

Title: Topological Quantum Field Theories Lecture 20231027

Speakers: Lukas Mueller

Collection: Topological Quantum Field Theories - mini-course

Date: October 27, 2023 - 2:00 PM

URL: <https://pirsa.org/23100081>

## 2) Basic properties

$$Z: \text{Bord}_n \longrightarrow \text{Vect}_\mathbb{C}, \quad \varepsilon: Z(\phi) \xrightarrow{\sim} \mathbb{C}$$

$$[M] \in \text{Bord}_n(\phi, \phi)$$

$$\sum_{\Sigma} \mu_{\Sigma} Z(\Sigma \sqcup \Sigma')$$

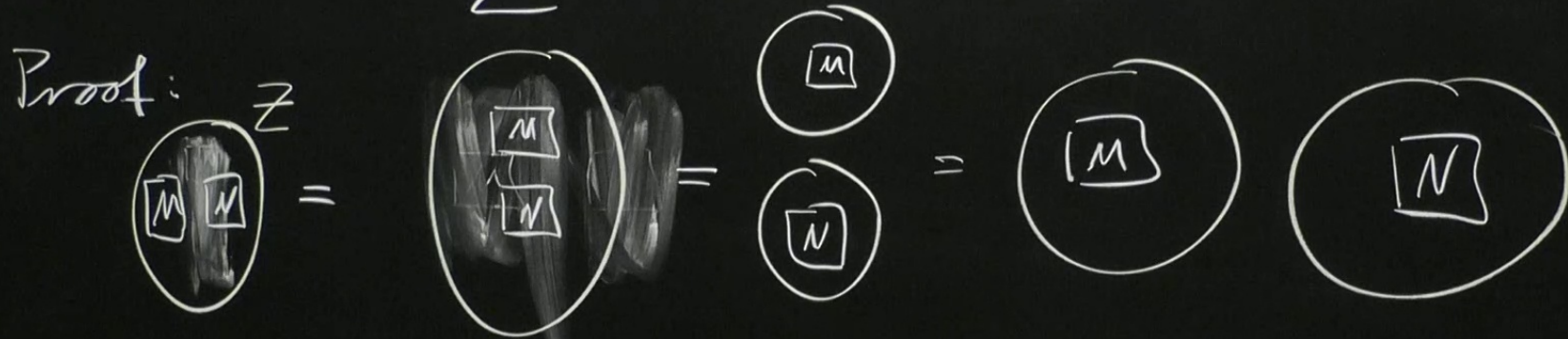
$$\mathbb{C} \xrightarrow{\varepsilon} Z(\phi) \xrightarrow{Z(M)} Z(\phi) \xrightarrow{\varepsilon^{-1}} \mathbb{C}$$

$$\downarrow \mu$$

$$Z(\Sigma) \otimes Z(\Sigma')$$

$$I_Z(M) = \varepsilon^{-1} \circ Z(M) \circ \varepsilon [1] \in \mathbb{C}$$

Prop 2.1.  $I_Z(M \perp N) = I_Z(M) \cdot I_Z(N)$



$$\begin{aligned}
 \varepsilon^{-1} \circ Z(M \perp N) \varepsilon [1] &= \varepsilon^{-1} Z(M \circ N) \varepsilon [1] \\
 &= \varepsilon^{-1} Z(M) \circ Z(N) \varepsilon [1] = \varepsilon^{-1} Z(M) \varepsilon \circ \varepsilon^{-1} Z(N) \varepsilon [1] \\
 &= I_Z(M) I_Z(N) \quad \square
 \end{aligned}$$

## 2) Basic properties

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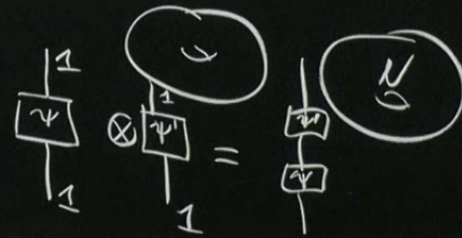
$$\mathbb{C} \xrightarrow{\varepsilon} Z(\phi) \xrightarrow{Z(M)} Z(\phi) \xrightarrow{\varepsilon^{-1}} \mathbb{C}$$

