Title: Topological Quantum Field Theories Lecture 20231013

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Collection: Topological Quantum Field Theories - mini-course

Date: October 13, 2023 - 2:00 PM

URL: https://pirsa.org/23100079

$X, Y, Z \qquad f : X \longrightarrow Y$

Rem 1.4

We will draw $X \xrightarrow{f} Y \xrightarrow{g} Z$ to denote the composition the composition $g \circ f$. We say the square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "h"'] & Y \arrow[d, "g"] \\ W \arrow[r, "k"'] & Z \end{tikzcd}$$

commutes if $g \circ f = k \circ h$

$X, Y, Z \qquad f : X \longrightarrow Y$

Rem 1.4

We will draw $X \xrightarrow{f} Y \xrightarrow{g} Z$ to denote the compositio

e.g. BG

$\mathrm{Obj}(BG) = *$

$\mathrm{Mor}(BG) = G$

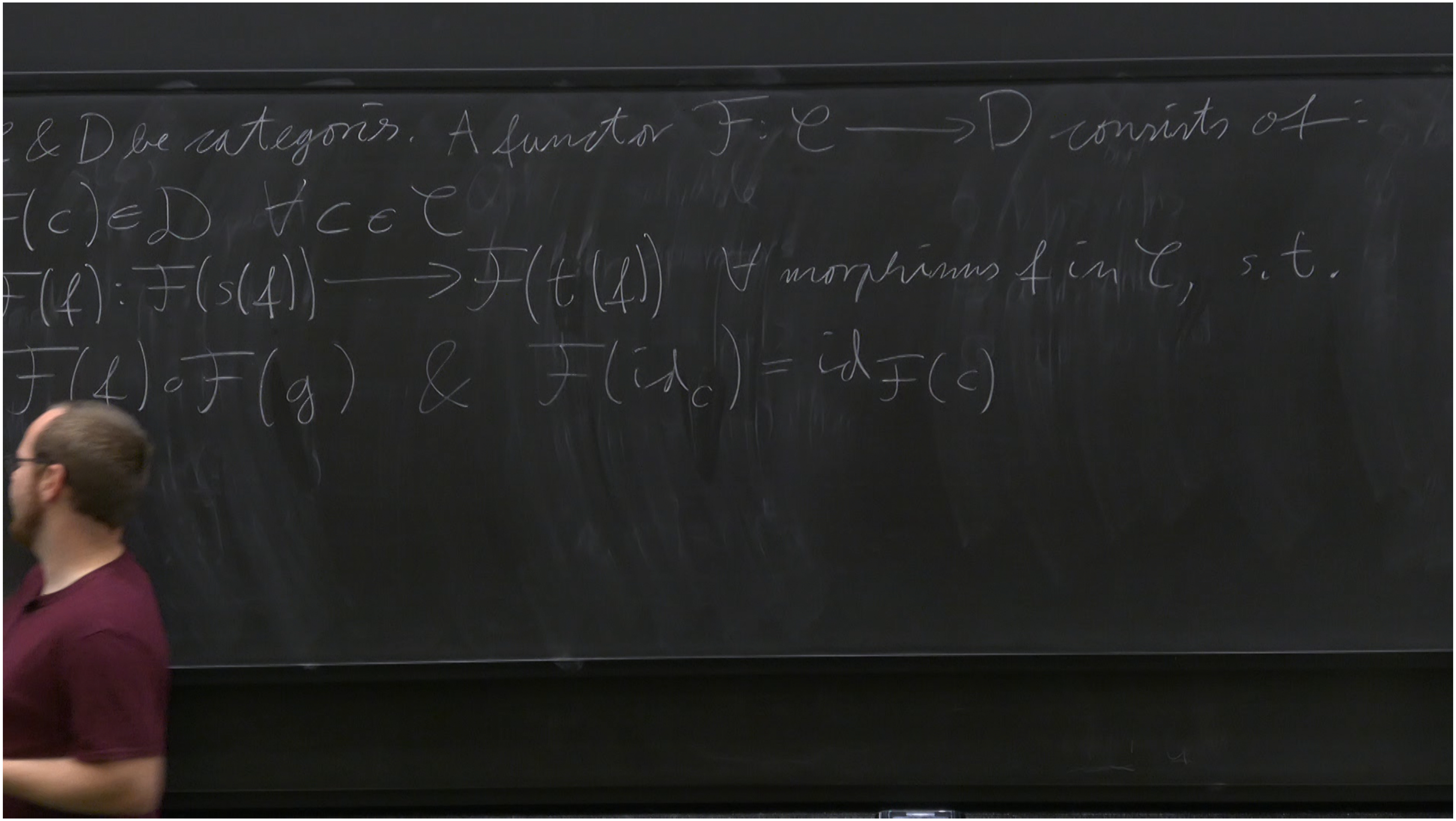
$$\begin{tikzcd} * \arrow[r, "g"] \arrow[d, "h"'] & * \arrow[d, "h"] \\ * \arrow[r, "g"'] & * \end{tikzcd}$$

