Title: Reflection positivity for extended topological field theories

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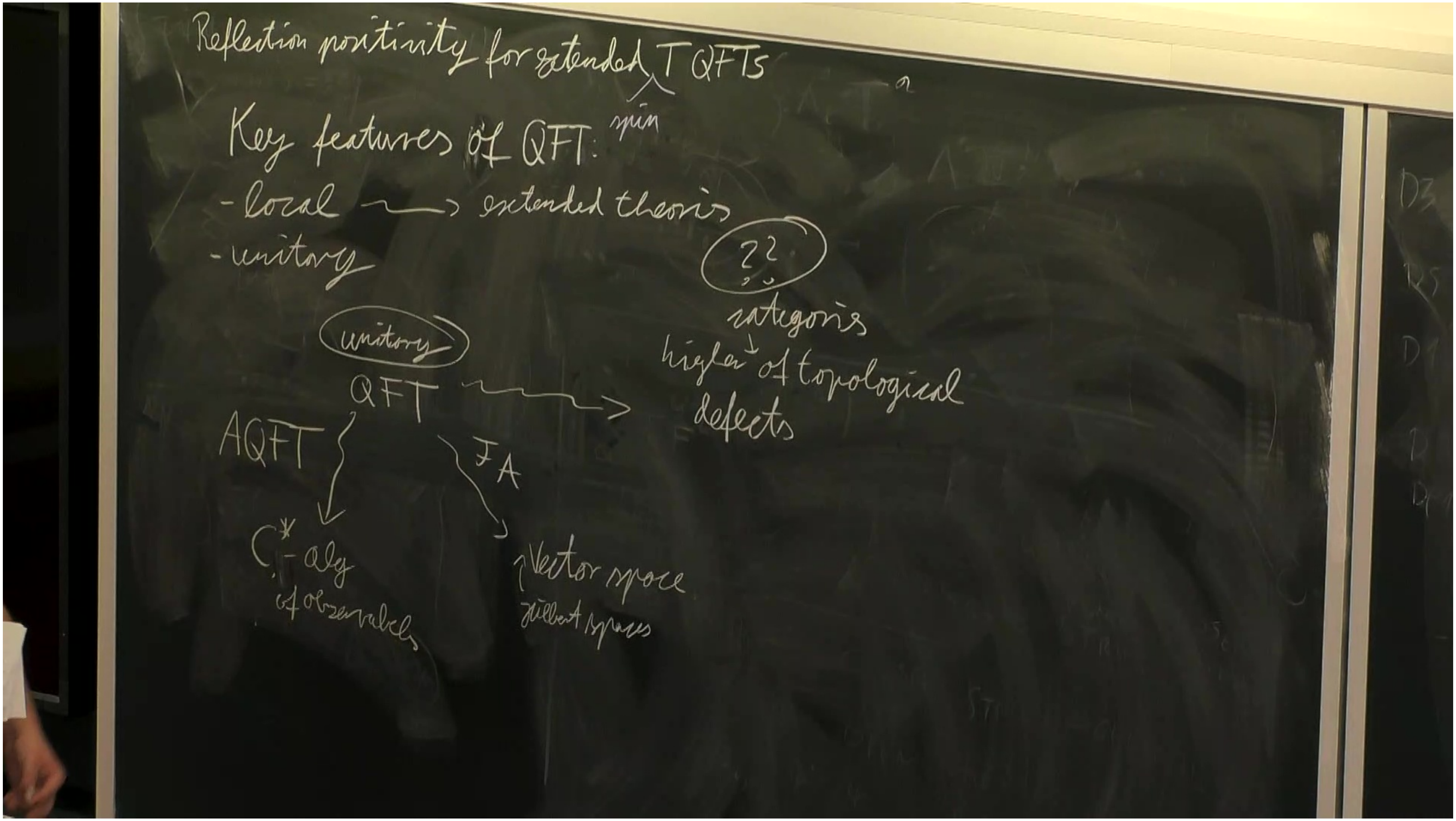
Series: Mathematical Physics