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$$\mu_{\Sigma, \Sigma'} Z(\Sigma \amalg \Sigma')$$

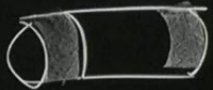
$$\downarrow Z$$

$$Z(\Sigma) \otimes Z(\Sigma')$$



$$\Sigma \in \text{Bord}_n \quad \text{MPC}(\Sigma) = \pi_0(\text{Diff}(\Sigma))$$

$$e \rightsquigarrow \Sigma \times [0,1] \xleftarrow{\text{inv}} \Sigma \times [0,1] =: \mathcal{M}(e)$$

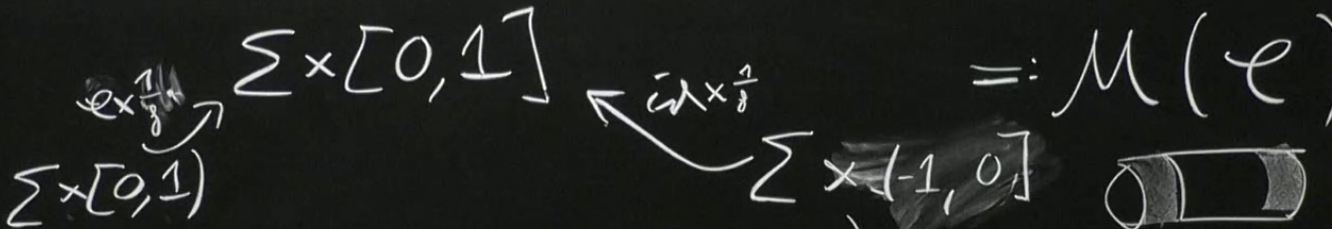
$\xrightarrow{\text{ex}} \Sigma \times [0,1]$ 


$$\mathcal{Z}(\mathcal{M}(e)): \mathcal{Z}(\Sigma) \rightarrow \mathcal{Z}(\Sigma)$$

Exercise: This gives an action of  $\text{MPC}(\Sigma)$  on  $\mathcal{Z}(\Sigma)$ .

$$\mathcal{M}(e): \Sigma \rightarrow \Sigma \in \text{Bord}_n$$

$$\Sigma \in \text{Bord}_n \quad \text{MPC}(\Sigma) = \pi_0(\text{Diff}(\Sigma))$$

$$e \rightsquigarrow \begin{array}{c} \xrightarrow{\text{ex} \frac{1}{g}} \\ \Sigma \times [0, 1] \end{array} \xleftarrow{\text{in} \frac{1}{g}} \Sigma \times [-1, 0] =: \mathcal{M}(e)$$


$$\mathcal{Z}(\mathcal{M}(e)) : \mathcal{Z}(\Sigma) \rightarrow \mathcal{Z}(\Sigma)$$

Exercise: This gives an action of  $\text{M}(\mathcal{G}(\Sigma))$  on  $\mathcal{Z}(\Sigma)$ .

$$\mathcal{M}(e) : \Sigma \rightarrow \Sigma \in \text{Bord}_n$$

Def. Let  $Z: \text{Bord}_n \rightarrow \text{vect}$  be TQFT and  $\Omega$  an oriented closed  $k < n$  dimensional manifold. The dimensional reduction  $\text{Red}_\Omega Z$  is the theory:

$$\text{Red}_\Omega Z: \text{Bord}_{n-k} \xrightarrow{\times \Omega} \text{Bord}_n \xrightarrow{Z} \text{vect}$$

$$M, \Sigma \rightarrow \Sigma' \mapsto (M \times \Omega, \Sigma \times \Omega \rightarrow \Sigma' \times \Omega)$$



Def. Let  $Z: \text{Bord}_n \rightarrow \text{Vect}$   
 $\Omega$  an oriented closed  $k < n$  dimensional  
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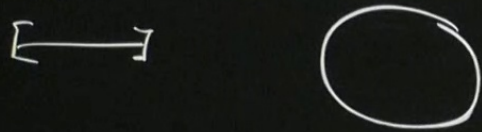
In particular,  $\forall \Sigma \in \text{Bord}_n \xrightarrow{\sim} \text{IDTQFT } \text{Red}_\Omega Z$   
which maps  $\text{pt}^+ \rightarrow Z(\Sigma)$