commutes iff $g \cdot h = h \cdot g$

Def. 1.5 Let $\mathcal{C}$ & D be categories. A functor $\mathcal{F}: \mathcal{C} \longrightarrow D$ -

- An object $\mathcal{F}(c) \in \mathcal{D}$ $\forall c \in \mathcal{C}$
- A morphism $\mathcal{F}(f): \mathcal{F}(s(f)) \longrightarrow \mathcal{F}(t(f))$ $\forall$ morphisms f

$$\mathcal{F}(f \circ g) = \mathcal{F}(f) \circ \mathcal{F}(g) \quad \& \quad \mathcal{F}(id_c) = id_{\mathcal{F}(c)}$$


& D be categories. A functor F: C ⟶ D consists of:
F(c) ∈ D ∀ c ∈ C
F(f): F(s(f)) ⟶ F(t(f)) ∀ morphisms f in C, s.t.
F(f) ∘ F(g) & F(id_c) = id_F(c)

Def. 1.5 Let $\mathcal{C}$ & $\mathcal{D}$ be categories. A functor $\mathcal{F}: \mathcal{C} \longrightarrow \mathcal{D}$

- An object $\mathcal{F}(c) \in \mathcal{D}$ $\forall c \in \mathcal{C}$
- A morphism $\mathcal{F}(f) : \mathcal{F}(s(f)) \longrightarrow \mathcal{F}(t(f))$ $\forall$ morphisms f

$$\mathcal{F}(f \circ g) = \mathcal{F}(f) \circ \mathcal{F}(g) \quad \& \quad \mathcal{F}(id_c) = id_{\mathcal{F}(c)}$$

Example 1.6

$$U : \mathrm{Vect}_{\mathbb{K}} \longrightarrow \mathrm{set} \qquad \rho : BG \longrightarrow \mathrm{Vect}_{\mathbb{K}}$$

$$V \longmapsto V \qquad * \longmapsto V$$

$$f : V \to W \longmapsto f : V \to W \qquad * \xrightarrow{g} * \longmapsto \rho(g) : V \to V$$

1.1.5 Let $\mathcal{C}$ & $\mathcal{D}$ be categories. A functor $\mathcal{F}: \mathcal{C} \longrightarrow \mathcal{D}$ consists of:

n object $\mathcal{F}(c) \in \mathcal{D}$ $\forall c \in \mathcal{C}$

morphism $\mathcal{F}(f): \mathcal{F}(s(f)) \longrightarrow \mathcal{F}(t(f))$ $\forall$ morphisms f in $\mathcal{C}$, s.t.

