Title: A Nonsmooth Approach to Einstein's Theory of Gravity

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Abstract: While Einstein's theory of gravity is formulated in a smooth setting, the singularity theorems of Hawking and Penrose describe many physical situations in which this smoothness must eventually breakdown. In positive-definite signature, there is a highly successful theory of metric and metric-measure geometry which includes Riemannian manifolds as a special case, but permits the extraction of nonsmooth limits under curvature and dimension bounds analogous to the energy conditions in relativity: here sectional curvature is reformulated through triangle comparison, while Ricci curvature is reformulated using entropic convexity along geodesics of probability measures.This lecture explores recent progress in the development of an analogous theory in Lorentzian signature, whose ultimate goal is to provide a nonsmooth theory of gravity. We highlight how the null energy condition of Penrose admits a nonsmooth formulation as a variable lower bound on timelike Ricci curvature. We discuss an example showing the null energy condition does not survive the same sort of limits that uniform bounds on the timelike Ricci curvature respect.

Zoom link: https://pitp.zoom.us/j/92777399316?pwd=QmNaM1J0elgyRGdmaGxCTTQxT2Y4QT09

A nonsmooth approach to Einstein's theory of gravity

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• Einstein's gravity is formulated on smooth Lorentzian manifolds, but often predicts such manifolds are geodesically incomplete.

• Due to e.g. black hole (Penrose) or big bang (Hawking) type singularities a nonmsooth theory is highly desirable

Example (Inspiring positive signature developments)

In metric(-measure) geometry with positive signature, there are theories of

• sectional curvature bounds based on triangle comparison (Aleksandrov. . .)

• pointed Gromov-Hausdorff limits of manifolds under lower Ricci and upper dimensional bounds (Fukaya, Gromov, Cheeger-Colding, . . .)

• Ricci lower bounds via displacement convexity of entropy (Bakry-Emery, Lott-Sturm-Villani, Ambrosio-Gigli-Savare, . . .)

Can something similar be done in Lorentzian geometry?

• tidal forces (Kunzinger-Sämann '18)

• convergence of spaces (Müeller 22+, Minguzzi-Suhr 22+)

• Einstein eq (M 20, Mondino-Suhr 23, Cavalletti-Mondino 20+, Braun

Elliptic v hyperbolic geometry (c.f. BBCGMORS octet)

ELLIPTIC: $\mathbf{R}^n$ equipped with Euclidean norm $\|v\|_E := (\sum v_i^2)^{1/2}$

- $\|v + w\|_E \le \|v\|_E + \|w\|_E$

HYPERBOLIC: Minkowski space $\mathbf{R}^n$ equipped with the *hyperbolic 'norm'*

$$\|v\|_F := \begin{cases} (v_1^2 - \sum_{i\ge 2} v_i^2)^{1/2} & v \in F := \left\{ v \in \mathbf{R}^n \mid v_1 \ge (\sum_{i \ge 2} v_i^2)^{1/2} \right\} \\ -\infty & \textit{else} \end{cases}$$

- $\|v + w\|_F \ge \|v\|_F + \|w\|_F$

the future $F \subset \mathbf{R}^n$ is a convex cone; $v \in F$ called *causal* or *future-directed*

- v is *timelike* if $v \in F \setminus \partial F$
- v is *lightlike (or null)* if $v \in \partial F \setminus \{0\}$

(• v is *spacelike* iff $\pm v \notin F$ and *past-directed* if $-v \in F$)

- smooth curves are called *timelike (etc.)* if all tangents are timelike (etc.)

A crash course in differential geometry: action principles

Manifold M^n with symmetric nondegenerate smooth tensor field $g_{ij} = g_{ji}$

RIEMANNIAN: $(g_{ij}) > 0$ defines Euclidean norm on each tangent space

- its geometry is also encoded in the (symmetric) distance function

$$d(x,y)^q := \inf_{\sigma(0)=x,\sigma(1)=y} \int \|\dot\sigma_t\|^q_{E_g} dt \qquad q > 1$$

LORENTZIAN: $g \sim (+1, -1, \ldots, -1)$ defines hyperbolic norm on $T_x M$

- its asymmetric geometry is also encoded in the time-separation function

$$\ell(x,y)^q := \sup_{\sigma(0)=x,\sigma(1)=y} \int \|\dot\sigma_t\|^q_{F_g} dt \qquad 0 < q < 1$$

- $(-\infty)^q := -\infty$ so $\ell(x,y) = -\infty$ unless a causal curve links x to y
- extremizers are independent of q; they are called *geodesics*
- $\ell(x,z) \geq \ell(x,y) + \ell(y,z)$ (analog of the triangle inequality d satisfies)

Concave p-energy: trading linearity for ellipticity

Additional conditions are imposed to ensure $\ell \neq +\infty$ and extremizers exist
• complete or *proper* (boundedly compact) in the Riemannian case
• *globally hyperbolic* in the Lorentzian case (i.e. compact diamonds, future F varies continuously over M, no closed future-directed curves)

