Title: Analyticity properties of 2d Ising Field Theories

Speakers: Hao-Lan Xu

Series: Quantum Fields and Strings

Date: October 03, 2023 - 2:00 PM

URL: https://pirsa.org/23100073

Abstract: In this talk, I will discuss the analyticity properties of 2d Ising field theories (IFTs). I will start with a short introduction to 2d Ising field theory, which is the continuous limit of the 2d Ising model on square lattice. Then the different spectrum scenarios for high-T and low-T domains will be introduced. Generally speaking, an IFT which sits not at the critical temperature and has a non-vanishing external field is neither solvable nor integrable. However, it's possible to look into the analytical properties of various quantities in the theory space, then further non-perturbative information can be extracted. I will focus on the analyticity properties for mass of the first excitation, and discuss its critical behaviours and dispersion relations in both ordered and disordered phase. Finally, if time allowed, I will switch to the analyticity properties of the analytical structure of S-matrices, and show various related interesting phenomenons together with unsolved problems

References:
[1], Ising field theory in a magnetic field: Analytic properties of the free energy, P. Fonseca and A. Zamolodchikov, hep-th/0112167 [hep-th].
[2], Ising Spectroscopy II: Particles and poles at T > Tc, A. Zamolodchikov, 1310.4821 [hep-th].
[3], 2D Ising Field Theory in a magnetic field: the Yang-Lee singularity, H. Xu and A. Zamolodchikov, 2203.11262 [hep-th].
[4], On the S-matrix of Ising field theory in two dimensions, B. Gabai and X. Yin, 1905.00710 [hep-th]
[5], Ising field theory in a magnetic field: phi^3 coupling at T > Tc, H. Xu and A. Zamolodchikov, 2304.07886 [hep-th]
[6], Corner Transfer Matrix Approach to the Yang-Lee Singularity in the 2D Ising Model in a magnetic field, V.V.Mangazeev, B.Hagan and V.V.Bazhanov, 2308.15113 [hep-th]
[7], Ising Field Theory in a Magnetic Field: Extended analyticity properties of M1, H. Xu, in preparation.

Zoom link: https://pitp.zoom.us/j/97062411964?pwd=TWFHU0I5UGw3eXZjZzRHUEFnbjlydz09

Analyticity properties of 2d Ising Field Theories

Hao-Lan Xu

C.N. Yang Institute of Theoretical Physics, SUNY Stony Brook

October 3, 2023

Outline

- Basics on 2d Ising field theories (IFTs):
  Definition, state functions, symmetries and scenarios.
- Analyticity properties of scaling functions:
  Dispersion relations in high-T and low-T.
- Extended analyticity conjectures:
  Dispersion relations connecting both phases.
- Polology of Ising field theory: (optional)
  Evolution and analyticity properties of scattering.
- Summary and outlooks.
- Appendices.

Basics

- Ising model: classical spins (up and down) sitting on lattices (square, triangle, etc.) with interaction between nearest sites ($J > 0$):

$$\mathcal{H} = -J \sum_{\langle i,j \rangle} \sigma_i \sigma_j + H \sum_i \sigma_i \,, \quad \mathcal{Z} = \sum_{\{\sigma_i\}} e^{-\beta \mathcal{H}} \,.$$

- Why Ising model? It describes important universality classes in nature: Vapor/liquid phase transition, ferromagnetic transition near Curie point, etc.
- Current understanding of Ising models:
  $d = 4 - \epsilon$: famous picture of Wilsionian RG with φ^4.
  $d = 3$: numerical solutions near criticality (perturbative RG, Monte-Carlo, numerical conformal bootstrap, etc).
- $d = 2$: Onsager gave the solution at $H = 0$ (Onsager 1944)[1].
  Yang and Lee established the theorem of circle and zeros (Yang & Lee 1952)[2].
  However, for generic J and H: no solution available in closed form.
  Also, when not at criticality: conformal symmetry or integrability broken.
- "How much can we understand 2d Ising" is still an interesting question.

[1] Lars Onsager. "Crystal Statistics. I. A Two-Dimensional Model with an Order-Disorder Transition". In: *Phys. Rev.* 65 (3-4 1944), pp. 117–149. DOI: 10.1103/PhysRev.65.117. URL: https://link.aps.org/doi/10.1103/PhysRev.65.117.

[2] Chen-Ning Yang and Tsung-Dao Lee. "Statistical theory of equations of state and phase transitions. I. Theory of condensation". In: *Physical Review* 87.3 (1952), p. 404.

