s}$


Q2
ℂ
)Q2(u+i/2) = 1/u^L
~ u^s, Q2 ~ u^(-L+1-S)
-i/2
-3i/2

$$\begin{vmatrix} Q_a(u+i) & Q_a(u) & Q_a(u-i) \\ Q_1(u+i) & Q_1(u) & Q_1(u-i) \\ Q_2(u+i) & Q_2(u) & Q_2(u-i) \end{vmatrix} = 0 \quad , a=1,2.$$

$$Q_a(u+i)P_1(u) - Q_a(u)P_2(u) + Q_a(u-i)P_3(u) = 0$$

2.

2.

$$Q_a(u+i)\,D_1(u) - Q_a(u)\,D_2(u) + Q_a(u-i)\,D_3(u) = 0$$

spin $\frac{1}{2}$ $\qquad \overset{\shortparallel}{T}(u)$

$$Q_1(u)\,Q_2(u-i) - (1 \leftrightarrow 2) = \left(u-\tfrac{i}{2}\right)^L \qquad \left(u+\tfrac{i}{2}\right)^L$$

$$\left(u-\tfrac{i}{2}\right)^L Q_a(u-i) - Q_a(u)\,T(u) + \left(u+\tfrac{i}{2}\right)^L Q_a(u+i) = 0$$

* Why "QSC"

equations of motion

– Classical integrability $\dot{L} = [L, M]$

$\mathrm{tr}\, L = 0$

– Classical spec. curve $\det(\lambda - L) = \lambda^2 - \mathrm{tr}L\, \lambda + \det L$

– Quantize: $\det(\lambda - \hat{L}D)\, Q_a(u) = 0$

$D = e^{i\partial_u}$