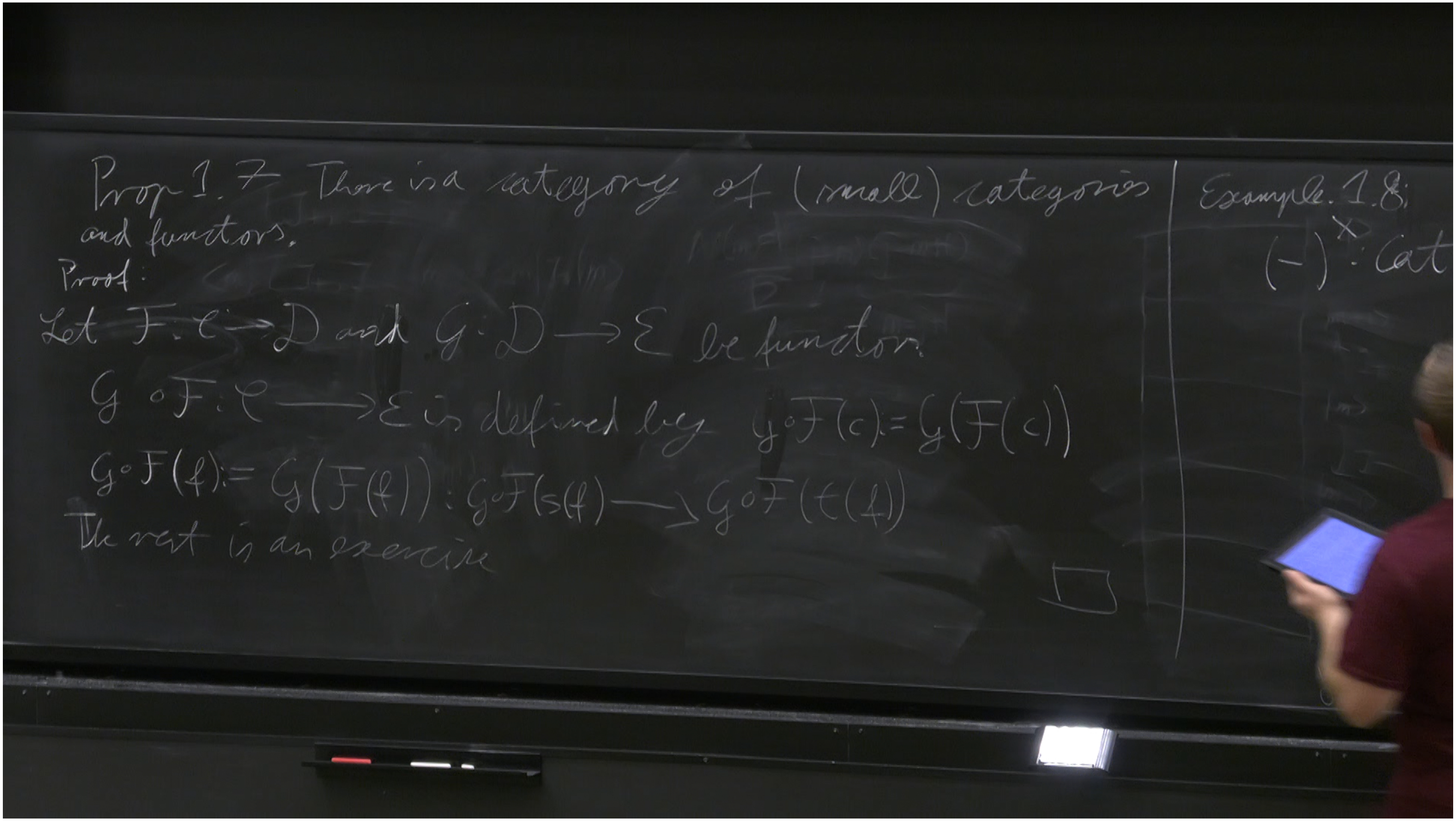
$\mathcal{F}(f \circ g) = \mathcal{F}(f) \circ \mathcal{F}(g)$ & $\mathcal{F}(id_c) = id_{\mathcal{F}(c)}$

ample 1. 6

$$\begin{aligned} \mathrm{Vect}_{\mathbb{K}} &\longrightarrow \mathrm{set} \\ V &\longmapsto V \\ f: V \to W &\longmapsto f: V \to W \end{aligned}$$

$$\begin{aligned} \rho: BG &\longrightarrow \mathrm{Vect}_{\mathbb{K}} \\ * &\longmapsto V \\ * \xrightarrow{g} * &\longmapsto \rho(g): V \to V \end{aligned}$$

$\rho(gh) = \rho(g)\rho(h)$ $\rho(e) = id_V$ ρ is a re


Prop 1.7 There is a category of (small) categories and functors.
Proof:
Let F: C → D and G: D → E be functors.
G∘F: C → E is defined by G∘F(c) := G(F(c))
G∘F(f) := G(F(f)) : G∘F(s(f)) → G∘F(t(f))
The rest is an exercise
Example 1.8:
(−)^× : Cat

ries

Example 1.8:

$$(-)^{\times} : \mathrm{Cat} \longrightarrow \mathrm{Grpd}$$

$$\mathcal{C} \longmapsto \mathcal{C}^{\times}$$

$\mathcal{F} : \mathcal{C} \to \mathcal{D}$

$f : x \to y \in \mathcal{C}$ is an isomorphism

$\Longrightarrow \mathcal{F}(f)$ is iso in $\mathcal{D}$

$$\mathcal{F}(f) \circ \mathcal{F}(f^{-1}) = \mathcal{F}(f \circ f^{-1})$$

$$= \mathcal{F}(\mathrm{id}_{t(f)}) = \mathrm{id}_{\mathcal{F}(t(f))}$$

$f: V \to W \longmapsto f: V \to W$

Example 1.9

Let $\mathcal{C}$ be a category. Then we define $\mathcal{C}^{op}$ to be the category with the same objects as $\mathcal{C}$ $X, Y, \ldots$ and morphisms