The Ising Field Theories

- In the continuous limit, Ising models at critical point $\Longrightarrow$ Ising conformal field theories (ICFTs). Non-critical Ising models $\Longrightarrow$ Ising field theories (IFTs):

$$\mathcal{A}_{\text{IFT}} = \mathcal{A}_{\text{CFT}}^{\text{Ising}} + \tau \int \varepsilon(x)\, d^2x + h \int \sigma(x)\, d^2x \,,$$

  as relevant deformations away from Ising CFT at UV.

- Continuous limit: lattice spacing $\to 0$, while spin-spin correlator normalized.

- $\varepsilon(x) \sim \sigma_i \sigma_{i+1}$: energy operator, while temperature perturbation $\tau = \frac{m}{2\pi} \propto 1 - \frac{T}{T_c}$;

- $\sigma(x) \sim \sigma_i$: spin operator, while external magnetic field $h \propto H$.

- From abstract CFT point of view, Ising CFT is defined as a conformal field theory with $\mathbb{Z}_2$ symmetry, which has only 2 local relevant scalar operators ($\Delta < d$).

- We would call the $\mathbb{Z}_2$ even operator $\varepsilon(x)$, and the $\mathbb{Z}_2$ odd operator $\sigma(x)$. Their conformal dimensions Δ_σ and Δ_ε would determine the critical scaling laws.

2d Ising CFT as minimal model $(3,4)$

- In 2d, exist a set of solvable diagonal CFTs: minimal models, which are labelled by co-prime integers (p,q). The minimal model $(3,4)$ describes 2d Ising CFT.
- In ICFT, $c_{\text{Ising}} = \frac{1}{2}$. The conformal dimensions are $(h_\varepsilon, \overline{h}_\varepsilon) = (\frac{1}{2}, \frac{1}{2})$ and $(h_\sigma, \overline{h}_\sigma) = (\frac{1}{16}, \frac{1}{16})$, with $[m] = 1$ and $[h] = \frac{15}{8}$.
- Dimensionless combinations: scaling parameters, which label the RG flows.

$$\xi = \frac{h}{|m|^{15/8}}, \quad \text{and} \quad \eta = \frac{m}{h^{8/15}},$$

they are related by $\xi = \eta^{-\frac{15}{8}}$ or $\eta = \xi^{-\frac{8}{15}}$ (up to signs), and both live in $\mathbb{C}$.

- With vanishing h, action of IFT is equivalent to the one of 2d Majorana fermions:

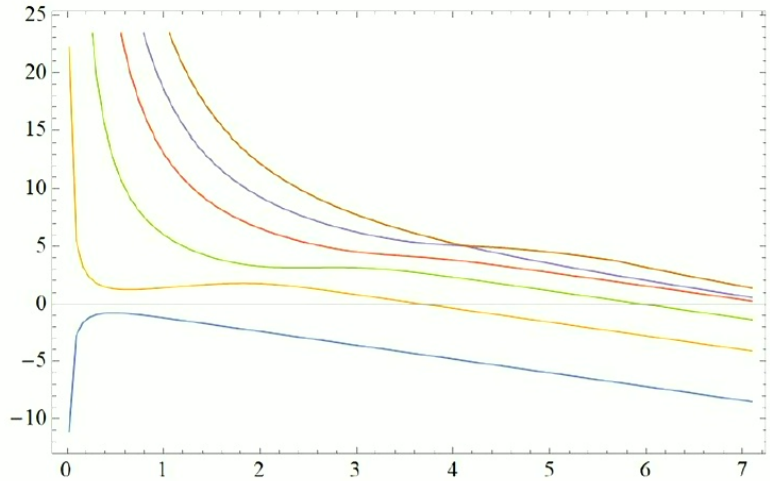
$$\mathcal{A}_{\text{FF}} = \frac{1}{2\pi}\int \left(\psi\bar{\partial}\psi + \bar{\psi}\partial\bar{\psi} + im\bar{\psi}\psi\right) d^2x = \mathcal{A}_{\text{CFT}}^{\text{Ising}} + \frac{m}{2\pi}\int \varepsilon(x)\, d^2x\,.$$

$|m|$ now is the fermion mass. Their Hilbert spaces are different by projection.

- Scenario of IFT and scenario of FF are different, in IFT $\varepsilon(x)$ and $\sigma(x)$ are local operators, while $\psi(x)$ and $\bar{\psi}(x)$ are mutually semi-local with respect to $\sigma(x)$.