$M, \Sigma \rightarrow \Sigma' \mapsto (M \times \Omega, \Sigma \times \Omega \rightarrow \Sigma' \times \Omega)$



Lemma/Fact:

Every connected compact <sup>oriented</sup> 1D manifold is  
diffeomorphic to either  $S^1$  or  $[0,1]$



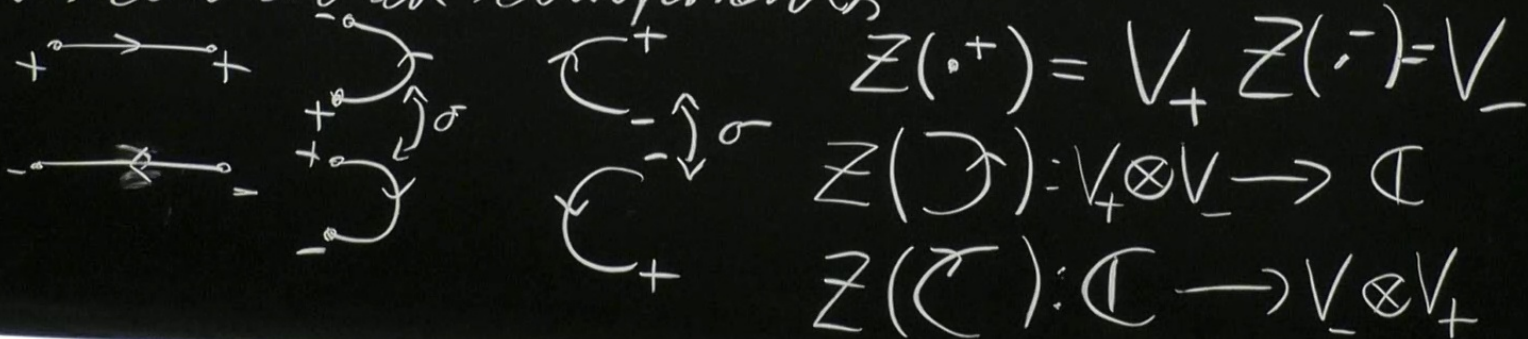
Every 1D cobordism is a disjoint union of  
its connected components

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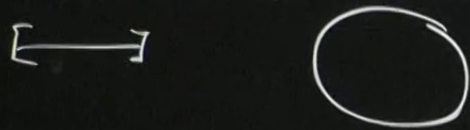


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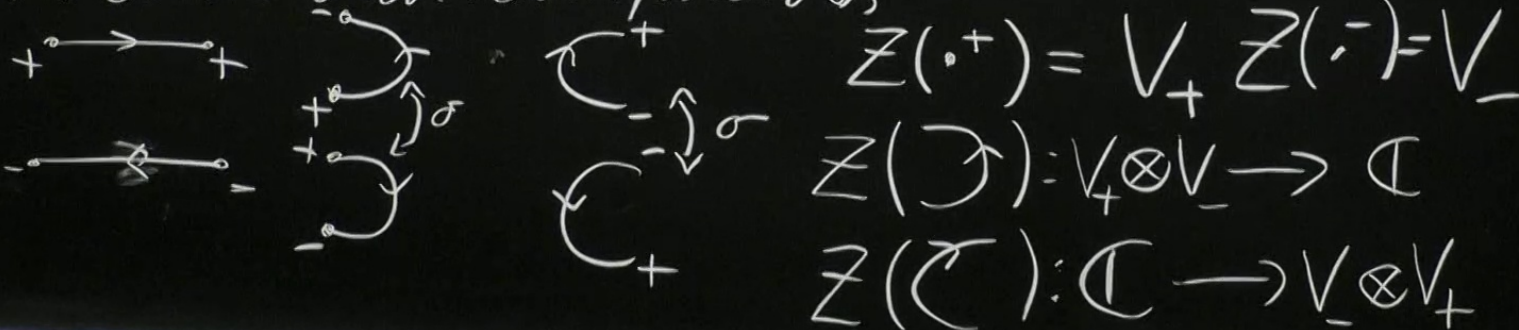