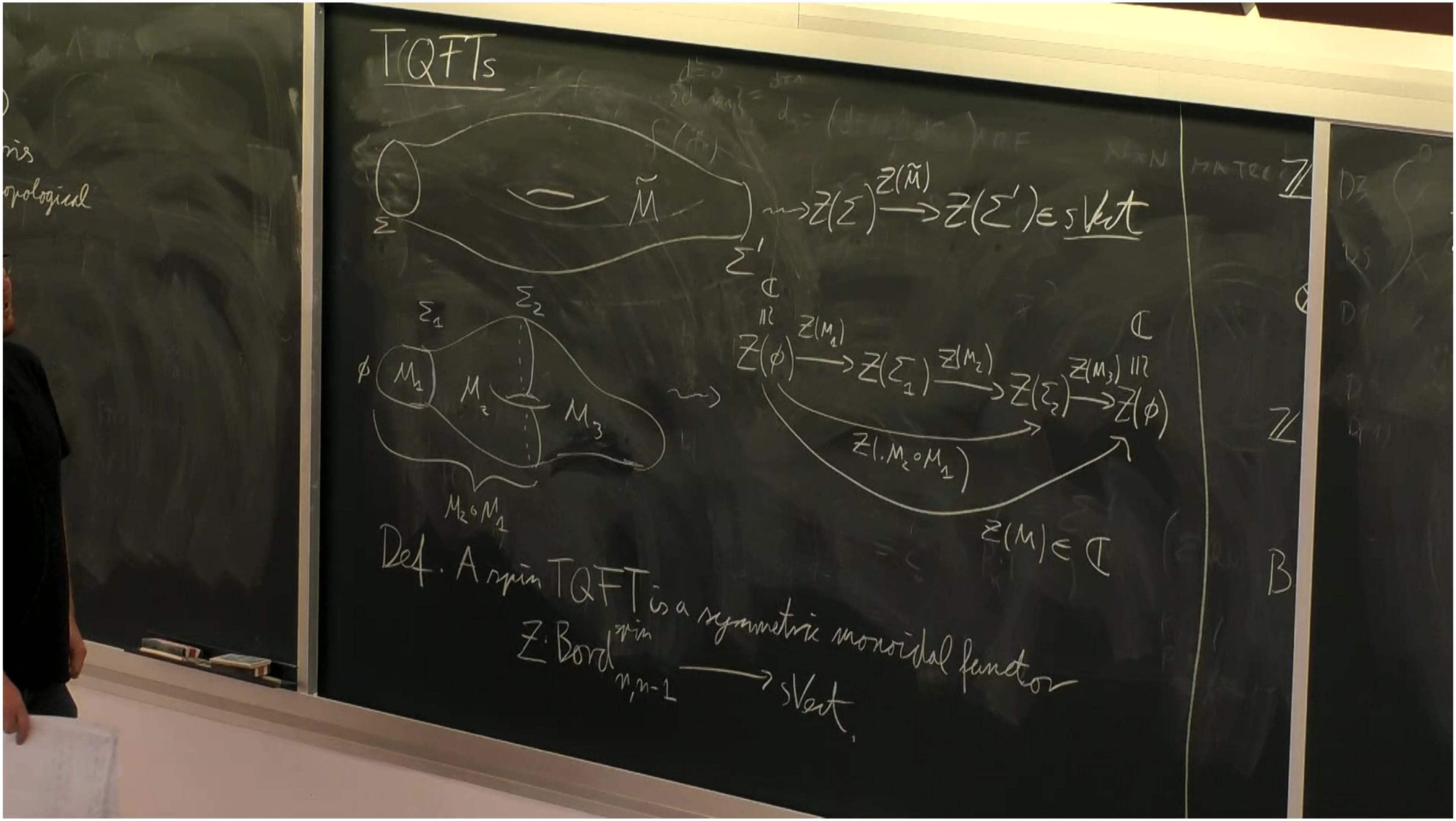
Date: October 05, 2023 - 1:30 PM

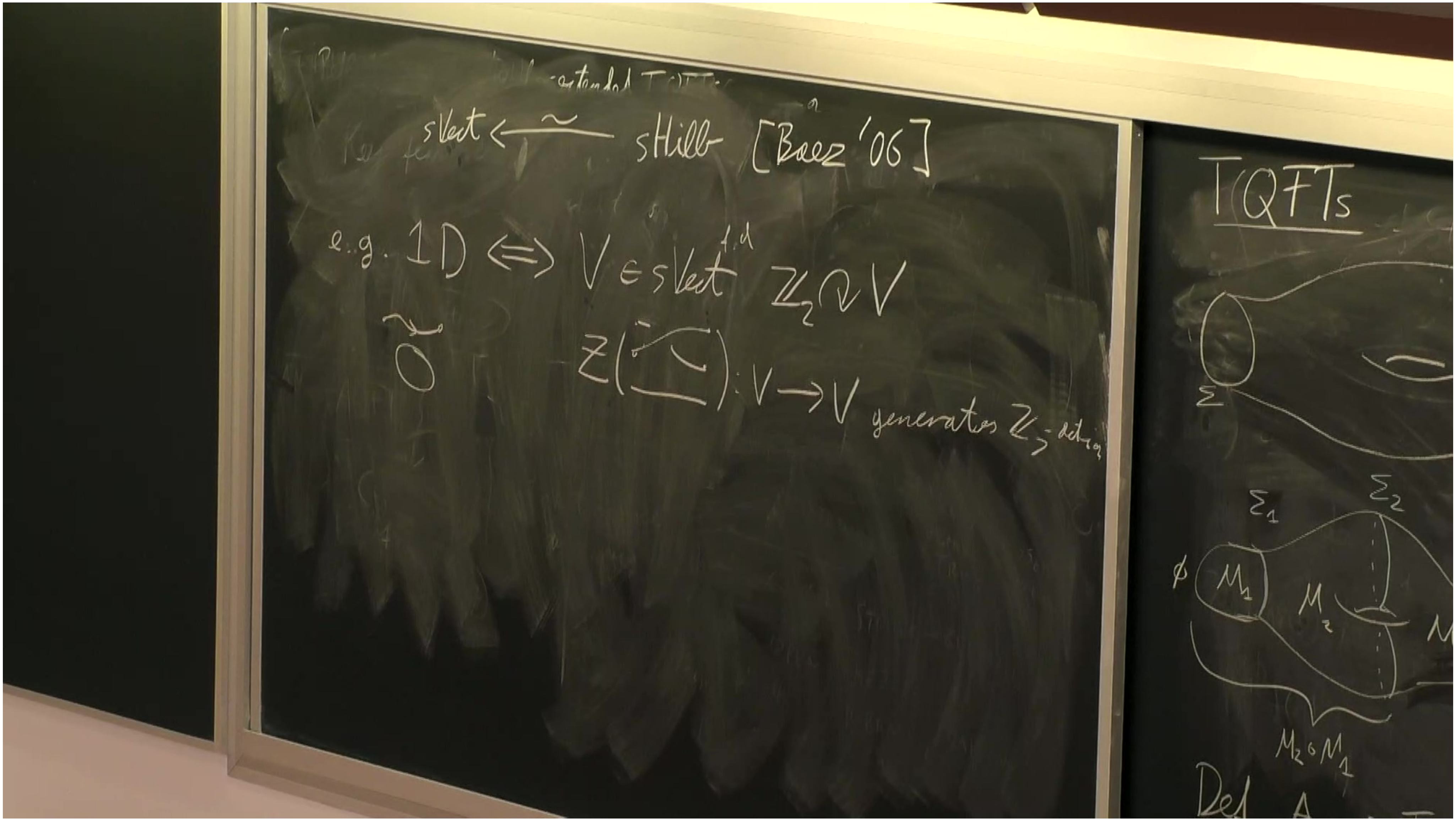
URL: https://pirsa.org/23100075

Abstract: In quantum field theories, locality and unitarity are essential properties. For functorial field theories, locality is manifested through compatibility with cutting and gluing of manifolds, which can be fully encoded in the definition of fully extended functorial field theories. However, unitarity