State functions and scaling functions

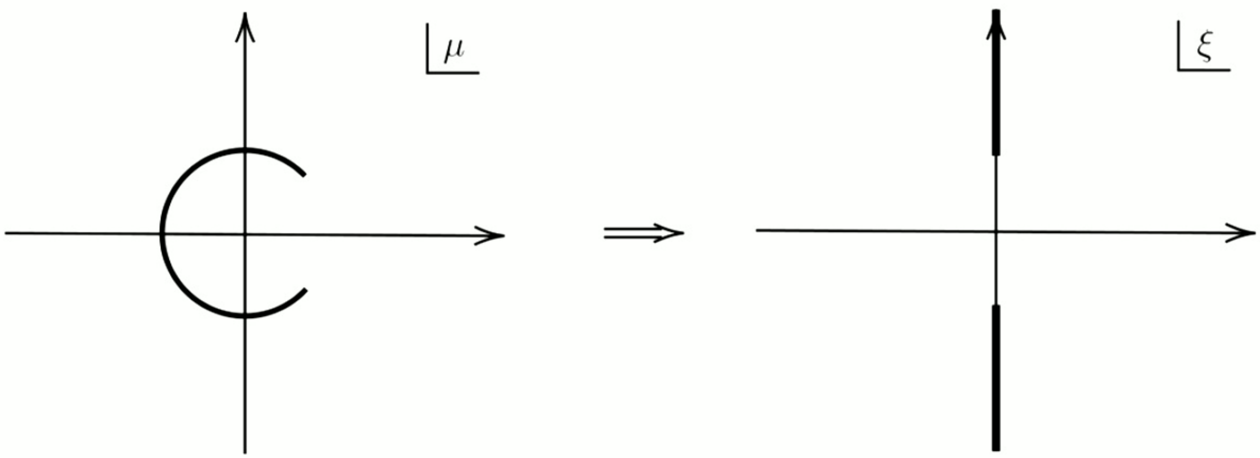
- State functions of 2d IFTs (of continuous limit): depend on (m, h).
- Examples: free energy density $F(m, h)$, the first mass gap $M_1(m, h)$ (inverse correlation length), poles of $S(\theta)$ and their residues, etc.
- Many of them are available using numerical methods (truncated method, etc.). For example, F and M_n are from slope and gaps of finite size spectrum $E_n(R)$.



- Scaling functions: dimensionless functions depend on ξ or η, describe the flow.
- i.e.: $\mathcal{G}(\xi) = \frac{1}{|m|^2} F(m, h) - \frac{\eta^2}{8\pi} \log \eta^2$ and $\hat{M}_1(\xi) = \frac{1}{|m|} M_1(m, h)$.

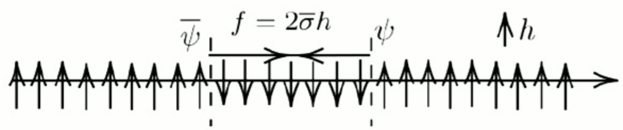
Phases and Scenarios: $T > T_c$

- At $T > T_c$: disordered phase and unbroken $\mathbb{Z}_2$ symmetry: $h \leftrightarrow -h$. The scaling functions are even in ξ and would depend on $\xi^2 \sim h^2$.
- Yang-Lee theorem: the lattice partition function $\mathcal{Z}$ has zeros distributed on the unit circle of fugacity $\mu = e^{-2\beta H}$ plane. The circle becomes an arc in high-T phase.
- In the continuous limit, the zeros of $\mathcal{Z}$ condensed into a branch cut of $F = -\log \mathcal{Z}$, known as Yang-Lee (YL) branch cut. The edges become YL edge singularities.
- Properties of YL branch cut and singularity will be discussed later.