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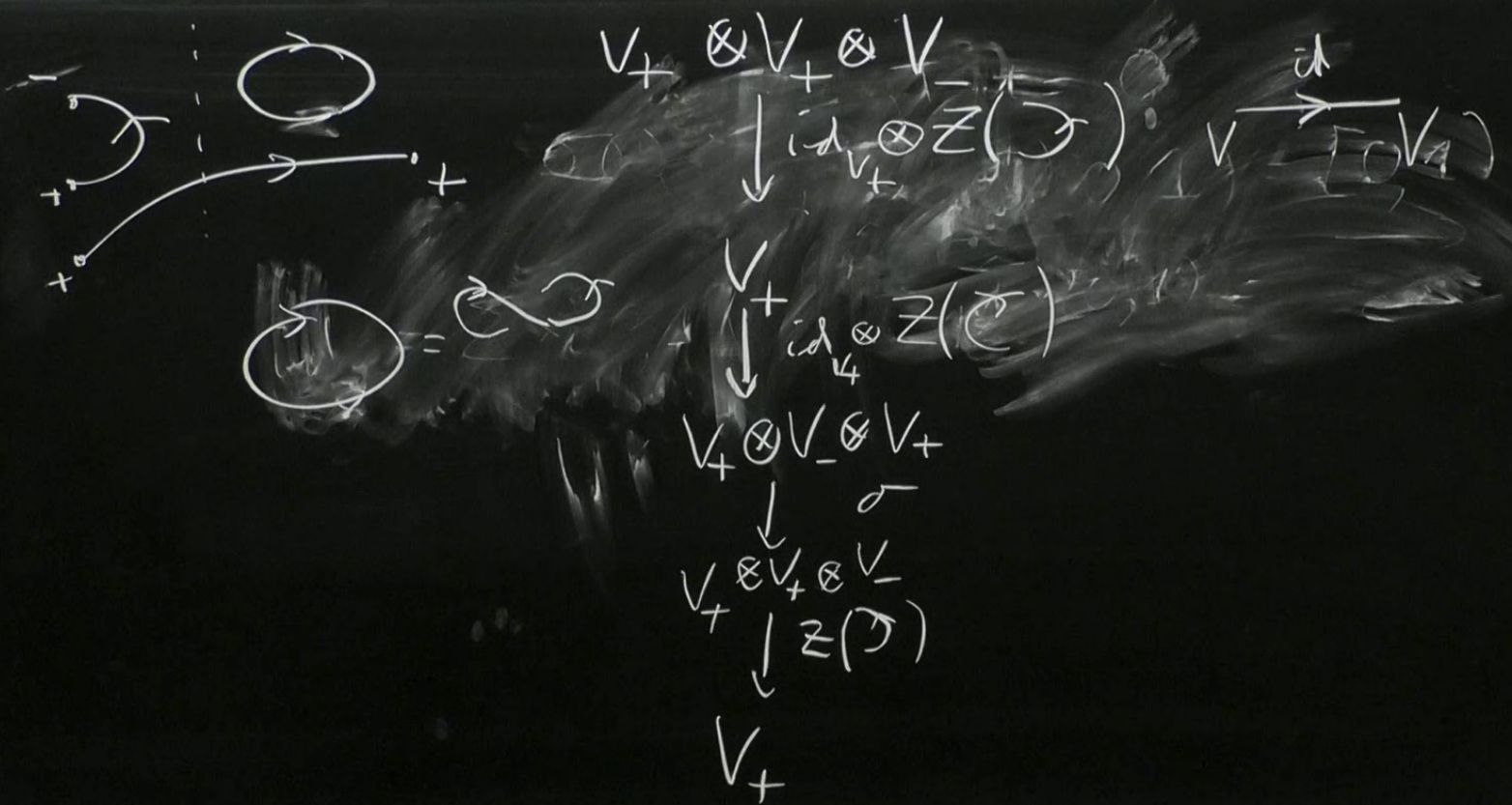
$$Z(\cdot^+) = V_+ \quad Z(\cdot^-) = V_-$$

$$Z(\sigma) = V_+ \otimes V_- \rightarrow \mathbb{C}$$

$$Z(\sigma) = \mathbb{C} \rightarrow V_- \otimes V_+$$

CAUTION  
 WE BELIEVE WE LISTED THE HAZARDOUS MATERIALS,  
 BUT WE CANNOT BE RESPONSIBLE FOR THE ACCIDENT.  
 IF YOU EXPERIMENTALIST HAS ANY  
 QUESTIONS PLEASE CONTACT US.

