Title: Topological Quantum Field Theories Lecture 20231027

Speakers: Lukas Mueller

Collection: Topological Quantum Field Theories - mini-course

Date: October 27, 2023 - 2:00 PM

URL: https://pirsa.org/23100081

2) Basic properties

$$Z : \mathrm{Bord}_n \longrightarrow \mathrm{Vect}_{\mathbb{C}}, \quad \varepsilon : Z(\phi) \xrightarrow{\sim} \mathbb{C}$$

$$\mu_{\Sigma,\Sigma'} : Z(\Sigma \sqcup \Sigma') \xrightarrow{\wr} Z(\Sigma) \otimes Z(\Sigma')$$

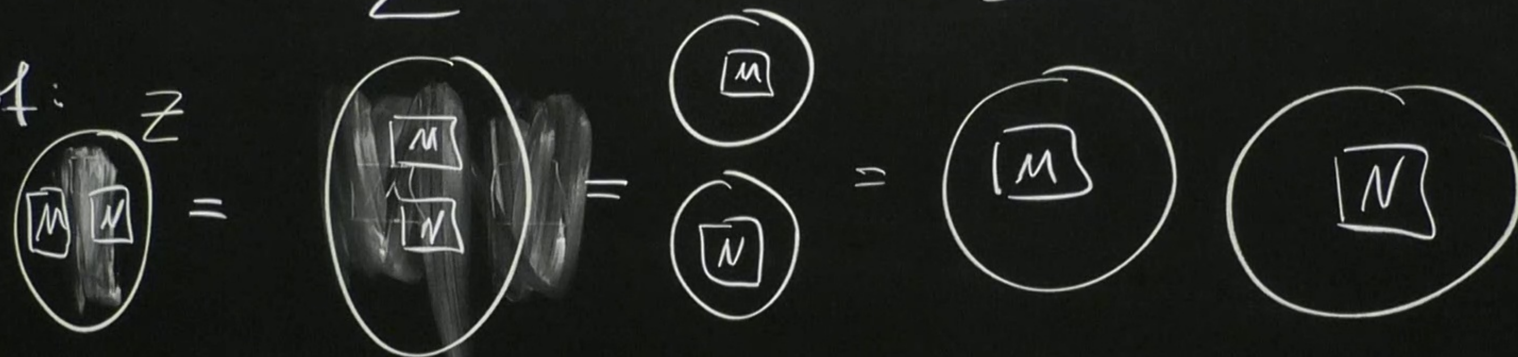
$$[M] \in \mathrm{Bord}_n(\phi, \phi)$$

$$\mathbb{C} \xrightarrow{\varepsilon} Z(\phi) \xrightarrow{Z(M)} Z(\phi) \xrightarrow{\varepsilon^{-1}} \mathbb{C}$$

$$I_Z(M) = \varepsilon^{-1} \circ Z(M) \varepsilon [1] \in \mathbb{C}$$

Prop 2.1. $I_Z(M \amalg N) = I_Z(M) \cdot I_Z(N)$

Proof:



$$\varepsilon^{-1} \circ Z(M \amalg N)\,\varepsilon[1] = \varepsilon^{-1} Z(M \circ N)\,\varepsilon[1]$$

$$= \varepsilon^{-1} Z(M) \circ Z(N)\,\varepsilon[1] = \varepsilon^{-1} Z(M)\,\varepsilon \circ \varepsilon^{-1} Z(N)\,\varepsilon[1]$$

$$= I_Z(M)\, I_Z(N) \qquad \square$$

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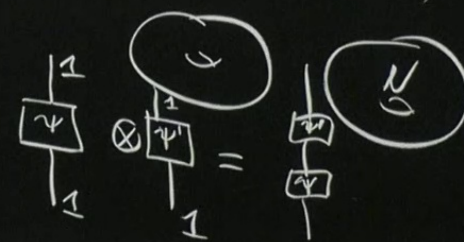
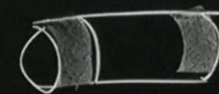


$\Sigma \in \mathrm{Bord}_n \qquad MPC(\Sigma) = \pi_0(\mathrm{Diff}(\Sigma))$

$$e \rightsquigarrow \Sigma \times [0,1] =: M(e)$$

$$\Sigma \times [0,1) \xrightarrow{e \times \frac{1}{8}\cdot} \Sigma \times [0,1] \xleftarrow{id \times \frac{1}{8}} \Sigma \times (0,1]$$



$$Z(M(e)) : Z(\Sigma) \longrightarrow Z(\Sigma)$$

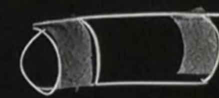
Exercise: This gives an action of $MCG(\Sigma)$ on $Z(\Sigma)$.

