Title: Topological Quantum Field Theories Lecture 20231006

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Collection: Topological Quantum Field Theories - mini-course

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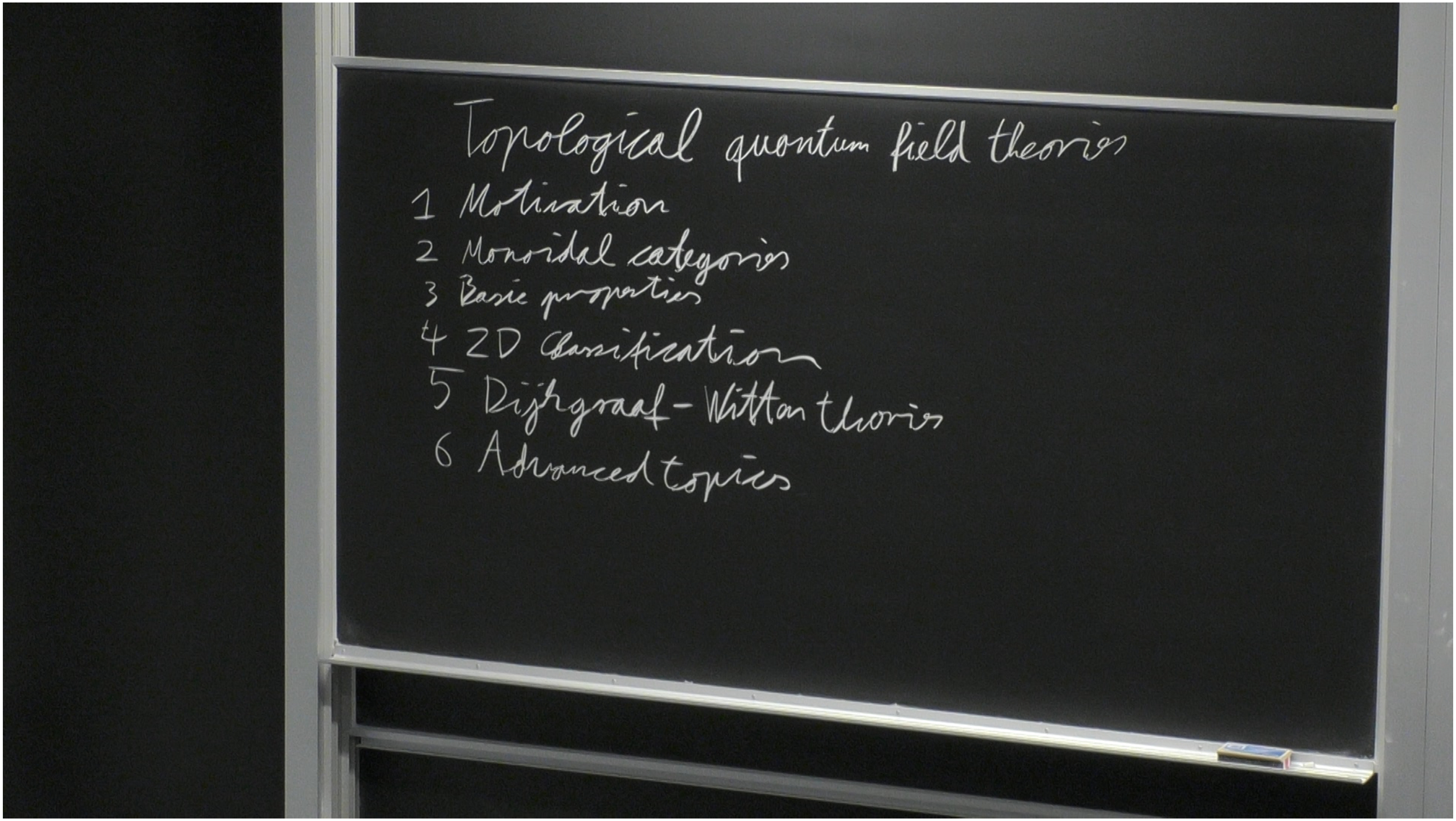