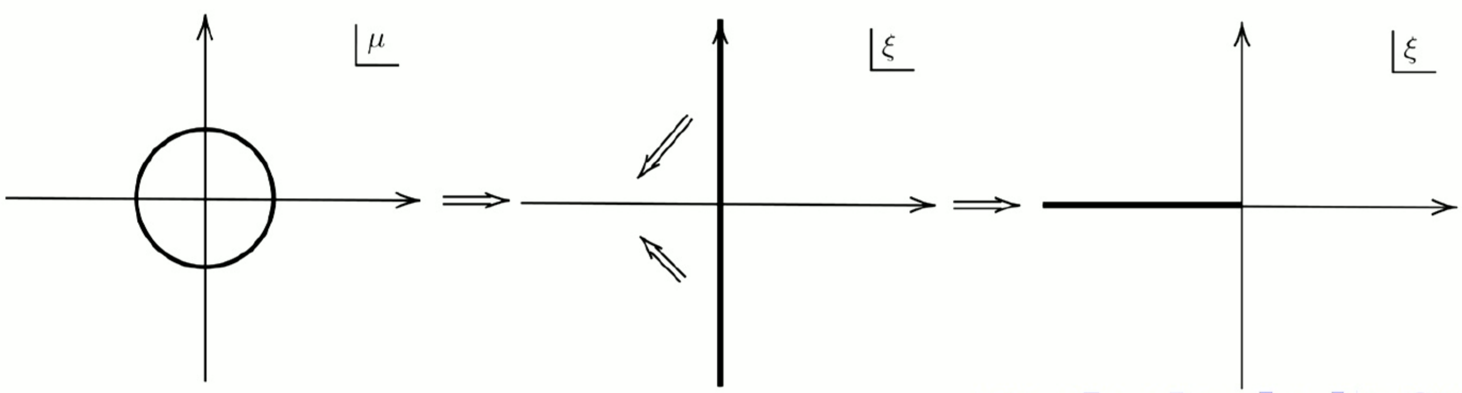


Phases and Scenarios: $T < T_c$

- At $T < T_c$: ordered phase with broken $\mathbb{Z}_2$ symmetry.
  VEV of spin density: $\langle\sigma\rangle = \pm\bar{\sigma} = \pm\bar{s}|m|^{1/8}$, with $\bar{s} = 1.35783834\ldots$.
- Double degenerate vacuum at $h = 0$, degeneracy lifted at $h \neq 0$.
  At small h: stable vacuum (spins aline along h) and metastable vacuum.
- In $1+1$ d as a field theory: meson spectrum (McCoy-Wu scenario), fermions as domain walls and h provides binding force. String tension $f = 2\bar{\sigma}h$.

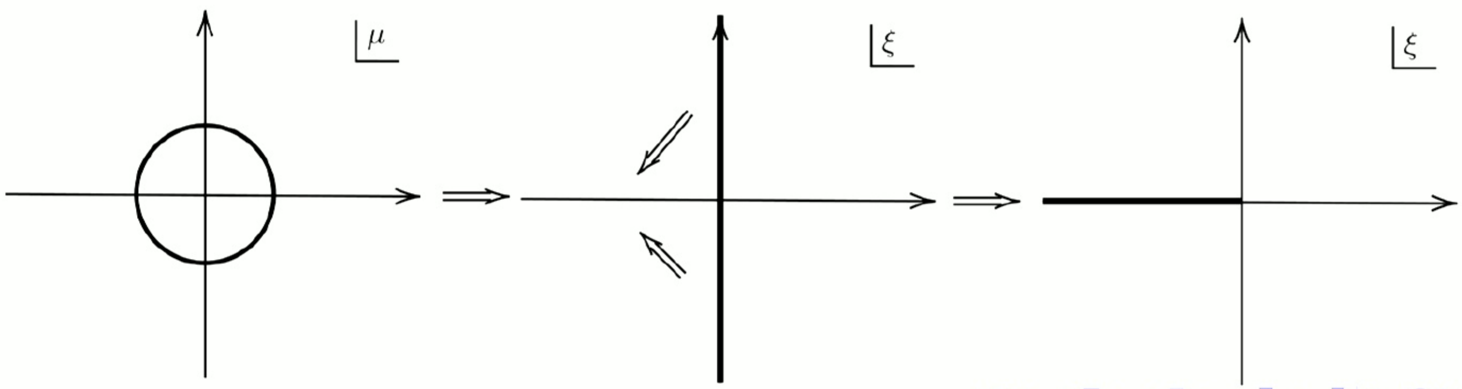


- Along negative ξ-axis: Fisher-Langer's branch cut of scaling functions.



Phases and Scenarios: $T < T_c$ and Fisher-Langer's branch cut

- At $T < T_c$: Yang-Lee theorem gives the full circle.
  The zeros of lattice partition function sit on the unit circle in the fugacity plane.
- $|\mu| < 1$ domain and $|\mu| > 1$ domain correspond to different choice of VEV.
  i.e.: $|\mu| > 1$ is the one with $\langle\sigma\rangle = +\bar{\sigma}$ and $|\mu| < 1$ is with $\langle\sigma\rangle = -\bar{\sigma}$.
- In the continuous limit, zeros of $\mathcal{Z}$ would condense into a natural bound of analyticity for thermodynamic functions. On the complex ξ-plane, the "wall" is the imaginary axis separating $\Re e\,\xi > 0$ and $\Re e\,\xi < 0$, and in each domain different functions $M_1\,, F\,, \cdots$ can be defined. They are sitting in different phase.
- However, for functions defined from $\Re e\,\xi > 0$ it's possible to do analytically continuation till the full complex ξ-plane, and would leave a discontinuity along the real negative axis: the Fisher-Langer's branch cut.