$$M(e) : \Sigma \longrightarrow \Sigma \in \mathrm{Bord}_n$$

$\Sigma \in \mathrm{Bord}_n$ $\qquad MCG(\Sigma) = \pi_0(\mathrm{Diff}(\Sigma))$

$$\varphi \rightsquigarrow \Sigma \times [0,1] =: M(\varphi)$$

$$\Sigma \times [0,1) \xrightarrow{\varphi \times \frac{1}{8}} \Sigma \times [0,1] \xleftarrow{\mathrm{id} \times \frac{1}{8}} \Sigma \times (-1,0]$$



$$Z(M(\varphi)) : Z(\Sigma) \longrightarrow Z(\Sigma)$$

Exercise: This gives an action of $MCG(\Sigma)$ on $Z(\Sigma)$.

$M(\varphi) : \Sigma \longrightarrow \Sigma \in \mathrm{Bord}_n$

Def. Let $Z : \mathrm{Bord}_n \longrightarrow \mathrm{Vect}$ be TQFT and Ω an oriented closed $k<n$ dimensional manifold. The dimentional reduction $\mathrm{Red}_\Omega Z$ is the theory:

$$\mathrm{Red}_\Omega Z : \mathrm{Bord}_{n-k} \xrightarrow{\times \Omega} \mathrm{Bord}_n \xrightarrow{Z} \mathrm{Vect}$$

$$M : \Sigma \to \Sigma' \longmapsto (M \times \Omega : \Sigma \times \Omega \longrightarrow \Sigma' \times \Omega)$$

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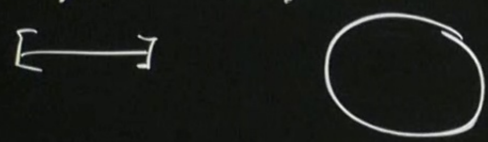
In particular, $\forall\, \Sigma \in \mathrm{Bord}_n \rightsquigarrow$ 1D TQFT $\mathrm{Red}_\Sigma Z$ which maps $pt^+ \longmapsto Z(\Sigma)$

Lemma/Fact:

Every connected compact oriented 1D manifold is diffeomorphic to either S^1 or $[0,1]$



Every 1D cobordism is a disjoint union of its connected components

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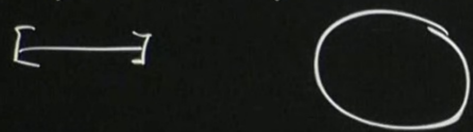

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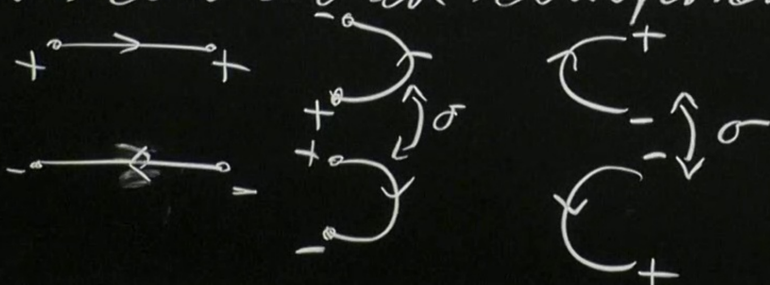

$Z(\cdot^+) = V_+ \quad Z(\cdot^-) = V_-$

$Z(\supset): V_+ \otimes V_- \longrightarrow \mathbb{C}$

$Z(\subset): \mathbb{C} \longrightarrow V_- \otimes V_+$

Lemma/Fact:

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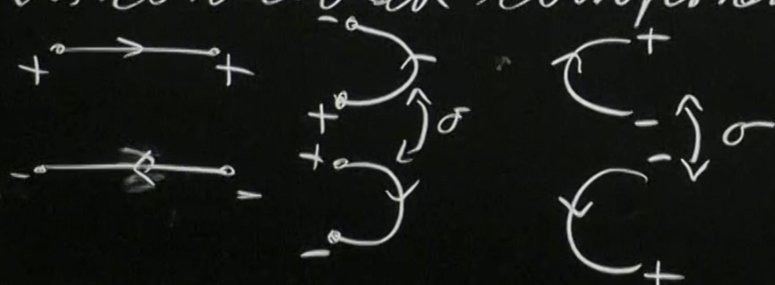