Topological quantum field theories
1 Motivation
2 Monoidal categories
3 Basic properties
4 2D Classification
5 Dijkgraaf-Witten theories
6 Advanced topics

$$- \langle \mathcal{O}_1 \cdots \mathcal{O}_r \rangle_g = \frac{1}{Z_{M,g}} \int_{\phi \in \mathcal{F}_M} D\phi \, \mathcal{O}_1(\phi) \cdots \mathcal{O}_r(\phi) \, e^{i S_g[\phi]}$$

partition function: $Z_{M,g} = \int D\phi \, e^{i S_g[\phi]}$

A theory is topological if:

$$\frac{\delta \langle \mathcal{O}_1 \cdots \mathcal{O}_r \rangle_g}{\delta g} = 0$$

e.g. $S[A] = \int \mathrm{Tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right)$

- 3D Quantum gravity

$\mathcal{H}$ " $= \mathcal{L}^2(\mathcal{F}_\Sigma)$ "

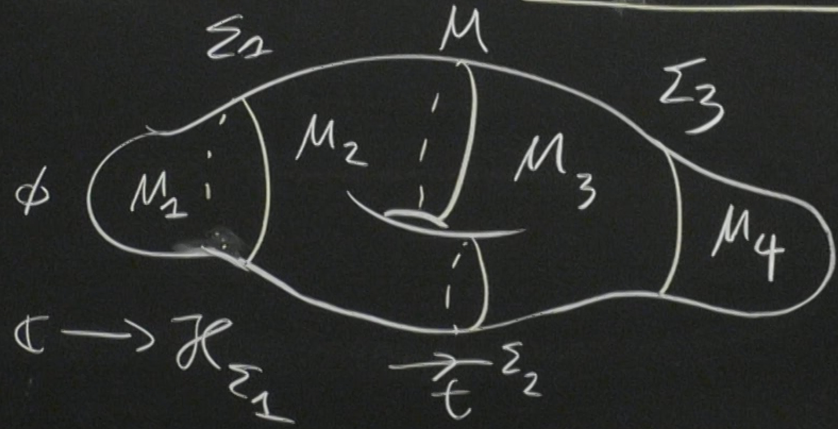
$$\text{"}\mathcal{H}_\Sigma\text{"} = \mathcal{L}^2(\mathcal{F}_\Sigma)\text{"}$$

$\Sigma \leftarrow$ (n-1)-time slice

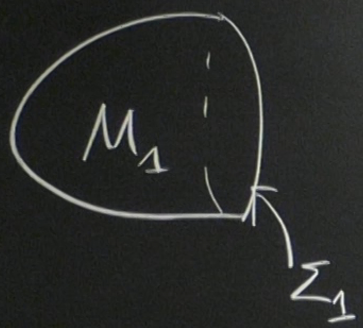
$$\mathcal{H}_{\Sigma \amalg \Sigma'} = \mathcal{L}^2(\mathcal{F}_{\Sigma \amalg \Sigma'}) = \mathcal{L}^2(\mathcal{F}_\Sigma \times \mathcal{F}_{\Sigma'})$$

$$= \mathcal{L}^2(\Sigma) \otimes \mathcal{L}^2(\Sigma')$$

$$\boxed{\mathcal{H}_{\Sigma \amalg \Sigma'} \cong \mathcal{H}_\Sigma \otimes \mathcal{H}_{\Sigma'}}$$