or reflection positivity (its Euclidean version) has so far only been defined for non-extended or invertible field theories. In this talk, I will address the challenge of defining reflection positivity for extended topological field theories, proposing a definition based on a version of higher dagger categories.

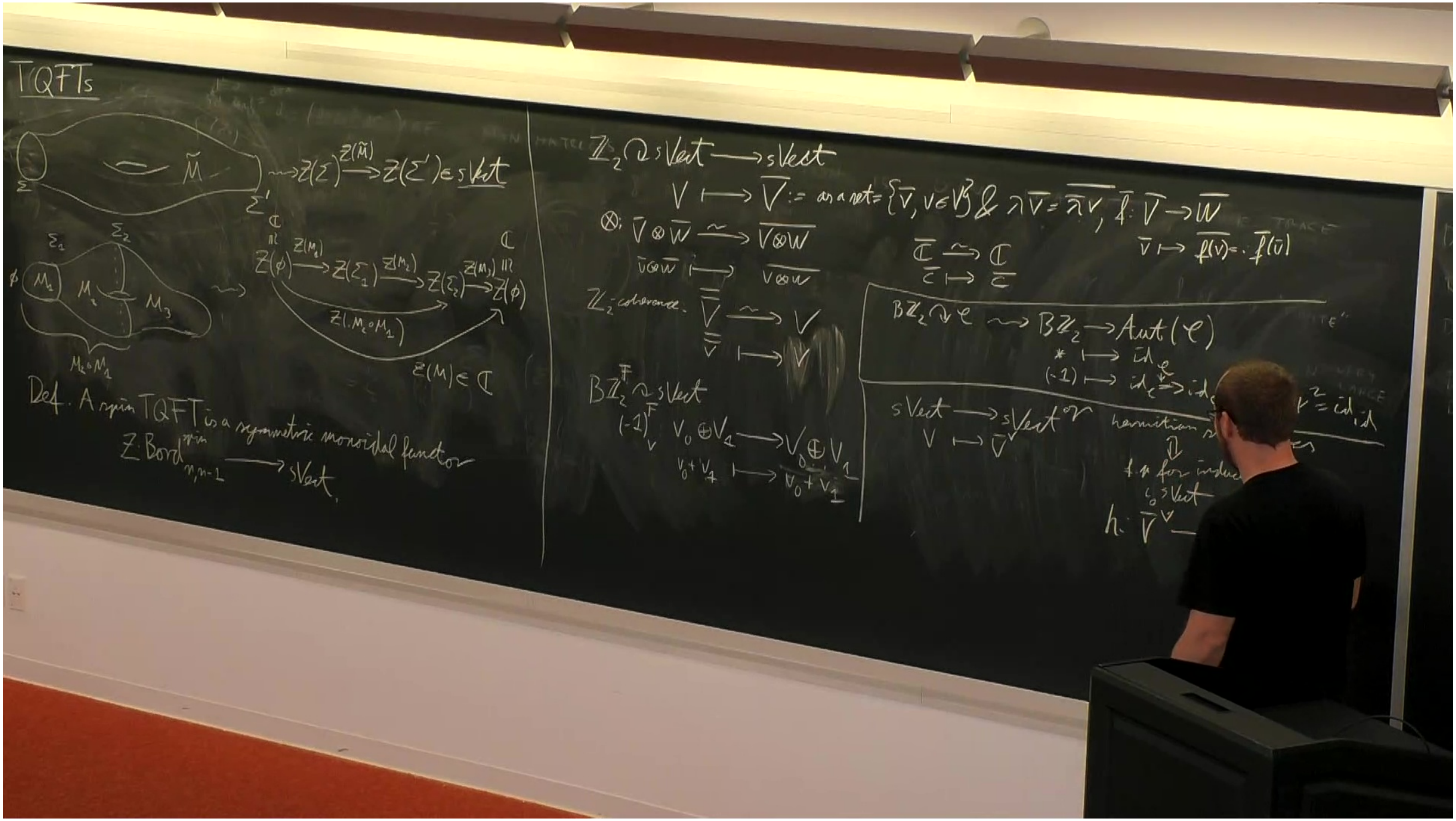
Zoom link: https://pitp.zoom.us/j/93806769415?pwd=TVJFbWlJK0JyZCtjbHMvOWYxVUlRUT09