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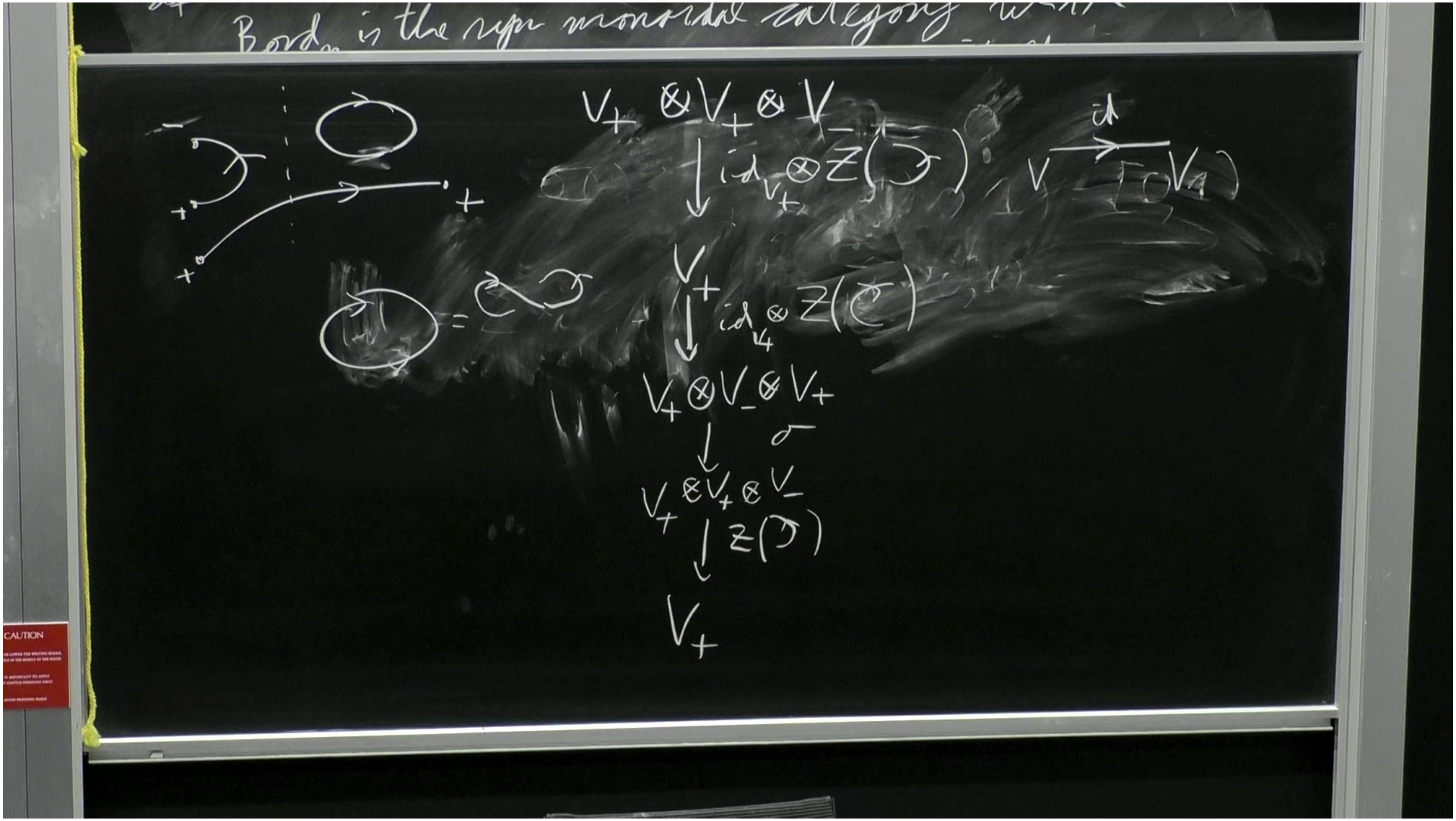
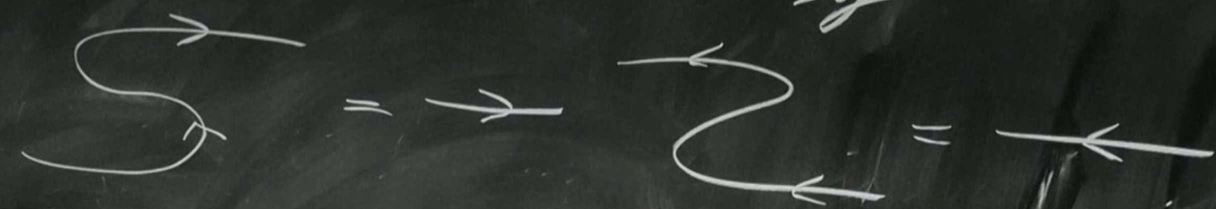

is the sym monoidal category
$V_+ \otimes V_+ \otimes V_-$
$id_{V_+} \otimes Z($
V_+
$id_{V_+} \otimes Z($
$V_+ \otimes V_- \otimes V_+$
σ
$V_+ \otimes V_+ \otimes V_-$
$Z($
V_+
id