Concave Hamiltonian $H(w) = \frac{1}{p}\|w\|^p_{F^*}$ and Lagrangian $L(v) = \frac{1}{q}\|v\|^q_F$ satisfy $DH = (DL)^{-1}$ if $p^{-1} + q^{-1} = 1$ (here $p < 0$ since $0 < q < 1$)
• note $-L = (-H)^*$ jumps to $+\infty$ across future cone boundary ∂F
  (but $-H$ diverges continuously at the boundary of the dual cone F^*)
• nonsmooth already on smooth Lorentzian manifolds

Beran Braun Calisti Gigli M. Ohanyan Rott Sämann (octet):
extremizers of p-Dirichlet energy $u \mapsto \int_M H(du) d\mathrm{vol}_g$ rel. to compactly supported perturbations satisfy a new degenerate elliptic nonlinear PDE
• trade linearity of d'Alembertian for ellipticity of p-d'Alembertian!
• gives new (elliptic) approach to Eschenburg-Galloway splitting theorem

The Riemann curvature tensor

Given (timelike) geodesics $(\sigma_s)_{s\in[0,1]}$ and $(\tau_t)_{t\in[0,1]}$ with $\sigma_0 = \tau_0$ and $\dot\sigma_0 - \dot\tau_0 \in F \setminus \partial F$,

$$\ell(\sigma_s, \tau_t)^2 = \|s\dot\sigma_0 - t\dot\tau_0\|^2_{F_g} - \frac{Sec}{6} s^2 t^2 + O((|s| + |t|)^5)$$

where sectional curvature $Sec = R(\dot\sigma_0, \dot\tau_0, \dot\sigma_0, \dot\tau_0)$ is quadratic in $\dot\sigma_0 \wedge \dot\tau_0$ and measures the leading order correction to Pythagoras

- polarization of this quadratic form gives the *Riemann* tensor $R(\cdot,\cdot,\cdot,\cdot)$
- its trace $\mathrm{Ric}_{ik} = g^{jl} R_{ijkl}$ yields the *Ricci* tensor; $\mathrm{Ric}(v,v)$ measures the correction to Pythagoras averaged over all triangles including side v
- second trace $R = g^{ik}\mathrm{Ric}_{ik}$ yields the *scalar curvature*; in the elliptic case it gives leading order correction to the area of a sphere of radius r (and the volume of a ball of radius r)
- $d\mathrm{vol}_g(x) = \sqrt{|\det(g)|}\, d^n x$ in coordinates; (in the Riemannian case it coincides with the n-dimensional Hausdorff measure associated to d)

General relativity: Einstein's gravity and field equation

Gravity not a force, but rather a manifestation of curvature of spacetime
"Spacetime tells matter how to move" (along timelike/null geodesics...)

Field equation "Matter tells spacetime how to bend"

$$\begin{aligned} \textit{geometry} &= \textit{physics} \\ \text{curvature} &= \text{flux of energy and momentum} \\ \mathrm{Ric}_{ij} - \frac{1}{2} R g_{ij} &= 8\pi T_{ij} \qquad (\text{replaces } \Delta\phi = \rho \text{ and } F = -\nabla\phi) \end{aligned}$$

- just integrate this local conservation law for $T_{ij}(x)$ to find $g_{ij}\ldots$

What if matter distribution is unknown?

Energy conditions and singularity theorems

WEC (weak energy condition): $T(v,v) \geq 0$ for all future $v \in F$ (physical)
SEC (strong energy condition): $\mathrm{Ric}(v,v) \geq 0$ for all future $v \in F$ (less ")
NEC (null energy condition): " ≥ 0 for all lightlike $v \in \partial F$

[Cosmological constant (dark matter): $\geq (n-1)Kg(v,v)$]

Hawking '66 (big bang type) singularity theorem:
SEC + mean curvature bound $H_\Sigma \geq h > 0$ on a suitable hypersurface Σ implies finite-time singularities along all timelike geodesics through Σ
Cavalletti-Mondino '20+: genuinely nonsmooth version

Kasue '83: $\Omega \subset \mathbf{R}^n$ with $H_{\partial\Omega} > h > 0$ bounds radius of largest ball in Ω
Burtscher-Ketterer-M.-Woolgar '20: extend to $CD(K,N)$ setting; in $RCD(K,N)$ setting equality only if $\Omega =$ ball or cone

Penrose '65 (stellar collapse type) singularity theorem
NEC + trapped codimension-2 compact surface S + suitable noncompact hypersurface Σ imply finite-time singularity along some null geodesic
Open: genuinely nonsmooth version?

A nonsmooth null energy condition

(Idea: reformulate the null energy condition in a timelike way

Lemma

Any smooth Riemannian manifold admits $k \in C(M)$ such that $\mathrm{Ric}(v, v) \geq k(x)g(v, v)$ for all $v \in T_x M$.