Reflection positivity for extended TQFTs
spin
Key features of QFT:
- local ⇝ extended theories
- unitary
??
categories
higher of topological defects
unitary
QFT
AQFT
FA
C*-alg
of observables
Vector space
Hilbert spaces


TQFTs
M̃
Z(M̃)
⇝ Z(Σ) → Z(Σ') ∈ sVect
Σ
Σ'
Σ1
Σ2
∅
M1
M2
M3
M2∘M1
ℂ
Z(M1)
Z(M2)
Z(M3)
Z(∅) → Z(Σ1) → Z(Σ2) → Z(∅)
Z(M2∘M1)
Z(M) ∈ ℂ
Def. A spin TQFT is a symmetric monoidal functor
Z: Bord^spin_{n,n-1} → sVect,
ℤ
B



sVect ← sHilb [Baez '06]
e.g. 1D ⟺ V ∈ sVect f.d. Z2 ↻ V
Z( ) : V → V generates Z2-action
TQFTs
Σ
Σ1
Σ2
∅
M1
M2
M2 ∘ M1
Def


TQFTs
Def. A spin TQFT is a symmetric monoidal functor
Z: Bord^spin_{n,n-1} ⟶ sVect.
Z₂ ↻ sVect ⟶ sVect
V ⟼ V̄ := as a set = {v̄, v∈V} & λv̄ = \overline{λ̄v}, f̄: V̄ → W̄
Z₂-coherence.
BZ₂ ↻ 𝒞 ⇝ BZ₂ → Aut(𝒞)
sVect ⟶ sVect^op
hermitian

$\mathbb{Z}_2 \curvearrowright \mathrm{Bord}^{spin}_{n,n-1}$

$P \; Spin_n$

$\downarrow$

M

$P \times_{Spin_n} \mathbb{R}^n \xrightarrow[\sim]{\alpha} TM$

$\curvearrowleft Spin_n$

$\overline{(P,\alpha)} := (\bar P, \bar\alpha)$ $\quad \bar P = P \times_{Spin_n} Pin^+_n \setminus P$

$\mathbb{Z}_2 \downarrow O_2$

$B\mathbb{Z}_2^c$: spin flip $c \in Spin_n$ $\quad \bar\alpha : \bar P \times_{Spin} \mathbb{R}^n \xrightarrow{\sim} TM$