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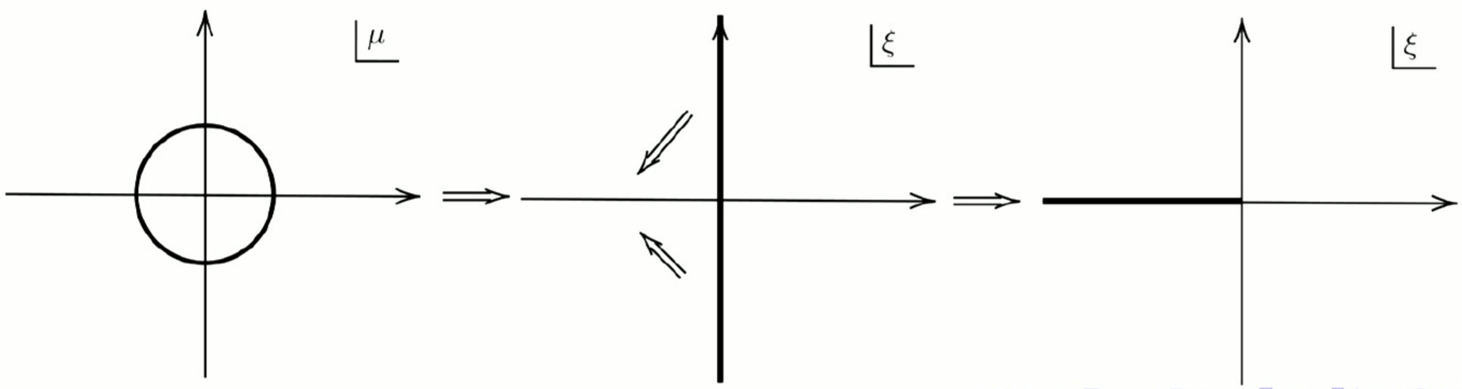


Special points

When ξ or η takes some special values, one finds the IFTs become integrable or solvable (on the full trajectory or in some limit). The special points are:

- $h = 0$, correspond to the Onsager's solution of free fermions, $|m|$ would be the mass of fermions, with free energy $F(m, 0) = \frac{m^2}{8\pi} \log m^2$ (in unit of J).
- $m = 0$ with nonvanishing h, which becomes the integrable E_8 field theory[3].
- When $m < 0$ and take h to be a pure imaginary. The Yang-Lee critical point is located at $\xi = \pm i\xi_0$, with $\xi_0^2 \approx 0.035846(4)$. Near which: infrared integrable".

Now focus on the Yang-Lee critical point ("edge" of condensing zeros):

- IR fixed point: non-unitary minimal model $(2, 5)$, with $c_{\text{YL}} = -\frac{22}{5}$[4].
  Its relevant deformation:

$$\mathcal{A}_{\text{SYLM}} = \mathcal{A}_{\text{CFT}}^{\text{YL}} + \lambda \int \phi(x)\, d^2x\,,$$

  is called the scaling Yang-Lee model (SYLM), which is massive and integrable[5].
  $\phi(x)$ with scaling dimension $(-\frac{1}{5}, -\frac{1}{5})$ is the only primary of Yang-Lee CFT.

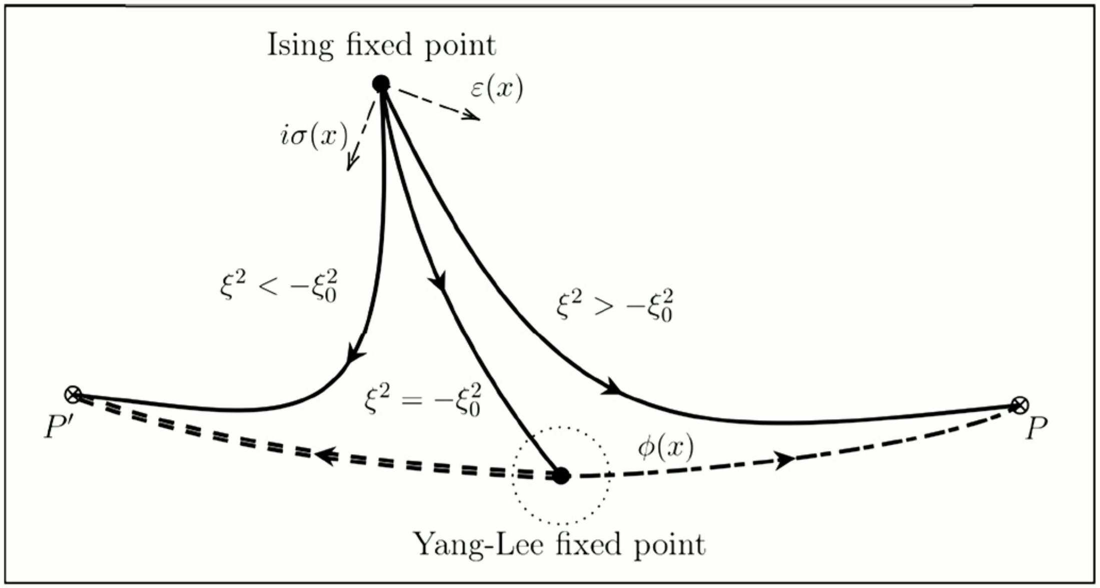
[3] Aleksandr B Zamolodchikov. "Integrals of motion and S-matrix of the (scaled) T= T c Ising model with magnetic field". In: *International Journal of Modern Physics A* 4.16 (1989), pp. 4235–4248.