Proof.

$k(x) = \inf_{v \in T_x M} \frac{\mathrm{Ric}(v,v)}{g(v,v)}$. □

RMK: Can't hold in Lorentzian setting, even for $v \in F$, unless NEC holds.

Theorem ((M. 23+) Not only sufficient, but necessary)

NEC $\Leftrightarrow \exists k \in C(M)$ such that every timelike vector $v \in T_x M$ satisfies

$$\mathrm{Ric}(v, v) \geq k(x)g(v, v).$$

i.e. NEC holds iff manifold admits a variable lower bound on timelike Ricci

What about the nonsmooth setting?

(please forget the foregoing)

Definition (Time-separation function)

On a set M of events, a *time-separation function* refers to $\ell : M \times M \longrightarrow \{-\infty\} \cup [0, \infty)$ satisfying the reverse triangle inequality and antisymmetry: $\forall x, y, z \in M$

$$\ell(x, y) \geq \ell(x, z) + \ell(z, y) \quad (1)$$

$$\min\{\ell(x, y), \ell(y, x)\} > -\infty \Leftrightarrow x = y. \quad (2)$$

Remark: $(1) + (2) \Rightarrow \ell(x, x) = 0$; (2) gives the arrow of time

Example (Minkowski space)

$M = R^{1,3}$ with $\ell(x, y) = \|y - x\|_F$

Example (Smooth globally hyperbolic Lorentzian manifolds)

Example (Causal spaces $(M, \leq, \ll)$ à la Kronheimer and Penrose '67)

A time-separation function gives a nested partial order $\leq$ and preorder $\ll$

$$M^2_{\leq} = \{(x,y) \in M^2 \mid \ell(x,y) \geq 0\}$$
$$M^2_{\ll} = \{ \quad '' \quad \mid \ell(x,y) > 0\}$$

Definition (Causal & timelike futures; causal diamonds and emeralds)

We say y lies in the *causal future* of x and write $x \leq y$ if $\ell(x,y) \geq 0$; we say y lies in the *timelike future* of x and write $x \ll y$ if $\ell(x,y) > 0$. Also

$$J^+(x) = \{y \in M \mid \ell(x,y) \geq 0\} = \textit{future} \qquad J^+(X) = \cup_{x \in X} J^+(x)$$
$$J^-(z) = \{y \in M \mid \ell(y,z) \geq 0\} = \textit{past} \qquad J^-(Z) = \cup_{z \in Z} J^-(z)$$
$$J(x,z) = J^+(x) \cap J^-(z) = \textit{diamond} \qquad J(X,Z) = J^+(X) \cap J^-(Z) = \textit{emerald}$$

and similarly $I^{\pm}(y)$ and $I(X,Z)$ but with strict inequalities $\ell > 0$.

Definition (Causal and timelike paths)

A path $s \mapsto \sigma(s) \in M$ is called *causal* if and only if $\ell(\sigma(s), \sigma(t)) \geq 0$ for all $s \leq t$, and *timelike* if and only if $\ell(\sigma(s), \sigma(t)) > 0$ for all $s < t$.

Definition (Lorentzian length of a causal path)

The (*negative*) ℓ-length of a causal path $\sigma : [a, b] \longrightarrow M$ is defined by

$$L_{-\ell}(\sigma) := \sup_{k \in \mathbf{N}} \sup_{a=t_0 \leq t_1 \leq \cdots \leq t_k = b} -\sum_{i=1}^{k} \ell(\sigma(t_{i-1}), \sigma(t_i))$$

$$\geq -\ell(\sigma(a), \sigma(b))$$

by the triangle inequality.

Definition (ℓ-path)

A path $\sigma : [0,1] \longrightarrow M$ is called an ℓ-path if and only if

$$\ell(\sigma(s),\sigma(t)) = (t-s)\ell(\sigma(0),\sigma(1)) > 0 \qquad \forall 0 \leq s < t \leq 1.$$

We denote the set of ℓ-paths by $\mathrm{TPath}^{\ell}(M)$.

- the above shows each ℓ-path minimizes $L_{-\ell}$ relative to its endpoints
- not all $L_{-\ell}$ minimizers are timelike, nor affinely parameterized

Definition

(M,ℓ) is *timelike ℓ-path space* if each $x \ll y$ are endpoints of an ℓ-path.

- Kunzinger and Sämann's *regular globally hyperbolic Lorentzian length spaces* provide a rich class of examples of timelike ℓ-path spaces
- to achieve this, they need a (metrizable) topology

a variation on Kunzinger & Sämann (hereafter K-S)

Definition (Metric spacetime)

A metric space (M, d) equipped with its metric topology and a time-separation function ℓ is called a *metric spacetime*

Definition (Causal curve)

A nonconstant causal *path* is called a causal *curve* if it is d-Lipschitz.