$\mathrm{id}_\Sigma : \Sigma \to \Sigma$

$[p \times c \times X] \longmapsto \alpha(p, c[X])$

$\mathbb{Z}_2 \curvearrowright sVect \longrightarrow$

$V \longmapsto$

$\otimes$: $\bar V \otimes \bar W \xrightarrow{\sim}$

$\bar v \otimes \bar w \longmapsto$

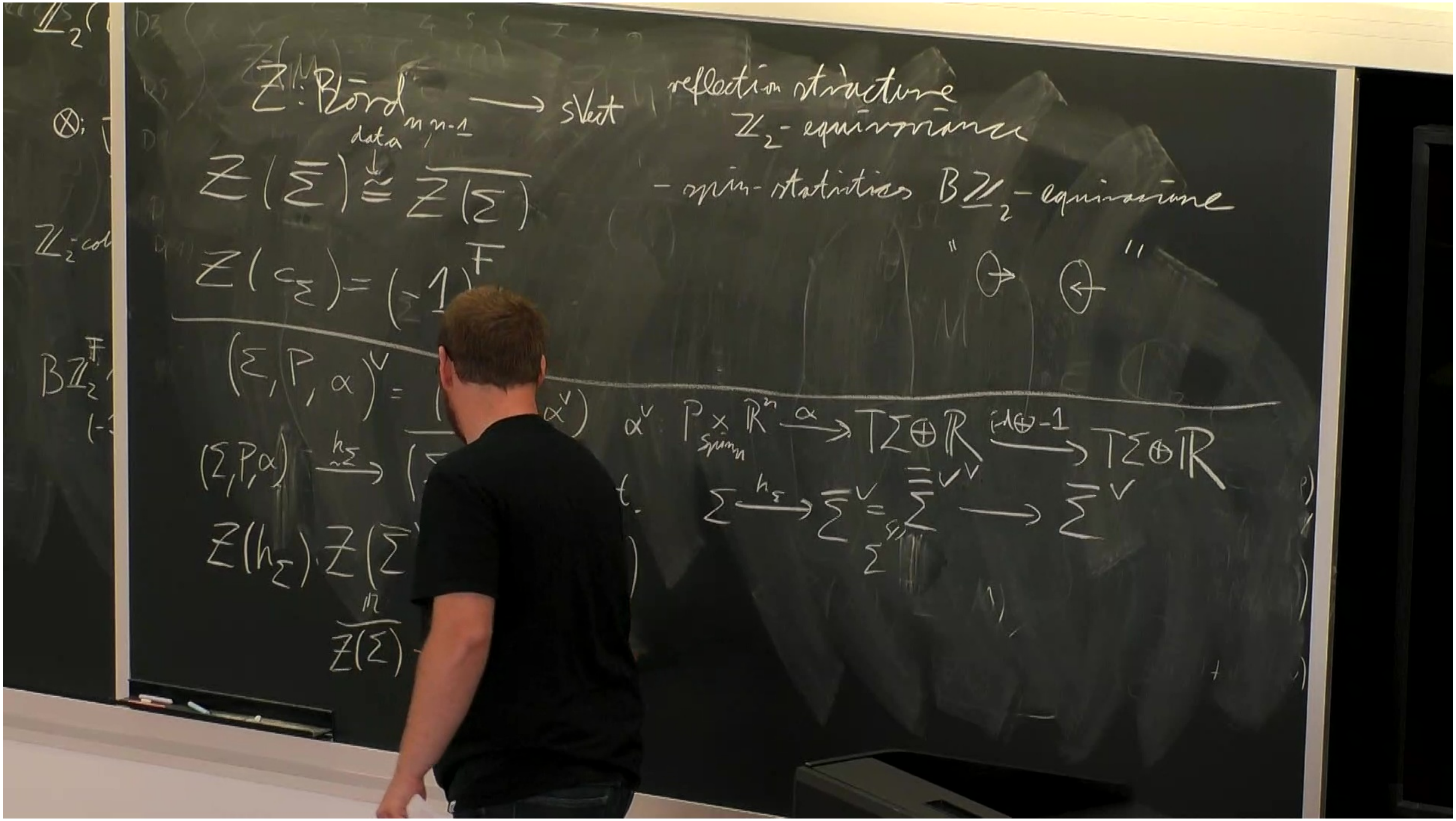
$\mathbb{Z}_2$-coherence: $\bar{\bar V} \longrightarrow$