Exercise:

convince yourself that all relations between cobordisms are induced by



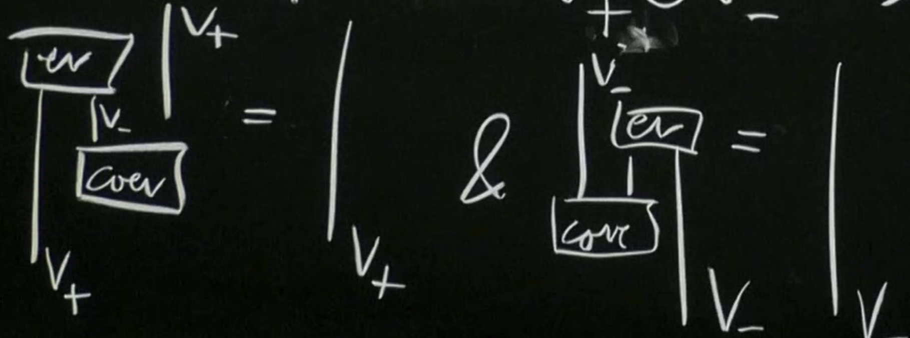
Lemma:

The category of 2D TQFTs is equivalent to the category with

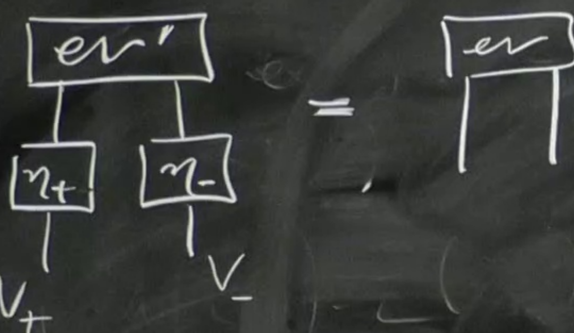
Obj: - A pair of vector spaces V_+ & V_-

- Linear map $ev: V_+ \otimes V_- \to \mathbb{C}$, $coev: \mathbb{C} \to V_- \otimes V_+$

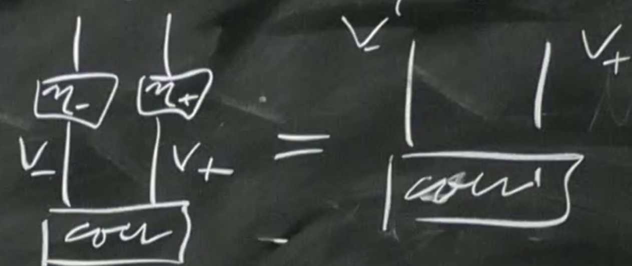
s.t.



Mor: $\eta_+ : V_+ \longrightarrow V_+'$ & $\eta_- : V_- \longrightarrow V_-'$



(2.3)



(2.4)

Proof:

- Obj: ✓

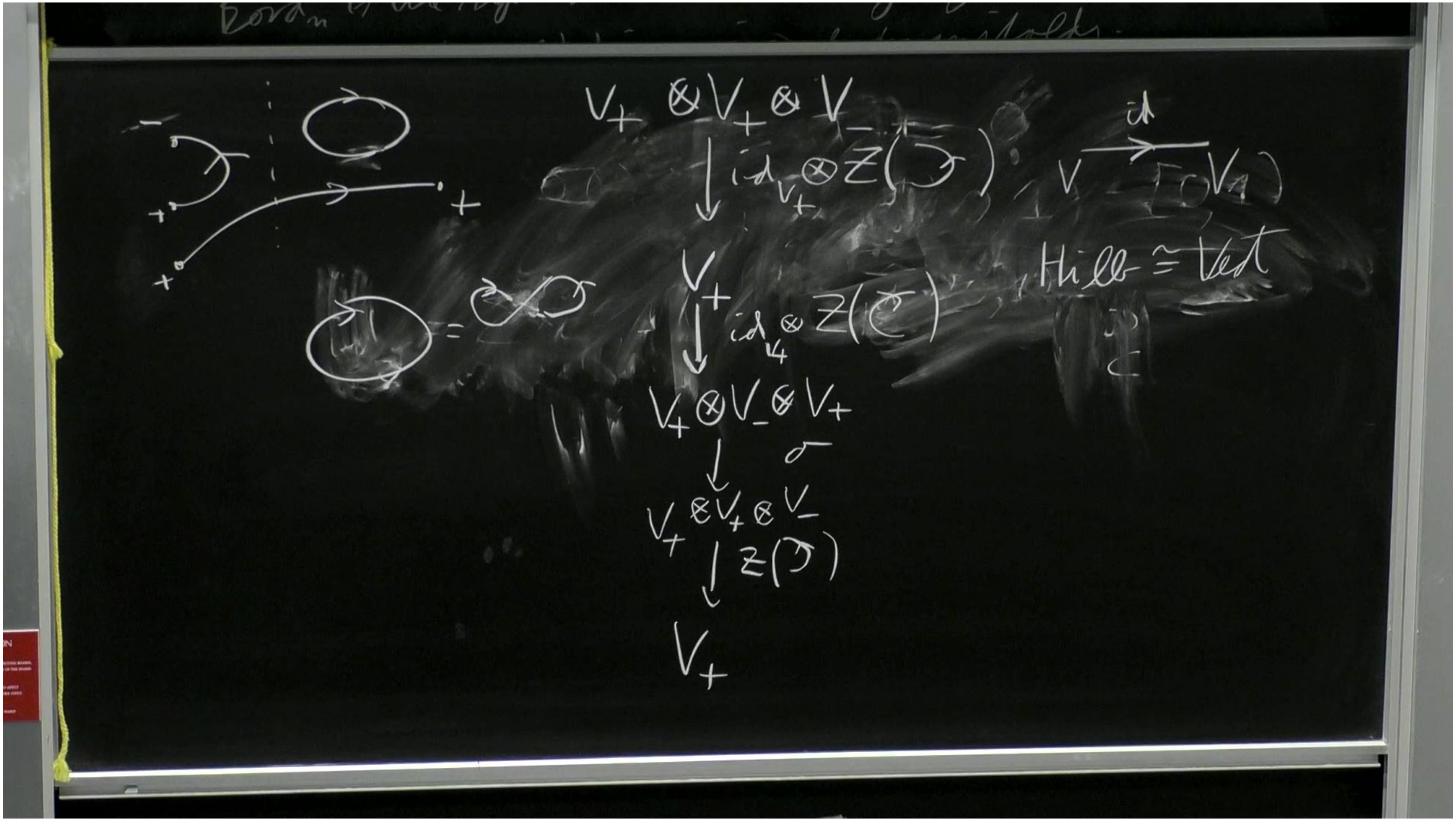
Mor: $\eta : Z \Longrightarrow Z'$

$$\begin{tikzcd} Z(\Sigma) \arrow[r, "\eta_\Sigma"] \arrow[d, "Z(M)"'] & Z'(\Sigma) \arrow[d, "Z'(M)"] \\ Z(\Sigma') \arrow[r, "\eta_{\Sigma'}"] & Z'(\Sigma') \end{tikzcd}$$

$\eta_+ : Z(pt^+) \longrightarrow Z'(pt^+)$

$\eta_- : Z(pt^-) \longrightarrow Z'(pt^-)$

$V \otimes V_+ \otimes V_-$


V+ ⊗ V+ ⊗ V−
id_V+ ⊗ Z( )
V+
id_V+ ⊗ Z( )
V+ ⊗ V− ⊗ V+
σ
V+ ⊗ V+ ⊗ V−
Z( )
V+
id
Hilb ≃ Vect

Lemma:

The category of 2DTQFS is equivalent to the category with

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- Linear map $ev: V_+ \otimes V_- \longrightarrow \mathbb{C}$, $coev: \mathbb{C} \longrightarrow V_- \otimes V_+$

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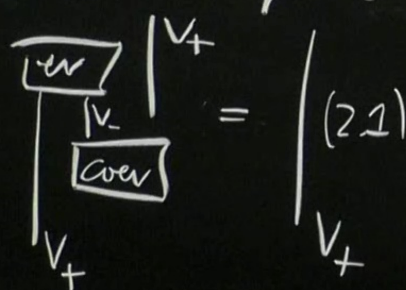
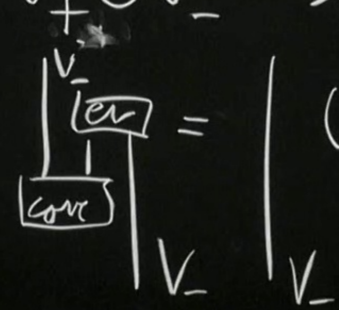