Definition (Non-total imprisoning)

A metric spacetime (M, d, ℓ) is *non-total imprisoning* if each compact $K \subset M$ has a bound $\sup L_d(\sigma) < \infty$ on d-length of causal curves σ in K.

Definition (Globally hyperbolic)

A metric spacetime (M, d, ℓ) is *globally hyperbolic* if it is non-total imprisoning and the causal diamond $J(x, y)$ is compact for each $x, y \in M$.

Definition (Timelike curve-connected; Lorentzian geodesic space)

A metric spacetime is *timelike curve-connected* iff each $x \ll y$ are connected by a timelike curve; it is a *Lorentzian geodesic space* iff each $x < y$ are connected by a causal curve σ with $L_{-\ell}(\sigma) = -\ell(\sigma(0), \sigma(1))$.

Theorem (M. 23+: Characterizing Lorentzian length spaces "LLS")

Assuming globally hyperbolicity, a metric spacetime (M, d, ℓ) is an LLS iff it is (a) a timelike curve-connected (b) Lorentzian geodesic space; (c) ℓ is upper semicontinuous; (d) $\ell_+ = \max\{\ell, 0\}$ is continuous and (e) $I^{\pm}(x)$ both nonempty $\forall x \in M$.

- modelled on manifolds without boundary
- In such spaces, K-S showed that metric topology coincides with the order topology induced by $\ll$; this implies g.h. LLS's are independent of d!
- Burtscher & Garcia-Hevelling 21+ characterize global hyperbolicity of an LLS via existence of Cauchy time functions (and surfaces)

- Unfortunately, its not clear that all ℓ-paths are continuous!

Definition (Regular(ly localizable))

An LLS is *regular* (or *regularly localizable*) if for any $L_{-\ell}$-minimizing causal curve, $L_{-\ell}(\sigma|_{[a,b]}) = 0$ with $\sigma|_{[a,b]}$ non-constant implies $L_{-\ell}(\sigma) = 0$.

Lemma (M. 23+)

In a globally hyperbolic regular LLS, each ℓ-path is continuous.

Corollary (Relating ℓ-paths to $L_{-\ell}$-extremizers)

In a globally hyperbolic regularly localizable Lorentzian length space:
(a) Every ℓ-path becomes a d-Lipschitz $L_{-\ell}$-minimizing curve after a continuous increasing (not necessarily Lipschitz) reparameterization.
(b) K-S: Conversely, every $L_{-\ell}$-minimizing curve with timelike separated endpoints becomes an ℓ-path after a similar reparameterization.

• For convenience, we deal only with metric spacetimes (M, d, ℓ) which are closed Lorentzian geodesic subsets of globally hyperbolic regular Lorentzian length spaces (= g.h.r. LLS).

Now that timelike geodesics exist:
• given a triple $x \ll y \ll z$ of timelike related events, we can compare the Lorentzian length of a bisector to that of the Minkowski triangle with the same Lorentzian sidelengths

• and similarly for generalized bisectors (i.e. ratios other than 1 : 1)

- K-S define $\text{T-Sec}(M, d, \ell) \geq 0$ if our generalized bisector is longer (and $\text{T-Sec}(M, d, \ell) \leq 0$ if it is shorter) for all such timelike triangles
- they define $\pm$ $\text{T-Sec}(M, d, \ell) \geq k \in \mathbf{R}$ analogously by comparing to timelike triangles in constant curvature Lorentzian spaces
- they also give causal sectional curvature bounds and show such bounds prevent branching of ℓ-geodesics:

Definition (timelike nonbranching)

(M, ℓ) *timelike nonbranching* if for all $\tilde{\sigma}, \sigma \in \mathrm{TPath}^{\ell}$ with $\sigma|_{[\frac{1}{3},\frac{2}{3}]} = \tilde{\sigma}|_{[\frac{1}{3},\frac{2}{3}]}$ then $\tilde{\sigma} = \sigma$;

- Andersson-Howard '03, Alexander-Bishop '08 shows consistency of these definitions with smooth timelike sectional curvature bounds on Lorentzian manifolds
- Kunzinger-Steinbauer '22, Minguzzi-Suhr '22+ show stability of similar bounds
- Beran-Ohanyan-Rott-Solis '22+: $\text{T-Sec}(M, d, \ell) \geq 0$ and existence of a timelike line implies geometric splitting of (M, d, ℓ)