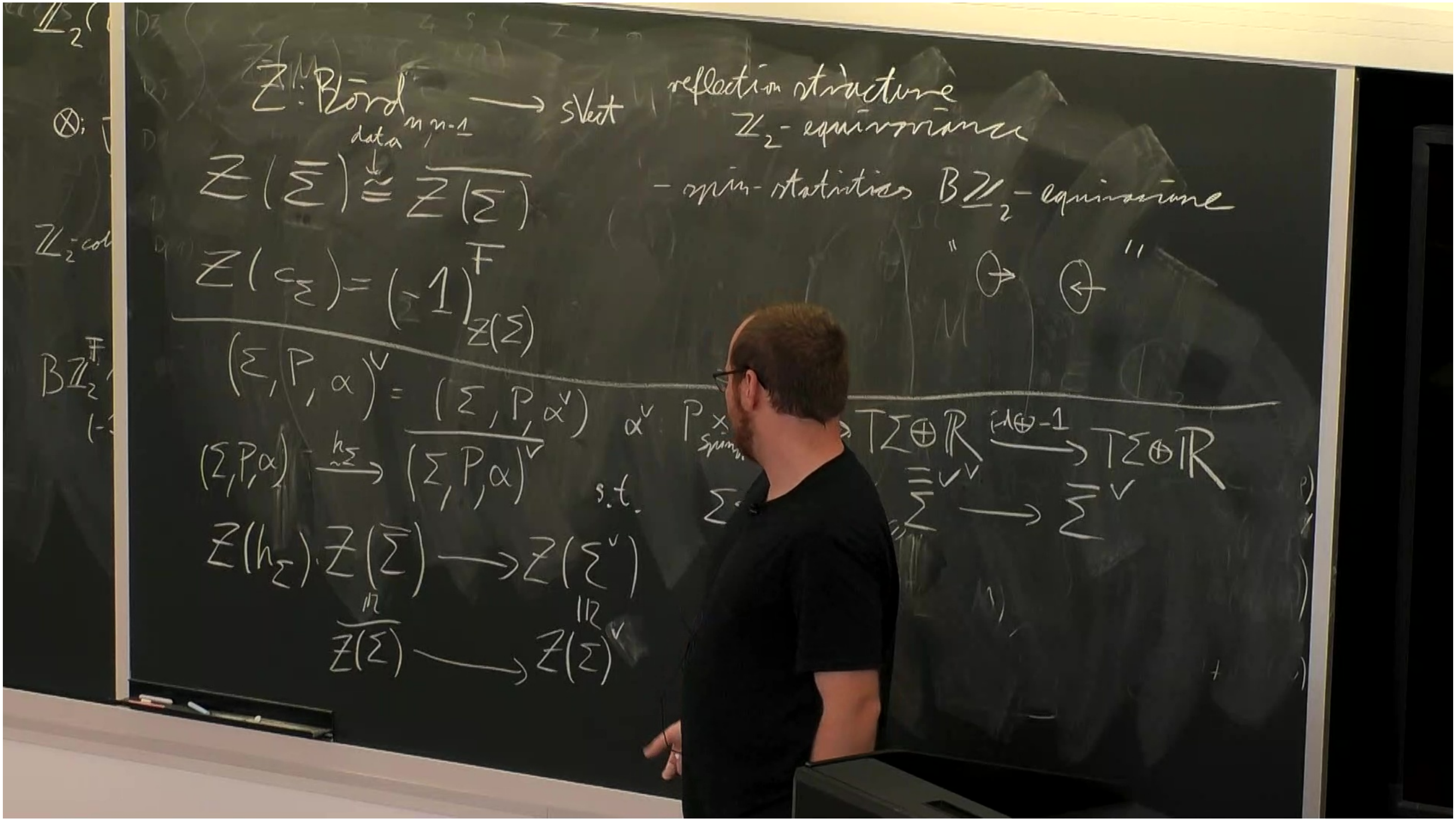
$\bar{\bar v}$

$B\mathbb{Z}_2^F \curvearrowright sVect$

$(-1)^F_V : V_0 \oplus V_1 \longrightarrow$

$v_0 + v_1 \longmapsto$


Z : Bord_{n,n-1} ⟶ sVect
data
reflection structure
Z_2 - equivariance
Z(Σ) ≅ Z(Σ)
- spin-statistics BZ_2 - equivariance
Z(c_Σ) = (-1)^F
(Σ, P, α)^∨ =
α^∨
α^∨ : P ×_{Spin_n} R^n ⟶ TΣ ⊕ R ⟶ TΣ ⊕ R
(Σ, P, α) ⟶ (
Σ ⟶ Σ^∨ = Σ^∨∨ ⟶ Σ^∨
Z(h_Σ) · Z(Σ
Z(Σ)


$Z : \mathrm{Bord}_{n,n-1}^{\mathrm{data}} \longrightarrow \mathrm{sVect}$
reflection structure
$\mathbb{Z}_2$- equivariance
$Z(\bar{\Sigma}) \cong \overline{Z(\Sigma)}$
- spin-statistics $B\mathbb{Z}_2$- equivariance
$Z(c_\Sigma) = (-1)^F_{Z(\Sigma)}$
$(\Sigma, P, \alpha)^\vee = (\Sigma, P, \alpha^\vee)$
$(\Sigma, P, \alpha) \xrightarrow{h_\Sigma} \overline{(\Sigma, P, \alpha)^\vee}$
s.t.
$T\Sigma \oplus \mathbb{R} \xrightarrow{-1 \oplus -1} T\Sigma \oplus \mathbb{R}$
$\bar{\Sigma}^{\vee\vee} \longrightarrow \bar{\Sigma}^\vee$
$Z(h_\Sigma) \cdot Z(\bar{\Sigma}) \longrightarrow Z(\Sigma^\vee)$
$\overline{Z(\Sigma)} \longrightarrow Z(\Sigma)^\vee$

Def [FH]

Z in RP iff $Z(h_\Sigma)$ defines a super Hilbert space.