[4] John L Cardy. "Conformal invariance and the Yang-Lee edge singularity in two dimensions". In: *Physical review letters* 54.13 (1985), p. 1354.

[5] John L. Cardy and G. Mussardo. "S Matrix of the Yang-Lee Edge Singularity in Two-Dimensions". In: *Phys. Lett.* B225 (1989), pp. 275–278. DOI: 10.1016/0370-2693(89)90818-6.

Big picture and the goal of the Project: Ising Spectroscopy

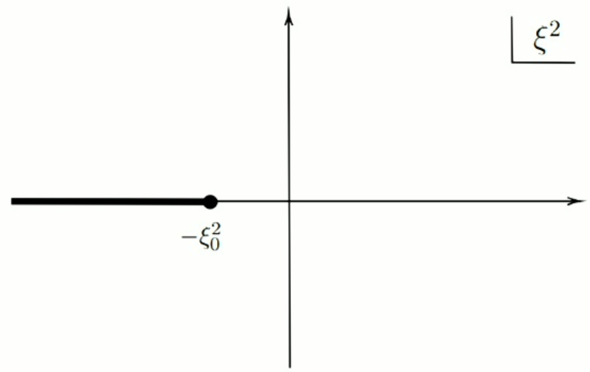
- Goal: Try to understand the space of Ising QFTs non-perturbatively.
- Near criticality and integrable points: able to use conformal perturbation theory and form-factor perturbation theory, and some exact results are available.
- Away from special points: some constraints from symmetries and scenarios.



- Analyticity structures of scaling functions would provide important information.

Analyticity properties at high-T.

- High-T standard analyticity conjecture: some universal scaling functions are analytical functions on the complex ξ^2-plane, except on the YL branch cut. i.e.: Free energy $\mathcal{G}(\xi)$[6], first mass $\hat{M}_1(\xi)$[7], effective φ^3 coupling[8].



- High-T dispersion relation of $\hat{M}_1(\xi^2) = M_1(m,h)/|m|$:

$$\hat{M}_1(\xi^2) = 1 + \xi^2 \int_{\xi_0^2}^{\infty} \frac{dx}{\pi} \frac{\Im m\, \hat{M}_1(-x+i0)}{x(x+\xi^2)} .$$

[6] P Fonseca and A Zamolodchikov. "Ising field theory in a magnetic field: analytic properties of the free energy". In: *Journal of statistical physics* 110.3-6 (2003), pp. 527–590.

[7] Hao-Lan Xu and Alexander Zamolodchikov. "2D Ising Field Theory in a magnetic field: the Yang-Lee singularity". In: *JHEP* 08 (2022), p. 057. DOI: 10.1007/JHEP08(2022)057. arXiv:2203.11262 [hep-th].

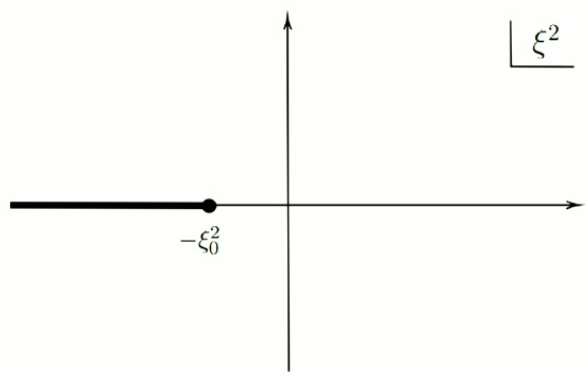
[8] Hao-Lan Xu and Alexander Zamolodchikov. "Ising Field Theory in a magnetic field: φ^3 coupling at $T > T_c$". In: (Apr. 2023). arXiv:2304.07886 [hep-th].