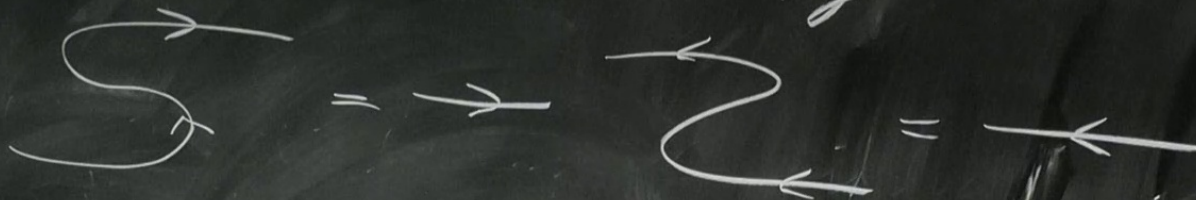
Bord<sub>n</sub> is the n-representable category



CAUTION  
DO NOT TOUCH THE BOARD  
IF YOU ARE NOT A TEACHER

Exercise

convince yourself that all relations between cobordisms are induced by



Lemma:

The category of IDT QFS is equivalent to the category with

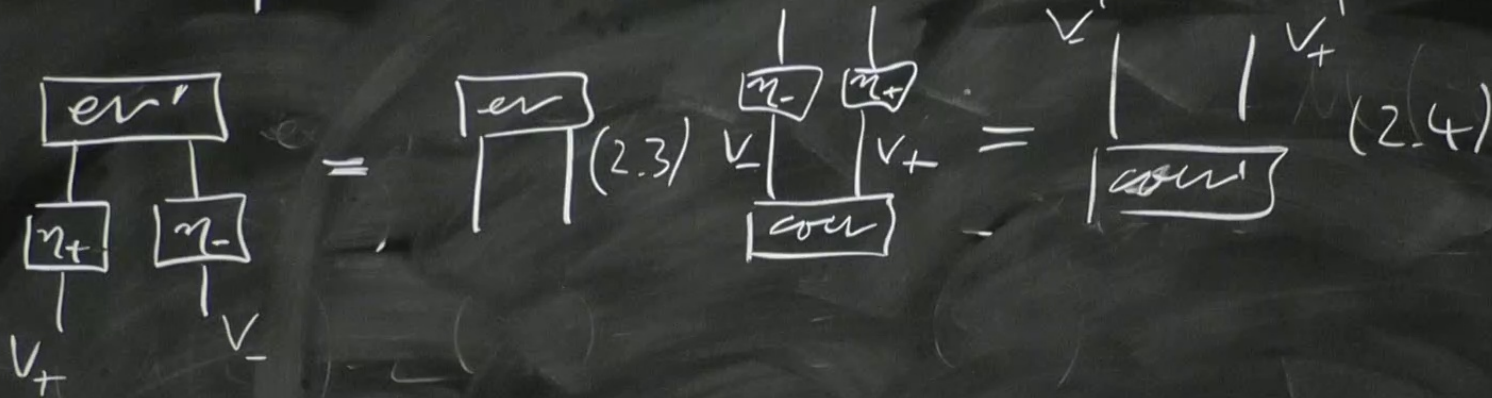
Obj: - A pair of vector spaces  $V_+$  &  $V_-$

- Linear map  $ev: V_+ \otimes V_- \rightarrow \mathbb{C}$ ,  $coev: \mathbb{C} \rightarrow V_- \otimes V_+$

s.t.

$$\begin{array}{c} \boxed{ev} \\ \downarrow \\ V_+ \end{array} \Big|_{V_+} = \Big|_{V_+} \quad \& \quad \begin{array}{c} \boxed{ev} \\ \downarrow \\ V_- \end{array} \Big|_{V_-} = \Big|_{V_-}$$