e.g. 1D $\Longleftrightarrow$ $V \in s\mathrm{Hilb}$. 2D fully extended RP $\Longleftrightarrow$ $s2\mathrm{Hilb}$

$$
\begin{tikzcd}
 & (s\mathrm{Bord}_n)^{\mathbb{Z}_2} \arrow[d] \arrow[rr] \arrow[d, phantom, "\circlearrowleft \mathbb{Z}_2"'] & & (s\mathrm{Vect})^{\mathbb{Z}_2} \arrow[d] & \\
\{\Sigma, h_\Sigma\} \arrow[ur] \arrow[r] & \mathrm{Bord}_{n,n-1} \arrow[rr] & & s\mathrm{Vect} \arrow[loop above, "\mathbb{Z}_2"] & s\mathrm{Hilb}^u \arrow[ul] \arrow[l]
\end{tikzcd}
$$

RP

Def A $\dagger$-category:

$$\mathcal{C} : \dagger : \mathcal{C} \longrightarrow \mathcal{C}^{op} \quad \dagger(c) = c \quad \dagger^2 = \mathrm{id}_{\mathcal{C}}$$

$$\dagger : \mathrm{Bord}_n \longrightarrow \mathrm{Bord}_n^{op}$$

$$\Sigma \longmapsto \Sigma$$

$$\Sigma_1 \xrightarrow{M} \Sigma_2 \longmapsto M^\dagger : \Sigma_2 \xrightarrow{h_{\Sigma_2}} \bar{\Sigma}_2^\vee \xrightarrow{\bar{M}^\vee} \bar{\Sigma}_1^\vee \xrightarrow{h_{\Sigma_1}} \Sigma_1$$

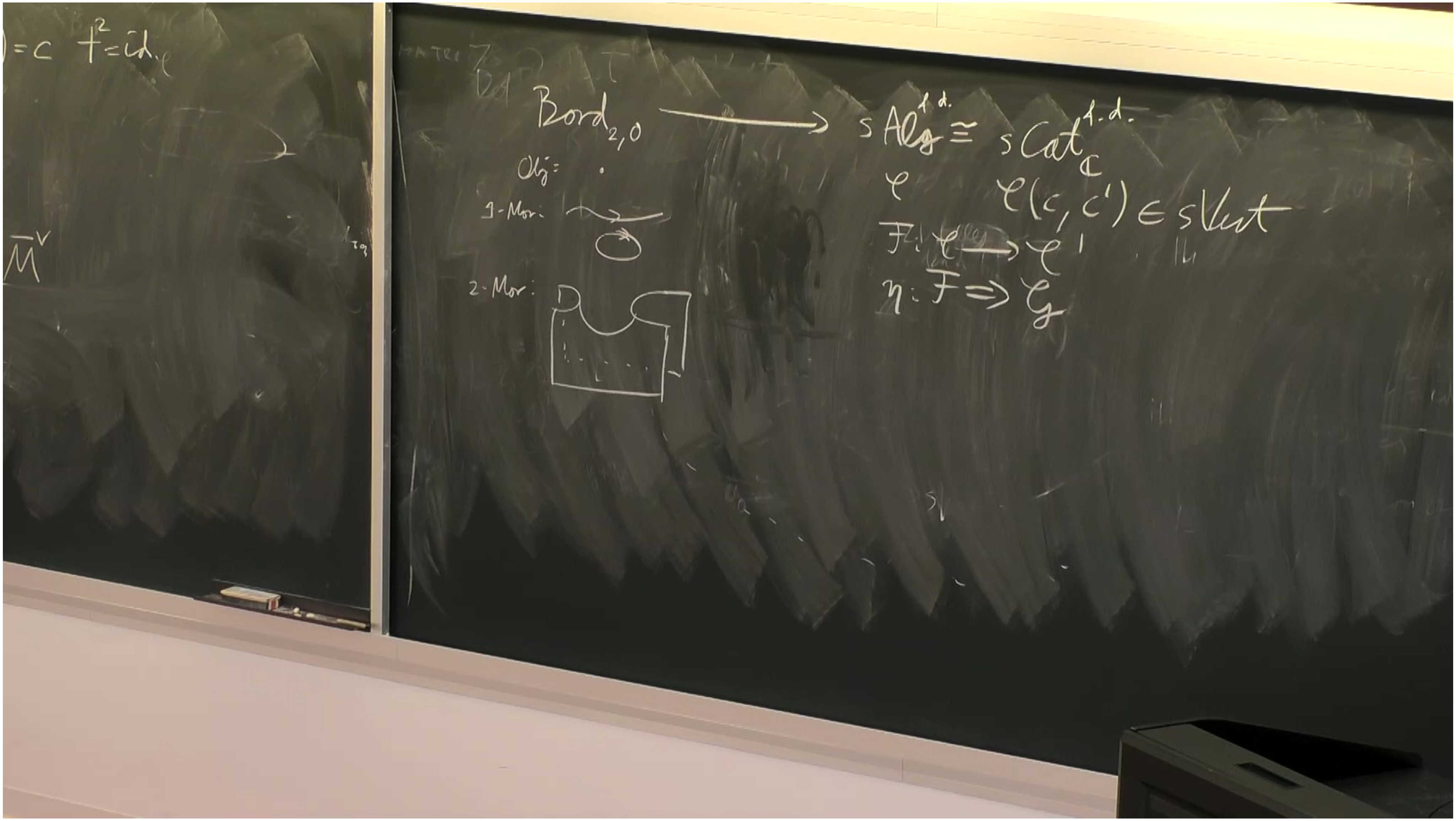
Thm [Stehouwer, Stehouwer-Steinebrunner]