Properties of the Yang-Lee branch cut.

- At pure imaginary ξ: not unitary, with reality properties due to "pseudo-hermiticity".

$$\exists\, S^2 = 1\,, \quad \text{so that} \quad H^\dagger = SHS\,.$$

- Thus spectrum bounded, with either unique ground state with real F or two vacua $|0_\pm\rangle$ with complex conjugate $F_\pm$. Similar properties work for M_1.
- The Yang-Lee branch cut represents line of complex first order phase transition.
- Mean field theory description of Yang-Lee branch cut: φ^3 theory with complex coupling[9][10]. Then the singularity at $\xi^2 = -\xi_0^2$ represents continuous phase transition, and the scaling behaviours should be controlled by the Yang-Lee CFT.



[9] John L Cardy. "Conformal invariance and the Yang-Lee edge singularity in two dimensions". In: *Physical review letters* 54.13 (1985), p. 1354.

[10] John L. Cardy and G. Mussardo. "S Matrix of the Yang-Lee Edge Singularity in Two-Dimensions". In: *Phys. Lett.* B225 (1989), pp. 275–278. DOI: 10.1016/0370-2693(89)90818-6.

Properties of the Yang-Lee singularity.

- When $\xi^2 = -\xi_0^2$: YLCFT as IR fixed point $\Longrightarrow$ controlling critical behaviours. For ξ^2 near $-\xi_0^2$: using effective action:

$$\mathcal{A}_{\text{eff}} = \mathcal{A}^*_{(2,5)} + \lambda(\xi^2) \int \phi(x) d^2x + \sum g_i(\xi^2) \int \mathcal{O}_i(x) d^2x \,,$$

  where irrelevant scalars $\mathcal{O}_i \in \mathcal{V}_{(2,5)}$, the Hilbert space of YLCFT.

- Effective couplings are regular near $-\xi_0^2$, say $g_i = g_i^{(0)} + (\xi^2 + \xi_0^2) g_i^{(1)} + \cdots$.
- Constant ξ_0^2 can be numerically measured with singular behaviours.
- Using dimensional analysis: singular expansions of scaling functions near $-\xi_0^2$. Since $[\lambda] = \frac{12}{5}$, the leading critical behaviours are:

$$\hat{M}_1(\xi^2) = (\xi^2 + \xi_0^2)^{\frac{5}{12}} \left(b_0 + \cdots\right), \quad \mathcal{G}(\xi^2) = (\xi^2 + \xi_0^2)^{\frac{5}{6}} \left(B_0 + \cdots\right),$$

  following $\lambda(\xi^2) = (\xi^2 + \xi_0^2)\lambda_1 + \cdots$.

- Beyond leading ones: irrelevant operator $\mathcal{O}_i$ with mass dimension Δ_i

$$\delta_i \hat{M}_1(\xi^2)/(\xi^2 + \xi_0^2)^{\frac{5}{12}} \propto (\xi^2 + \xi_0^2)^{\frac{5}{12}(\Delta_i - 2)} \,.$$

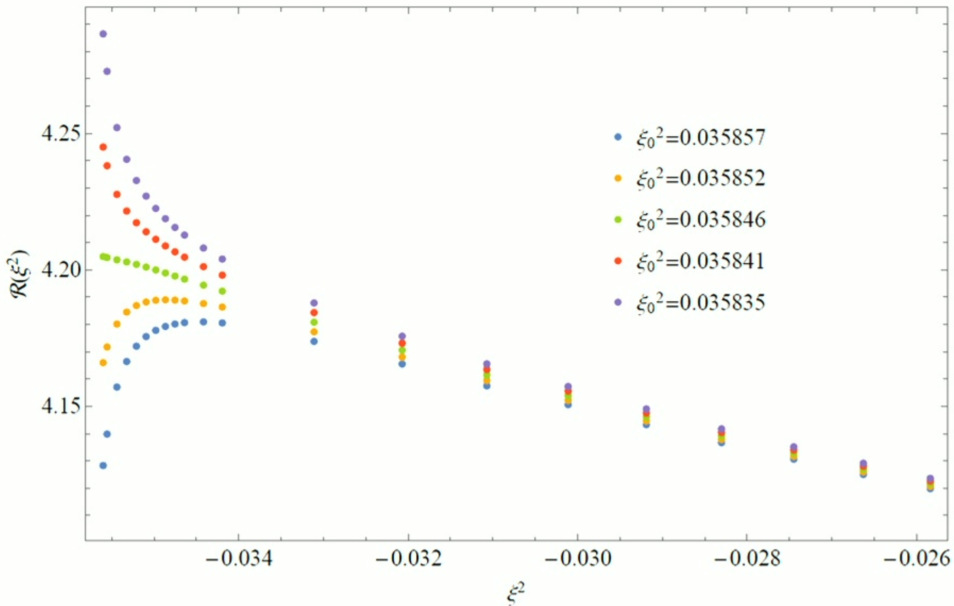
  from $g_i M_1^{\Delta_i - 2}$ is dimensionless, while $M_1 \sim (\xi^2 + \xi_0^2)^{\frac{5}{12}}$.