(2.1) &

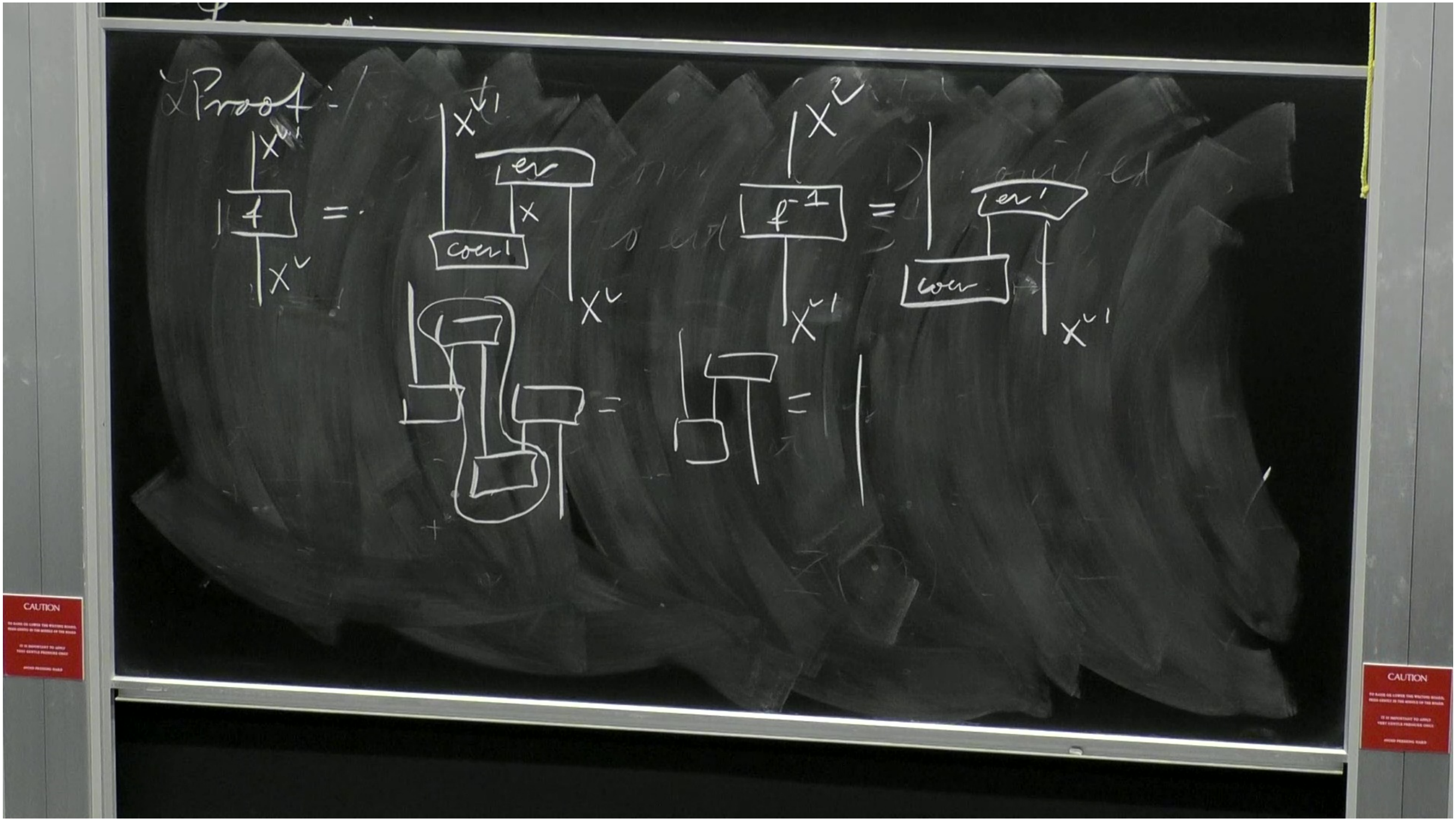


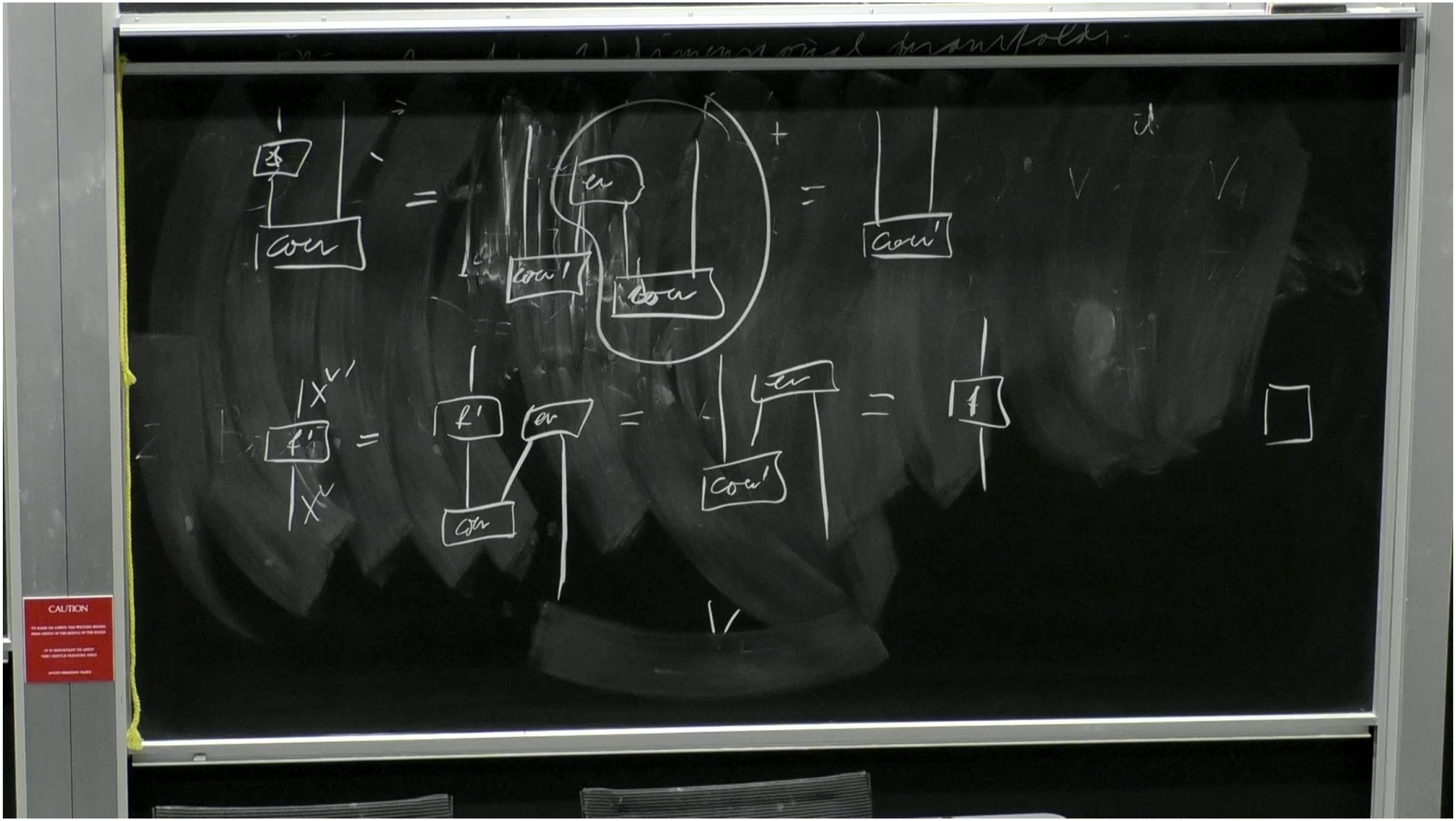
(2.2)

Def. Let $\mathcal{C}$ be a monoidal category.
A right dual to $X \in \mathcal{C}$ is an object $X^\vee \in \mathcal{C}$ together with morphisms $ev_X : X \otimes X^\vee \longrightarrow \mathbb{1}$, $coev_X : \mathbb{1} \longrightarrow X^\vee \otimes X$ satisfying (2.1) & (2.2).
Dual$(\mathcal{C})$ is the category with objects quadruples $(X, X^\vee, ev, coev)$ & morphisms are pairs $(f_+ : X \to X', f_- : X^\vee \to X'^\vee)$ satisfying (2.3) & (2.4)

Prop.