$f^{op}: Y \longrightarrow X$ for every morphism $f: X \longrightarrow Y$ in $\mathcal{C}$.

$Z \xrightarrow{g^{op}} Y \xrightarrow{f^{op}} X = Z \xrightarrow{(g \circ f)^{op}} X.$

$f: V \longrightarrow W$

$f^*: W^* \longrightarrow V^*$

$(-)^*: \mathrm{Vect} \longrightarrow \mathrm{Vect}^{op}$

$V \longmapsto V^* := \mathrm{Hom}(V, \mathbb{K})$

Ex. 1.10

$\mathrm{Hom}(c,-): \mathcal{C} \longrightarrow \mathrm{set}$

$d \longmapsto \mathrm{Hom}(c,d) = \{ f: c \to d \mid f \in \mathrm{Mor}\,\mathcal{C} \}$

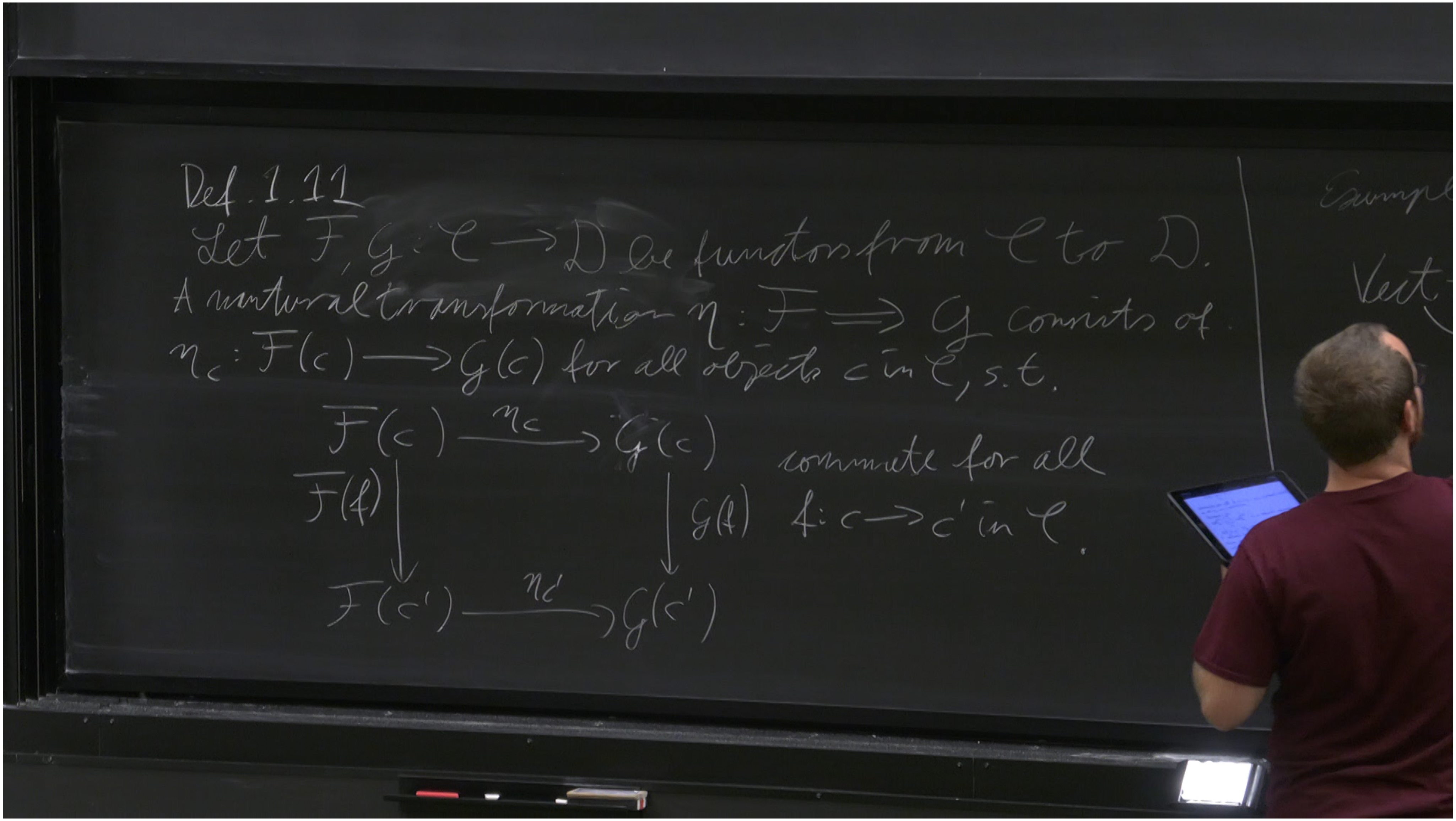
$f: d \to d' \longmapsto \mathrm{Hom}(c,d) \xrightarrow{f_*} \mathrm{Hom}(c,d')$