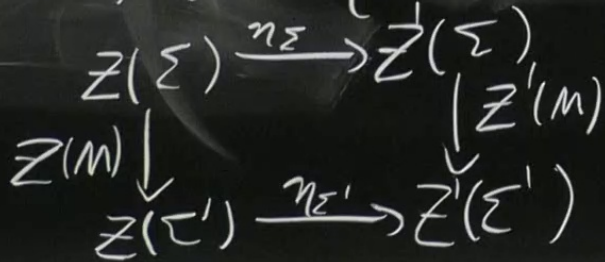
Mor:  $\eta_+ : V_+ \rightarrow V'_+$  &  $\eta_- : V_- \rightarrow V'_-$



Proof:

- Obj: ✓

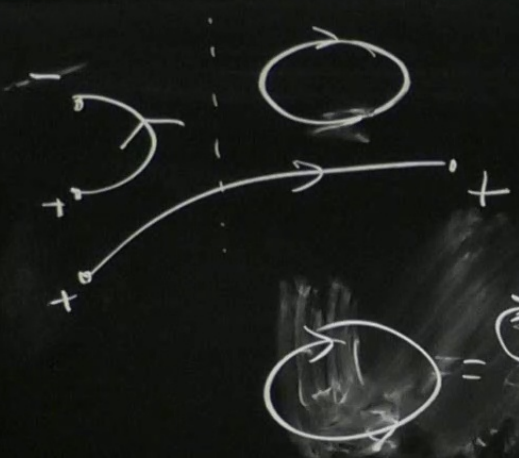
Mor:  $\eta : Z \Rightarrow Z'$



$\eta_+ : Z(pt^+) \rightarrow Z'(pt^+)$   
 $\eta_- : Z(pt^-) \rightarrow Z'(pt^-)$

$V \otimes V_+ \otimes V_-$

Diagram of the map  $\sigma$  and the map  $\sigma$



$$\begin{array}{c}
 V_+ \otimes V_+ \otimes V_- \\
 \downarrow \text{id}_{V_+} \otimes Z(\sigma) \\
 V_+ \\
 \downarrow \text{id}_{V_+} \otimes Z(\sigma) \\
 V_+ \otimes V_- \otimes V_+ \\
 \downarrow \sigma \\
 V_+ \otimes V_+ \otimes V_- \\
 \downarrow Z(\sigma) \\
 V_+
 \end{array}$$

$V \xrightarrow{\text{id}} \Gamma(V_+)$   
 $\text{Hilb} \cong \text{Vect}$



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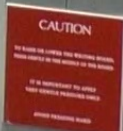
$$\begin{array}{c} \boxed{ev} \\ \downarrow \\ V_+ \end{array} \begin{array}{c} | \\ V_+ \\ | \\ \boxed{coev} \\ \downarrow \\ V_- \end{array} = \begin{array}{c} (2.1) \\ | \\ V_+ \end{array} \quad \& \quad \begin{array}{c} \boxed{ev} \\ \downarrow \\ V_- \end{array} \begin{array}{c} | \\ V_- \\ | \\ \boxed{coev} \\ \downarrow \\ V_+ \end{array} = \begin{array}{c} (2.2) \\ | \\ V_- \end{array}$$

CAUTION  
DO NOT TOUCH THE BOARD  
IF YOU ARE NOT A MEMBER OF THE BOARD  
IF YOU ARE A MEMBER OF THE BOARD  
PLEASE CONTACT THE BOARD

Def. Let  $\mathcal{C}$  be a monoidal category.

A right dual to  $X \in \mathcal{C}$  is an object  $X^\vee \in \mathcal{C}$  together with morphisms  $ev_X: X \otimes X^\vee \rightarrow \mathbb{1}$  and  $coev_X: \mathbb{1} \rightarrow X^\vee \otimes X$  satisfying (2.1) & (2.2)

$\text{Dual}(\mathcal{C})$  is the category with objects quadruples  $(X, X^\vee, ev, coev)$  & morphisms are pairs  $(f_+: X \rightarrow X', f_-: X'^\vee \rightarrow X^\vee)$  satisfying (2.3) & (2.4)



Prop.