Let $X^\vee$ and $X'^\vee$ be right dual to $X \in \mathcal{C}$. Then $\exists!$ isomorphism $X^\vee \xrightarrow{f} X'^\vee$, s.t. $(id, f) : (X, X^\vee) \to (X, X'^\vee)$ is a morphism in $\mathrm{Dual}(\mathcal{C})$

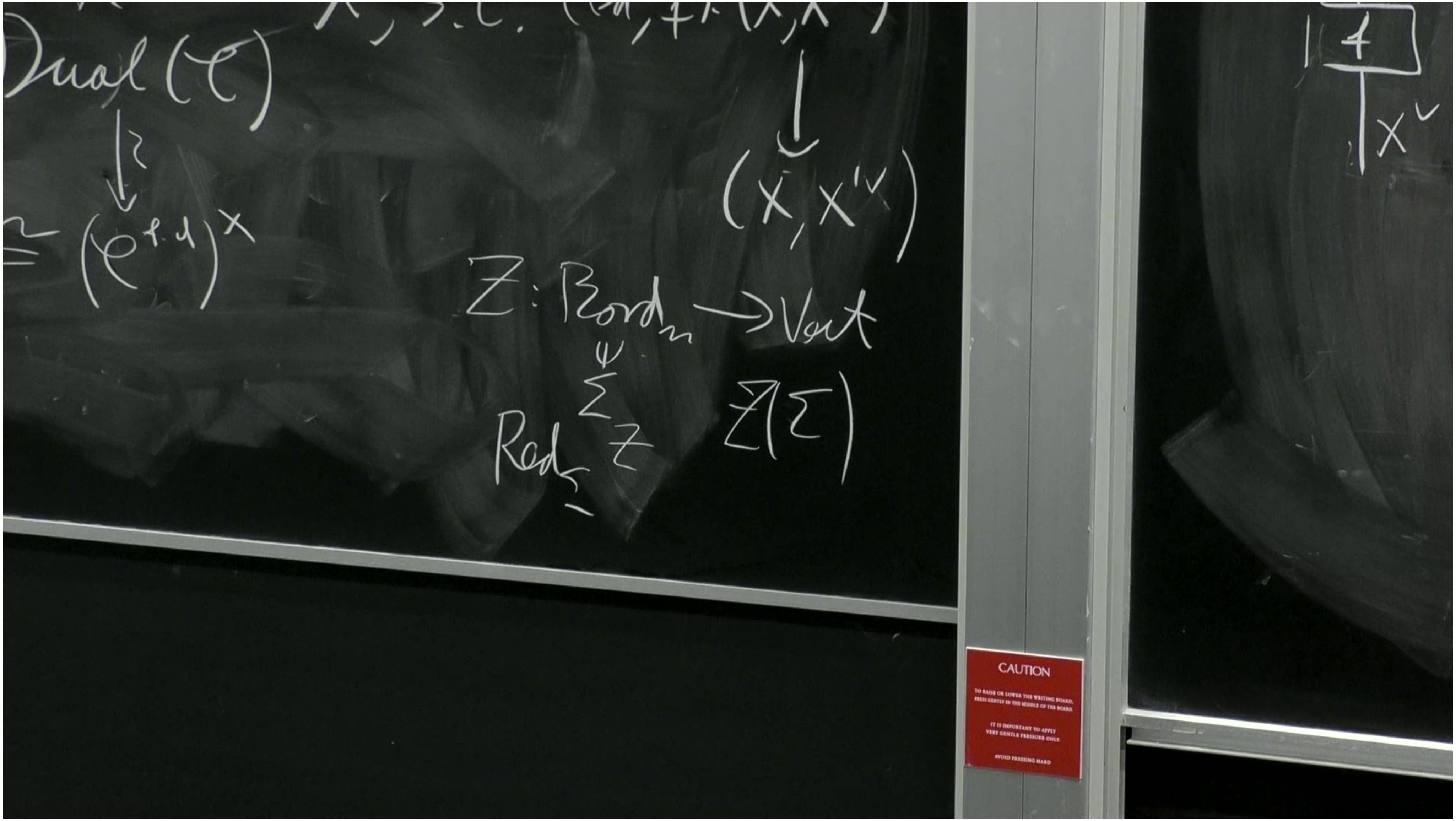

Proof:
f
ev
coev'
f^-1
ev'
coev


coev
ev
coev'
f'
f
CAUTION

Prop.

Let $X^\vee$ and $X'^\vee$ be right dual to $X \in \mathcal{C}$. Then $\exists!$ isomorphism $X^\vee \xrightarrow{f} X'^\vee$, s.t. $(\mathrm{id}, f) : (X, X^\vee) \to (X, X'^\vee)$ is a morphism in $\mathrm{Dual}(\mathcal{C})$

$$\mathrm{Dual}(\mathcal{C}) \xrightarrow{\ \wr\ } (\mathcal{C}^{r.d.})^X$$


Dual(C)
Z : Bord → Vect
Z(Σ)
CAUTION