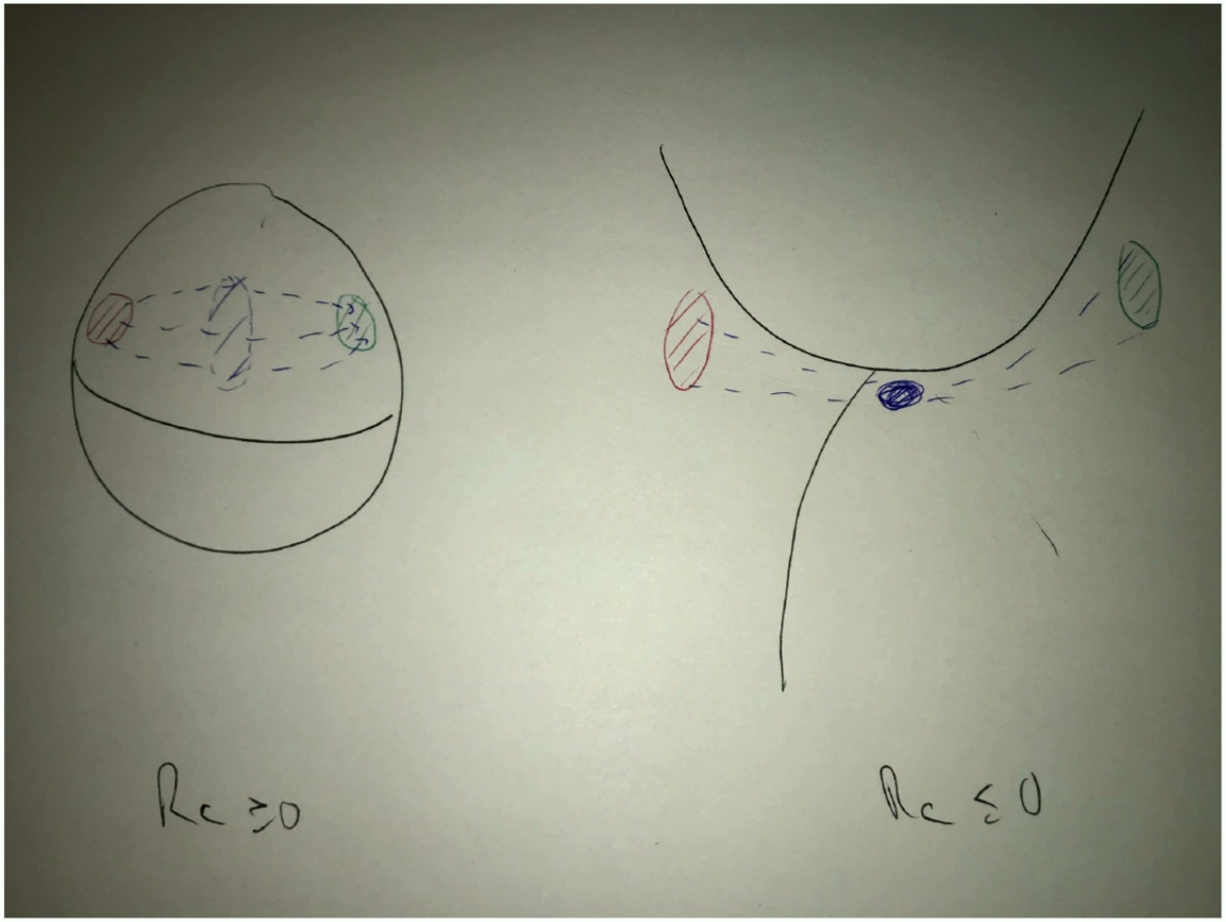
To pass from sectional to Ricci curvature / Einstein eq requires averaging:

Definition (Optimal transport distance between measures)

- Given metric spaces $(M^\pm, d^\pm)$, let $\mathcal{P}(M)$ denote the Borel probability measures on M and $\mathcal{P}_c(M)$ those with compact support.
- *Push-forward:* given $G : M^- \longrightarrow M^+$ Borel and $\mu^- \in \mathcal{P}(M^-)$, define $\mu^+ = G_\# \mu^- \in \mathcal{P}(M^+)$ by $\mu^+(B) = \mu^-(G^{-1}(B))$ for all $B \subset M^+$.
- Letting $\pi^\mp(x^-, x^+) = x^\mp$ denote the projection from $M^- \times M^+$ onto its left and right factors, set $\Gamma(\mu^-, \mu^+) = \{\gamma \in \mathcal{P}(M^- \times M^+) \mid \pi^\pm_\# \gamma = \mu^\pm\}$.
- Given $p \in [1, \infty)$ and $M = M^\pm$, the *p-Kantorovich-Rubinstein-Wasserstein distance d_p* between $\mu^\pm \in \mathcal{P}(M)$ defined by

$$d_p(\mu^-, \mu^+) := \inf_{\gamma \in \Gamma(\mu^+, \mu^-)} \left(\int_{M^2} d(x,y)^p d\gamma(x,y) \right)^{1/p} \quad (3)$$

is well-known to metrize convergence against functions growing no faster than $d(x, \cdot)^p$ provided (M, d) is *Polish* (i.e. complete and separable), in which case the inf is attained.
- If (M, d) is a geodesic space so is $(\mathcal{P}_c(M), d_p)$.