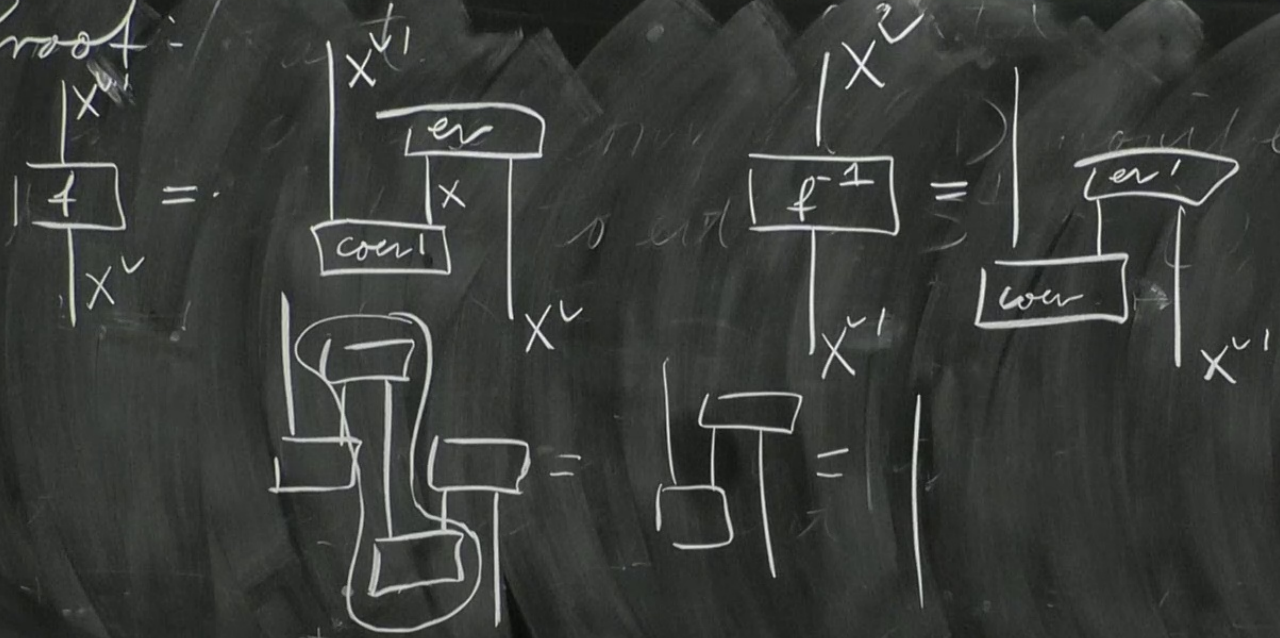
Let  $X^\vee$  and  $X^{\vee\vee}$  be right dual to  $X \in \mathcal{C}$ . Then

$\exists!$  isomorphism  $X^\vee \xrightarrow{\theta} X^{\vee\vee}$ , s.t.  $(\text{id}, \theta): (X, X^\vee)$