$h: c \to d \longmapsto c \xrightarrow{h} d \xrightarrow{f} d'$

Similarly

$\mathrm{Hom}(-,c): \mathcal{C}^{op} \longrightarrow \mathrm{set}$

$f \longmapsto f^*$


Def. 1.11
Let $\mathcal{F}, \mathcal{G} : \mathcal{C} \longrightarrow \mathcal{D}$ be functors from $\mathcal{C}$ to $\mathcal{D}$.
A natural transformation $\eta : \mathcal{F} \Longrightarrow \mathcal{G}$ consists of:
$\eta_c : \mathcal{F}(c) \longrightarrow \mathcal{G}(c)$ for all objects c in $\mathcal{C}$, s.t.
$\mathcal{F}(c) \xrightarrow{\eta_c} \mathcal{G}(c)$
$\mathcal{F}(f)$
$\mathcal{G}(f)$
$\mathcal{F}(c') \xrightarrow{\eta_{c'}} \mathcal{G}(c')$
commute for all
$f : c \longrightarrow c'$ in $\mathcal{C}$.
Example
Vect

Def. 1.11

Let $\mathcal{F}, \mathcal{G} : \mathcal{C} \longrightarrow \mathcal{D}$ be functors from $\mathcal{C}$ to $\mathcal{D}$.
A natural transformation $\eta : \mathcal{F} \Longrightarrow \mathcal{G}$ consists of
$\eta_c : \mathcal{F}(c) \longrightarrow \mathcal{G}(c)$ for all objects c in $\mathcal{C}$, s.t.

$$\begin{tikzcd} \mathcal{F}(c) \arrow[r, "\eta_c"] \arrow[d, "\mathcal{F}(f)"'] & \mathcal{G}(c) \arrow[d, "\mathcal{G}(f)"] \\ \mathcal{F}(c') \arrow[r, "\eta_{c'}"] & \mathcal{G}(c') \end{tikzcd}$$

commute for all $f : c \longrightarrow c'$ in $\mathcal{C}$.

$\boxed{\text{$\eta$ is a natural isomorphism if η_c is an isomorphism for all $c \in \mathcal{C}$.}}$

Examp

Vect

Example:

$$\begin{tikzcd} \mathrm{Vect} \arrow[r, "(-)^{**}"] \arrow[r, bend right=50, "\mathrm{id}"', ""{name=I}] & \mathrm{Vect} \arrow[from=I, phantom, "\Uparrow \eta"] \end{tikzcd}$$

$$V \longrightarrow V^{**}$$

$$v \longmapsto (f : V \to \mathbb{k}) \longmapsto f(v)$$

$$\rho, \rho' : BG \longrightarrow \mathrm{Vect}$$

$$\eta : \rho \Rightarrow \rho' \qquad f : V \longrightarrow V'$$

$$f \circ \rho(g) = \rho'(g) \circ f$$

Let $\mathcal{C}$ & $\mathcal{D}$ be categories. The- is a category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ with objects functors $F: \mathcal{C} \to \mathcal{D}$ & morphisms natural transformations

$$F \overset{\eta}{\Longrightarrow} G \overset{\eta'}{\Longrightarrow} \mathcal{H} \qquad (\eta \circ \eta')_c = F(c) \xrightarrow{\eta_c} G(c) \xrightarrow{\eta'_c} \mathcal{H}(c)$$