Rc ≥ 0
Rc ≤ 0

Definition (Causal and timelike measures)

In a Polish g.h.r LLS (M, d, ℓ), given $\mu, \nu \in \mathcal{P}_c(M)$ and $q \in (0,1]$ set

$$\Gamma_{\leq}(\mu,\nu) := \{\gamma \in \Gamma(\mu,\nu) \mid \gamma[M^2_{\leq}] = 1\} = \{\textit{causal measures}\}$$

$$\Gamma_{\ll}(\mu,\nu) := \{ \quad \text{"} \quad \mid \gamma[M^2_{\ll}] = 1\} = \{\textit{timelike measures}\}$$

Lemma (Lift time-separation from events to measures)

$$\ell_q(\mu,\nu) := \max_{\gamma \in \Gamma_{\leq}(\mu,\nu)} \left(\int_{M^2} \ell(x,y)^q d\gamma(x,y) \right)^{1/q} \qquad (4)$$

makes $(\mathcal{P}_c(M), \ell_q)$ into a timelike ℓ_q-path space. Not all such ℓ_q-paths are d_1-continuous; one will be if (μ,ν) is timelike q-dualizable:

Definition (timelike q-dualizability)

Let $\Gamma^q = \Gamma^q(\mu,\nu)$ denote the set of maximizers. Then

- (μ,ν) are *timelike q-dualizable* if $\Gamma^q_{\ll} := \Gamma^q \cap \Gamma_{\ll}(\mu,\nu)$ is non-empty
- (μ,ν) are *strongly* timelike q-dualizable if, in addition, $\Gamma^q \subset \Gamma_{\ll}(\mu,\nu)$

Definition (Polish / proper metric-measure spacetime)

A *metric-measure spacetime* refers to a Lorentzian geodesic closed subset (M, d, ℓ) of a g.h.r. LLS, equipped with a Borel measure $m \geq 0$, finite on bounded sets. It's called *Polish* if complete and separable, and *proper* if all bounded subsets $X \subset M$ are compact.

Example (Smooth metric-measure spacetimes)

Any smooth, connected, Hausdorff, time-oriented, n-dimensional Lorentzian manifold (M^n, g) of signature $(+ - \ldots -)$ is second-countable (Ozeki-Nomizu '61) and its topology comes from a complete Riemannian metric $\tilde{g}$ (Geroch '68). With the distance $d_{\tilde{g}}$ and time-separation function ℓ_g induced by $\tilde{g}$ and g respectively, is a proper g.h.r. LLS provided it has no closed causal curves and causal diamonds $J(x, y)$ are compact. Letting $V \in C^\infty(M)$ and vol_g denote its Lorentzian volume, setting $dm = e^{-V} d\mathrm{vol}_g$ makes it a proper metric-measure spacetime. We call such spaces *smooth metric-measure spacetimes*.

Synthetic timelike Ricci bounds

Desiderata:

- consistency (with the analogous smooth bounds)
- stability (preservation under suitable limits)
- consequences (e.g. Hawking-type singularity theorem)

Definition (Entropy)

We define the relative *entropy* by

$$H(\mu \mid m) := \begin{cases} \int_M \rho \log \rho \, dm & \text{if } \mu \in \mathcal{P}_c^{ac}(M) \text{ and } \rho := \frac{d\mu}{dm}, \\ +\infty & \text{if } \mu \in \mathcal{P}_c(M) \setminus \mathcal{P}^{ac}(M). \end{cases}$$

- our sign convention is opposite to that of the physicists' entropy

Entropic weak timelike curvature-dimension conditions

Definition (TCD versus wTCD; e.g. $K = 0 = 1/N$)

For $(K, N, q) \in \mathbf{R} \times (0, \infty] \times (0, 1]$ write $(M, d, \ell, m) \in wTCD^e_q(K, N)$ if and only if every strongly timelike q-dualizable finite entropy pair $\mu_0, \mu_1 \in \mathcal{P}_c(M)$ admit a maximizer $\gamma \in \Gamma^q_{\ll}$ and corresponding ℓ_q-path $(\mu_t)_{t\in[0,1]}$ along which the entropy $t \in [0, 1] \mapsto h(t) := H(\mu_t \mid m)$ is upper-semicontinuous and distributionally solves the semiconvexity inequality

$$h''(t) \geq \frac{h'(t)^2}{N} + K\|\ell\|^2_{L^2(\gamma)}.$$

Cavalletti-Mondino '20+ prove all limits of $TCD^e_q(K, N)$ space in a suitable (pointed measured weak) sense lie in $wTCD^e_q(K, N)$ if $N < \infty$

Pointed measured weak convergence [Cav.-Mondino 20+]