is a morphism in  $\text{Dual}(\mathcal{C})$

$\downarrow$   
 $(X, X^{\vee\vee})$

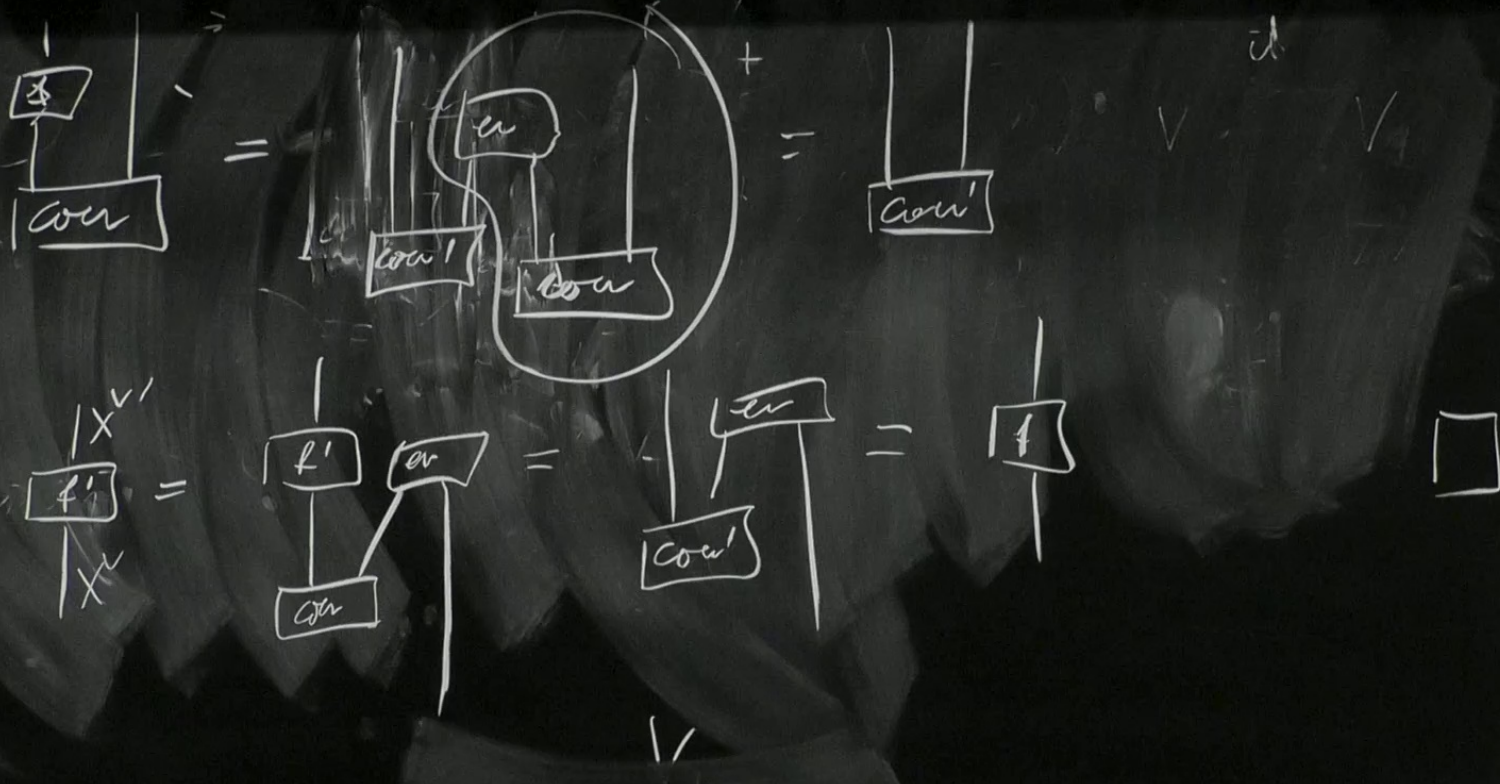
x Proof:



CAUTION  
 DO NOT TOUCH THE BOARD SURFACE.  
 IT IS NECESSARY TO USE  
 THE BOARD PROPERLY.

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1) dimensional manifolds



CAUTION  
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 OR THE BOARD SURFACE  
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Prop.

Let  $X^\vee$  and  $X^{\vee\vee}$  be right dual to  $X \in \mathcal{C}$ . Then

$\exists!$  isomorphism  $X^\vee \xrightarrow{f} X^{\vee\vee}$ , s.t.  $(\text{id}, f): (X, X^\vee)$

is a morphism in  $\text{Dual}(\mathcal{C})$

$$\downarrow \cong \\ (\text{id}, f): X$$

$$\downarrow \\ (X, X^{\vee\vee})$$

$\text{Dual}(C)$   
 $\downarrow$   
 $(e^{+1})^x$

$(x, x'')$

$Z: \text{Board}_n \rightarrow \text{Vect}$

$\downarrow$   
 $\sum Z$   
 $\text{Red}_5 Z$

$Z(Z)$

$\boxed{4}$   
 $\downarrow$   
 $x''$

**CAUTION**  
 TO RAISE OR LOWER THE WRITING BOARD,  
 PULL HANDLE IN THE MIDDLE OF THE BOARD.  
 IT IS IMPORTANT TO APPLY  
 VERY GENTLE PRESSURE ONLY.  
 AVOID PRESSING BOARD