As a picture

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left=60, "F"] \arrow[r, "G" description] \arrow[r, bend right=60, "\mathcal{H}"'] & \mathcal{D} \end{tikzcd}$$

$\Downarrow \eta$ $\Downarrow \eta'$

Exercise:

Make sense of:

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\eta"] & \mathcal{D} \arrow[r, bend left, "F'"] \arrow[r, bend right, "G'"'] \arrow[r, phantom, "\Downarrow\varepsilon"] & \mathcal{E} \end{tikzcd}$$

Exercise:
Make sense of:

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left=50, "\mathcal{F}"{name=F}] \arrow[r, bend right=50, "\mathcal{G}"'{name=G}] \arrow[r, phantom, "\Downarrow\eta"] & \mathcal{D} \arrow[r, bend left=50, "\mathcal{F}'"] \arrow[r, bend right=50, "\mathcal{G}'"'] \arrow[r, phantom, "\Downarrow\varepsilon"] & \mathcal{E} \end{tikzcd}$$

Def 1.14 A functor $\mathcal{F}: \mathcal{C} \xrightarrow{} \mathcal{D}$ is an equivalence if there exists a functor $\mathcal{F}^{-1}: \mathcal{D} \longrightarrow \mathcal{C}$ and natural isomorphisms $\mathcal{F}^{-1} \circ \mathcal{F} \overset{\sim}{\Longrightarrow} id_{\mathcal{C}}$ & $\mathcal{F} \circ \mathcal{F}^{-1} \overset{\sim}{\Longrightarrow} id_{\mathcal{D}}$.

Monoidal category:

$$V, W \rightsquigarrow V \otimes W$$

$$\Sigma, \Sigma' \rightsquigarrow \Sigma \amalg \Sigma'$$

Def 1.15 Let $\mathcal{C}, \mathcal{D}$ be categories. We define $\mathcal{C} \times \mathcal{D}$ with objects and morphisms given by $(c,d) \in \mathrm{Obj}(\mathcal{C}) \times \mathrm{Obj}(\mathcal{D})$ and $(f,g) \in \mathrm{Mor}(\mathcal{C}) \times \mathrm{Mor}(\mathcal{D})$ respectively.

$(f,g): (c,d) \to$

$(f,g):(c,d)\longrightarrow(c',d')$ $\quad f:c\to c'$ & $g:d\to d'$

Def. 1.16 A monoidal category consists of:

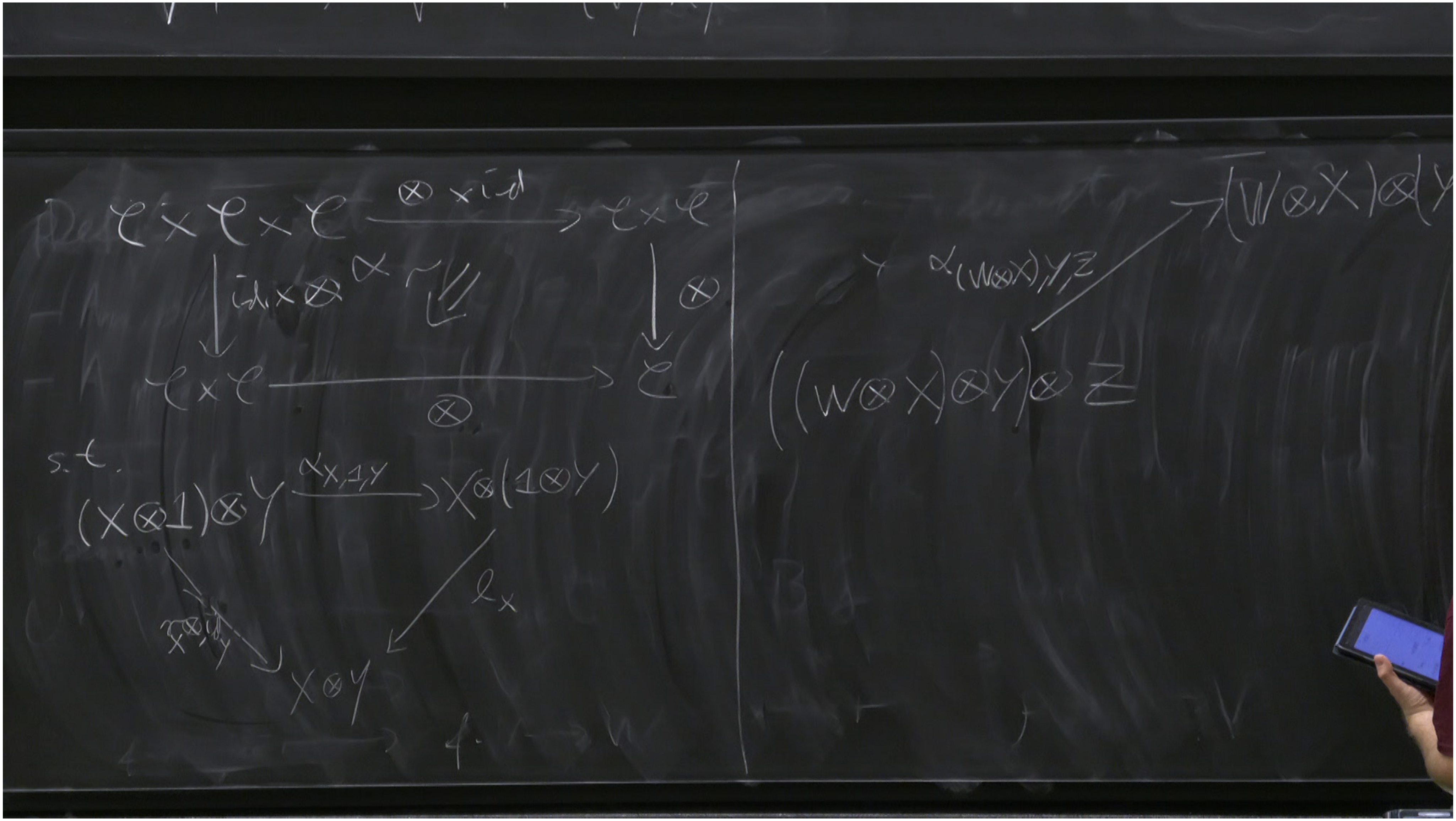
- a category $\mathcal{C}$, an object $1\in\mathcal{C}$
- a functor $\otimes:\mathcal{C}\times\mathcal{C}\longrightarrow\mathcal{C}$ and natural isomorphisms.

(l)

$$\begin{tikzcd} \mathcal{C} \arrow[r, "1\times\_"] \arrow[rr, bend right, "id"'] & \mathcal{C}\times\mathcal{C} \arrow[r, "\otimes"] \arrow[d, phantom, "\Downarrow\cong r"] & \mathcal{C} \\ & {} & \end{tikzcd}$$

$r_c: 1\otimes c \xrightarrow{\sim} c$

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\_\times 1"] \arrow[rr, bend right, "id"'] & \mathcal{C}\times\mathcal{C} \arrow[r, "\otimes"] \arrow[d, phantom, "\Downarrow\cong l"] & \mathcal{C} \\ & {} & \end{tikzcd}$$


C × C × C ⊗ × id → C × C
id × ⊗
α
⊗
C × C → C
⊗
s.t.
(X ⊗ 1) ⊗ Y α_{X,1,Y} → X ⊗ (1 ⊗ Y)
l_X
X ⊗ Y
α_{(W⊗X),Y,Z}
((W ⊗ X) ⊗ Y) ⊗ Z
(W ⊗ X) ⊗ (Y

$$
\begin{tikzcd}
& (W\otimes X)\otimes(Y\otimes Z) \arrow[dr, "\alpha_{W,X,Y\otimes Z}"] & \\
((W\otimes X)\otimes Y)\otimes Z \arrow[ur, "\alpha_{(W\otimes X),Y,Z}"] \arrow[d, "\alpha_{W,X,Y}\otimes id_Z"'] & & W\otimes(X\otimes(Y\otimes Z)) \\
(W\otimes(X\otimes Y))\otimes Z \arrow[rr, "\alpha_{W,X\otimes Y,Z}"] & & W\otimes((X\otimes Y)\otimes Z) \arrow[u, "id_W\otimes\alpha_{Y,Y,Z}"']
\end{tikzcd}
$$

- Pset with $\otimes := \times$
- Cat with $\otimes := \times$
- Vect with $\otimes := \otimes_K$

$\forall$ V, ... (small) categories

$\xrightarrow[\sim]{\alpha} V \otimes (W \otimes X)$

Exam