Fixing $x_j \in \operatorname{spt} m_j$ where m_j is a Radon measure, we say $(M_j, d_j, \ell_j, m_j, x_j) \to_{pmGL} (M_\infty, d_\infty, \ell_\infty, m_\infty, x_\infty)$ iff all $(M_j, d_j, \ell_j, m_j, x_j)$ embed d-continuously and ℓ-isometrically into a single proper g.h.r. LLS (X, d, ℓ) and after this embedding, $d(x_j, x_\infty) \to 0$ and the measures $m_j \to m_\infty$ converge weakly against continuous compactly supported test functions: i.e.

$$\lim_{j\to\infty} \int_X \phi dm_j = \int_X \phi dm_\infty \qquad \forall \phi \in C_c(X).$$

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$$\lim_{j\to\infty} \int_X \phi \, dm_j = \int_X \phi \, dm_\infty \qquad \forall \phi \in C_c(X).$$

- although the limit of $TCD^e_q(K,N)$ spaces is only $wTCD^e_q(K,N)$, Braun '22+ shows (q-essentially) timelike nonbranching $wTCD^e_q(K,N)$ spaces are $TCD^e_q(K,N)$. Hence a limit of timelike nonbranching $wTCD^e_q(K,N)$ spaces is $wTCD^e_q(K,N)$.

Positive energy ⇔ displacement convexity of entropy

DEF (N-Bakry-Emery modified Ricci tensor; cf. Erbar-Kuwada-Sturm'15)
Given $N \neq n$ and $V \in C^\infty(M^n)$ define

$$R_{ij}^{(N,V)} := R_{ij} + \nabla_i \nabla_j V - \frac{1}{N-n} (\nabla_i V)(\nabla_j V)$$

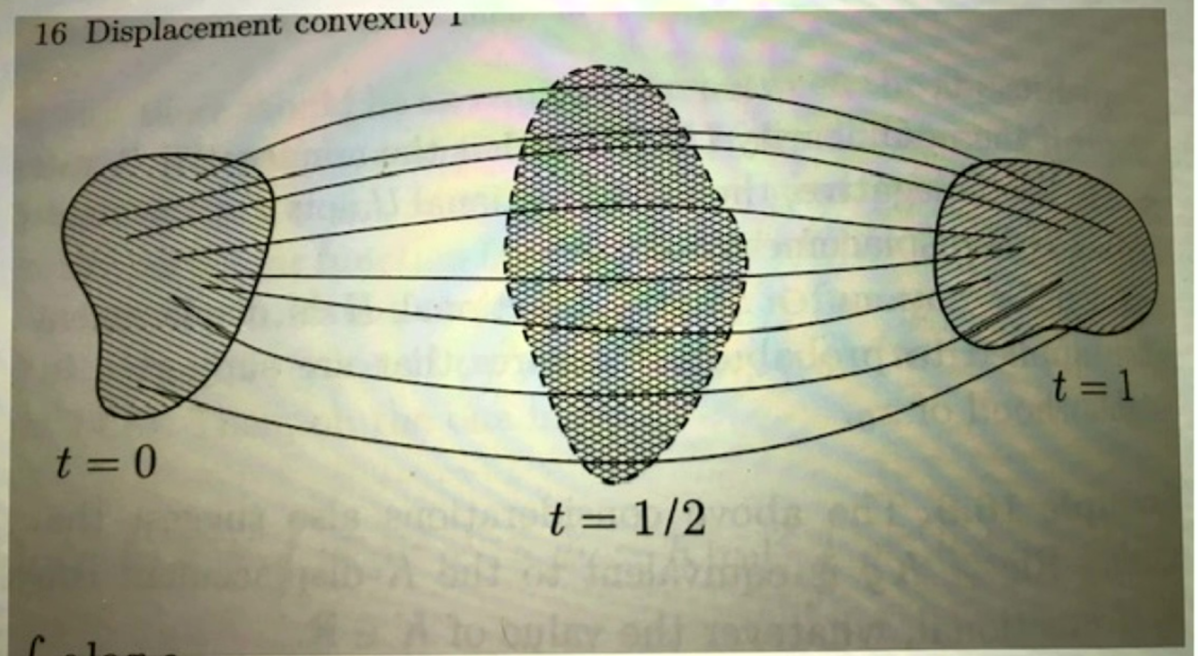
THM (M '20 Consistency) Fix $(K, N, q) \in \mathbf{R} \times (0, \infty] \times (0, 1)$ and a smooth metric-measure spacetime (M^n, g) with $dm = e^{-V} d\mathrm{vol}_g$. Then $(M, d_{\tilde{g}}, \ell_g, m) \in (w)TCD_q^e(K, N)$ if and only if either

(a) $N = n$, $V = const$ and $R_{ij} v^i v^j \geq K$ for all unit timelike $(v, x) \in TM$,
(b) $N > n$ and $R_{ij}^{(N,V)} v^i v^j \geq K$ for all unit timelike vectors $(v, x) \in TM$.