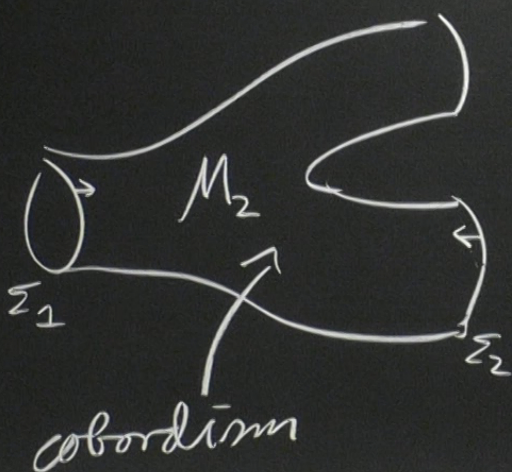


$\mathbb{C} \longrightarrow \mathcal{H}_{\Sigma_1}$ $\quad \to \mathbb{C}^{\Sigma_2}$



$$Z_{M_1} \in \mathcal{H}_{\Sigma_1} = \mathcal{L}^2(\mathcal{F}_{\Sigma_1})$$

$$\left(\phi_{\Sigma_1} \longmapsto \int_{\substack{\phi \in \mathcal{F}_{M_1} \\ \phi|_{\partial\Sigma_1} = \phi_{\Sigma_1}}} e^{iS[\phi]} \right) \in \mathcal{L}^2(\mathcal{F}_{\Sigma_1})$$



$$Z(M_2) : \mathcal{H}_{\Sigma_1} \longrightarrow \mathcal{H}_{\Sigma_2}$$

$$f(-) \longmapsto \phi_{\Sigma_2} \mapsto \int_{\substack{\phi \in \mathcal{F}(M_2) \\ \phi|_{\Sigma_2} = \phi_{\Sigma_2}}} e^{-iS[\phi]} f(\phi|_{\Sigma_1})$$

$$Z(M_3)\circ \boxed{Z(M_2)(f(-))}(\phi_{\Sigma_3}) = \int e^{iS[\phi]} Z(M_2)(f)[\phi_{M_3}|_{\Sigma_2}]$$

$$\phi_{M_3} \in F_{M_3}$$

$$\phi_{M_3}|_{\Sigma_3} = \phi_{\Sigma_3}$$

$$\underbrace{\int_{\phi_{M_3}|_{\Sigma_3}=\phi_{\Sigma_3}} \int_{\phi_{M_2}|_{\Sigma_2}=\phi_{M_3}|_{\Sigma_3}}} f(\phi_{M_2}|_{\Sigma_1}) e^{i\underbrace{(S_{M_3}[\phi_{M_3}]+S_{M_2}[\phi_{M_2}])}_{S_{M_3\circ M_2}[\phi_{M_3}\circ\phi_{M_2}]}}$$

$$\int_{\phi_{M_3\circ M_2}|_{\Sigma_3}=\phi_{\Sigma_3}} f(\phi|_{\Sigma_1}) e^{iS[\phi]} = Z(M_3\circ M_2)[f][\phi_{\Sigma_3}]$$

$$Z(M_3)\circ \boxed{Z(M_2)(f(-))}(\phi_{\Sigma_3}) = \int_{\substack{\phi_{M_3}\in \mathcal{F}_{M_3}\\ \phi_{M_3}|_{\Sigma_3}=\phi_{\Sigma_3}}} e^{iS[\phi]}\, Z(M_2)(f)[\phi_{M_3}|_{\Sigma_2}]$$

$$\underbrace{\int_{\phi_{M_3}|_{\Sigma_3}=\phi_{\Sigma_3}} \int_{\phi_{M_2}|_{\Sigma_2}=\phi_{M_3}|_{\Sigma_2}}}_{\int_{\phi_{M_3\circ M_2}|_{\Sigma_3}=\phi_{\Sigma_3}}} f(\phi_{M_2}|_{\Sigma_1})\, e^{i\underbrace{\left(S_{M_3}[\phi_{M_3}]+S_{M_2}[\phi_{M_2}]\right)}_{S_{M_3\circ M_2}[\phi_{M_3}\circ\phi_{M_2}]}}$$

$$\int f(\phi|_{\Sigma_1})\, e^{iS[\phi]} = Z(M_3\circ M_2)[f][\phi_{\Sigma_3}]$$

$$\boxed{Z(M_3\circ M_2) = Z(M_3)\circ Z(M_2)}$$

Guess for a definition:

An n-dimensional TQFT is the following data:

- A Hilbert space $\mathcal{H}_\Sigma$ for all compact oriented $(n-1)$ manifolds
- A linear map $Z(M): \mathcal{H}_\Sigma \longrightarrow \mathcal{H}_{\Sigma'}$ for all cobordisms $M: \Sigma \rightsquigarrow \Sigma'$

$\mathcal{H}_{\Sigma_1 \sqcup \Sigma_2} \cong \mathcal{H}_{\Sigma_1} \otimes \mathcal{H}_{\Sigma_2}$ $\quad \mathcal{H}_{\emptyset} \cong \mathbb{C}$

$Z(M_2) \quad \Sigma_2 \quad \}$

$$Z(M_2)\circ Z(M_1) = Z(M_2\circ M_1)$$

$\ulcorner S$ n-2-dim $\rightsquigarrow$ $\mathcal{H}_S$ - $\mathbb{C}$-linear category

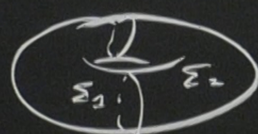
$*$ $\longmapsto$ $\mathbb{C}$-linear n-category $\lrcorner$

$Z(M_2) \quad \Sigma_2$

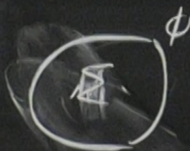
$$Z(M_2)\circ Z(M_1) = Z(M_2 \circ M_1)$$

[S n-2-dim $\rightsquigarrow$ $\mathcal{H}_S$ - $\mathbb{C}$-linear category

$*$ $\longmapsto$ $\mathbb{C}$-linear n-category]



$$\emptyset \xrightarrow{Z(\Sigma_1)} S \xrightarrow{Z(\Sigma_2)} \emptyset$$



$\mathcal{H}_\Sigma : \overline{\mathcal{F}_S} \longrightarrow \mathrm{Hilb}$ $\qquad$ $\mathcal{F}_S \longrightarrow \mathrm{Hilb}$

1 Monoidal categories

E. Riehl: Categories in context

Def. 1.1 A category $\mathcal{C}$ consists of

- a collection of objects $X, Y, \ldots$ $x, y, \ldots, c, c'$
- a collection of morphisms $f, g, \ldots$

$\mathrm{Obj}(\mathcal{C})$ for class of objects

$\mathrm{Mor}(\mathcal{C})$ for class of morphisms

s.t.

- Each f has specified domain and codomain

$f: X \longrightarrow Y$ We write $s(f) = X$ $t(f) = Y$

- Each object X has an identity morphism $id_X : X \longrightarrow X$
- Composition for all $f : X \longrightarrow Y \quad g : Y \longrightarrow Z$

$$g \circ f : X \longrightarrow Z$$

s.t $\quad g \circ id_{s(g)} = id_{t(g)} \circ g = g$

$$(h \circ g) \circ f = h \circ (g \circ f) \quad \forall f : X \to Y,\ g : Y \to Z \quad h : Z \longrightarrow U$$

Example 1.2

- Set: Objects: set & Mor: functions
- $Vect_{\mathbb{K}}$: Obj: Vector spaces over a field $\mathbb{K}$ Mor: linear maps

- Let G be a group. BG has one object $*$

$Mor(BG) = G$ $\quad * \xrightarrow{g} * \xrightarrow{h} * = h \circ g := h \cdot g$

$id_* = e$

Def 1.3 Let $\mathcal{C}$ be a category. A morphism $f: X \to Y$ is called an isomorphism if $\exists f^{-1}: Y \to X$, s.t.

$$f \circ f^{-1} = id_Y \quad f^{-1} \circ f = id_X.$$

$\mathcal{C}$ is called a groupoid if every morphism is an isomorphism.