RP TQFTs $\Longleftrightarrow$ $\dagger$-functors

Def

Z

e.g.


=c f² = id
Bord 2,0 → sAlg f.d. ≅ sCat f.d. ℂ
Obj: •
1-Mor:
2-Mor:
𝒞(c, c') ∈ sVect
F: 𝒞 → 𝒞'
η: F ⇒ G

$\mathcal{C} \longrightarrow \mathcal{C}^{or}$

$\mathrm{Aut}(\mathrm{Cat}) = \mathbb{Z}_2$

$\mathcal{C} \longmapsto \mathcal{C}^{or}$

$\mathrm{Aut}(\mathrm{BiCat}) = \mathbb{Z}_2 \times \mathbb{Z}_2$

$\mathrm{Aut}(\mathrm{AdjBiCat}) \underset{\text{conjecture}}{\cong} O_2 \supseteq O_2$

$O_2 = S_1 \rtimes \mathbb{Z}_2 \cong B\mathbb{Z} \rtimes \mathbb{Z}_2 \curvearrowright \mathrm{AdjBiCat}$

Def O_2-fixed bicategory
- F.P. for this O_2-action
- trivializations on objects and 1-Mor

$\mathcal{B} \longmapsto \mathcal{B}^{1\,or}$

$\mathcal{B} \xrightarrow{(-)^{RR}} \mathcal{B}$

$f \longmapsto f^{RR}$

Thm $\mathrm{Bord}_{n,n-2}$ is O_2-dagger

Thm $s\mathrm{Cat}_{\mathbb{C}}$ are O_2-dagger $\leftrightsquigarrow$ 3Hilbert space

Def. An extended RP-theory is an O_2-$\dagger$ functor

$$\mathrm{Bord}_{n,n-2} \longrightarrow s\mathrm{Hilb}$$

Thm

2D f.e. RP spin theories $\Longleftrightarrow$ 2sHilb

$$\mathcal{C} \xrightarrow{\dagger} \mathcal{C}^{op}$$

$f: c \to c'$

$g: c \to c'$

$\lambda|_c : \mathrm{End}(c) \longrightarrow \mathbb{C}$ $\quad \langle f, g \rangle = \lambda_c(f^\dagger \circ g)$

sCY-structure

is positive

$$N^{\sim}_{\mathcal{C}} = S_{\mathcal{C}} = \int^{d \in \mathcal{C}} \mathcal{C}(c,d)^* \otimes d$$

Baez '10?

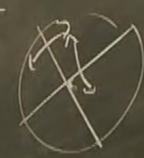
$$c \underset{g}{\overset{f}{\rightleftarrows}} c'$$

$\mathcal{C} \longrightarrow \mathcal{C}^{op}$

$\mathrm{Aut}(\mathrm{Cat}) = \mathbb{Z}_2$

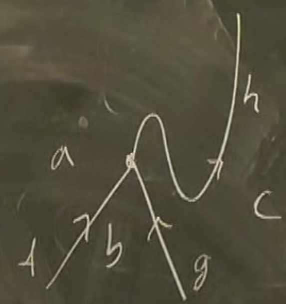
$\mathcal{C} \longmapsto \mathcal{C}^{op}$

$\mathrm{Aut}(\mathrm{BiCat}) = \mathbb{Z}_2 \times \mathbb{Z}_2$



$\mathrm{Aut}(\mathrm{AdjBiCat}) \underset{?}{=} O_2 \supseteq O_2$

conjecture



$O_2 = S_1 \rtimes \mathbb{Z}_2 \simeq B\mathbb{Z} \rtimes \mathbb{Z}_2 \curvearrowright \mathrm{AdjBiCat}$

$\mathcal{B} \longmapsto \mathcal{B}^{1\,op}$

$\mathcal{B} \xrightarrow{(-)^{RR}} \mathcal{B}$

$f \longmapsto f^{RR}$

Def O_2-t bicategory

- F.P. for this O_2-action
- trivializations on objects and 1-Mor

Thm

Thm

Def. An ... is an O_2-...

Bor...

Thm

2D f. ...