Mondino-Suhr '23 Use entropic convexity to say also when equality holds, giving a weak (but unstable) solution concept for Einstein field equation.

Akdemir-Cavalletti-Colinet-M.-Santarcangelo '21
$CD_p(K, N) \cap \{\text{nonbranching}\}$ is independent of $p > 1$

Lazy Gas Experiment (M. 94, Villani 09)



Action minimizing paths satisfy pressureless Euler equation.

Braun 23:

- $N = \infty$
- alternative definitions of $(w)TCD_q^{(*)}(K, N)$ based on convexity properties of a power-law entropy (instead of $H(\mu \mid m)$) along ℓ_q-paths

$$S_N(\mu) := -N \int_M (\frac{d\mu}{dm})^{1-\frac{1}{N}} dm$$

- equivalence of most of these various definitions to $TCD_q^e(K, N)$ assuming (q-essential) timelike nonbranching

Cavalletti-Mondino '22:

- asked for a synthetic formulation of the null energy condition (NEC)
- stronger physical motivation; more widely satisfied
- forms a key hypothesis in the Penrose singularity theorem for stellar collapse

Theorem (M' 23+)

Fix a smooth spacetime (M^n, g) with signature $(+ - \cdots -)$ and symmetric 2-tensor field Q. Then

$$Q(v,v) \geq 0 \quad \forall (v,x) \in TM \text{ with } g(v,v) = 0$$

holds if and only if each compact subdomain $X \subset M^n$ admits a timelike lower bound $K = K_X$ for Q, i.e.

$$Q(v,v) \geq K g(v,v) \quad \forall (v,x) \in TX \text{ with } g(v,v) > 0$$

Taking $Q = \mathrm{Ric}^{(N,V)}$ (or $Q_{ab} = 8\pi T_{ab}$ if Einstein holds) motivates

Definition (A synthetic null energy-dimension condition)

Given $(N, q) \in (0, \infty] \times (0, 1)$, a metric-measure spacetime (M, d, ℓ, m) satisfies $wNC_q^{(e)}(N)$ if and only if each compact subset $X \subset M$ admits a bound $K = K_X \in \mathbf{R}$ such that $J(X, X) \in wTCD_q^{(e)}(K, N)$.

- in other words, the null energy condition is equivalent to a variable lower (semicontinuous) bound $k(x)$ on the timelike Ricci curvature
- Consistency with smooth (NEC) + $(n \leq N)$: follows from theorem above
- for (q-essentially) timelike nonbranching spaces $wNC^e_q(N) = NC^*_q(N)$
- Consequences: many of Cavalletti & Mondino's nice properties (timelike Bishop-Gromov and Brunn-Minkowski inequalities, needle decomposition, etc) of nonsmooth $wTCD^{(e)}_q(K,N)$ spacetimes are therefore inherited directly by $wNC^{(e)}_q(N)$ spacetimes; c.f. Braun-M. (in progress)
- (In)stability: on the other hand, any stability result appears hopeless unless we are will to assume some uniformity in j of the lower bound $k(\cdot)$ along the sequence $(M_j, d_j, \ell_j, m_j, x_j)$
- BBCGMORS: infinitesmally Minkowski refinement of TCD^e_q, analogous to Ambrosio-Gigli-Savarè's infinitesmally Hilbertian refinement RCD of CD
- OPEN: it is natural to wonder if a Penrose singularity theorem can hold in this nonsmooth setting? (c.f. Graf '20 on $g \in C^1$ spacetimes (M^n, g), Ketterer '23+ entropic convexity derivation on $g \in C^\infty$ spacetimes)

Example (Instability displaying topology change)

Fix a slab $M := \{(x^1, \ldots, x^n) \in \mathbf{R}^n_1 \mid x^1 \in [-1, 1]\}$ of Minkowski space, with its usual metric g and $\tilde{g}$ but $dm_j(x) = e^{jg(x,x)} d\mathrm{vol}_g(x)$.
Then $(M, d_{\tilde{g}}, \ell_g, \hat{m}_j, 0) \to (M, d_{\tilde{g}}, \ell_g, m_\infty, 0)$ where $m_\infty = \frac{1}{2}(\delta_z + \delta_{-z})$ and $z = (1, 0, \ldots, 0)$. Here $d\hat{m}_j(x) = dm_j(x) / \int_M dm_j$.
Moreover $(M, d_{\tilde{g}}, \ell_g, \hat{m}_j, 0) \in NC^e_q(\infty)$ if and only if $j < \infty$ (and $\mathrm{spt}\, \mu_j$ is connected if and only if $j < \infty$)

Proof.

$$\mathrm{Ric}^{N,V}_x(w, w) = 0 + (-j) g(w, w) - \frac{1}{N - n} g(w, x)^2.$$

□

OPEN: Might the weak or dominant energy condition be stable?

A few references

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