Finding the location of Yang-Lee point.

- The regular behaviour of $\mathcal{R}_1 \Longrightarrow$ location of Yang-Lee critical point, as:

$$\mathcal{R}_1(\xi^2) = \hat{M}_1(\xi^2)/(\xi^2+\xi_0^2)^{\frac{5}{12}} = b_0 + b_1(\xi^2+\xi_0^2) + c_0(\xi^2+\xi_0^2)^{\frac{5}{6}} + \cdots$$

- Numerical result: $\xi_0^2 = 0.035846(4)$[11,12].



[11] Hao-Lan Xu and Alexander Zamolodchikov. "2D Ising Field Theory in a magnetic field: the Yang-Lee singularity". In: *JHEP* 08 (2022), p. 057. DOI: 10.1007/JHEP08(2022)057. arXiv:2203.11262 [hep-th].

[12] Vladimir V. Mangazeev, Bryte Hagan, and Vladimir V. Bazhanov. "Corner Transfer Matrix Approach to the Yang-Lee Singularity in the 2D Ising Model in a magnetic field". In: (Aug. 2023). arXiv:2308.15113 [hep-th].

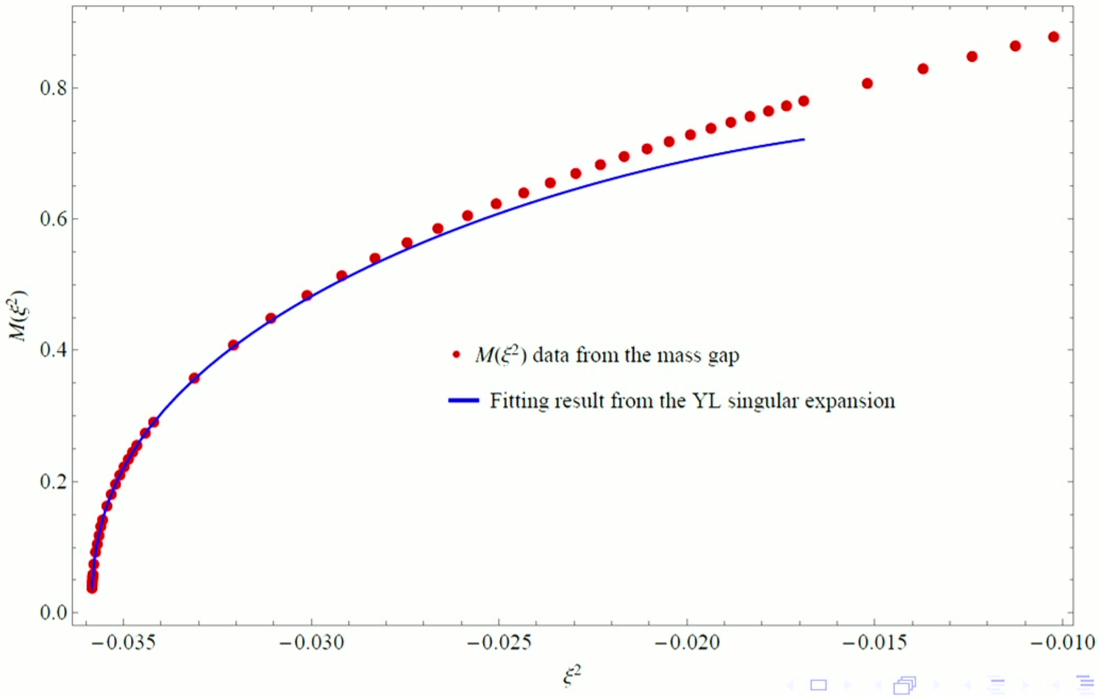
Singular behaviour of M_1 near Yang-Lee point.

- Singular expansion of $\hat{M}_1(\xi^2)$: (c_0 from $T\bar{T}$ with dimension 4.)

$$\hat{M}_1(\xi^2) = (\xi^2 + \xi_0^2)^{\frac{5}{12}} \left(b_0 + b_1(\xi^2 + \xi_0^2) + c_0(\xi^2 + \xi_0^2)^{\frac{5}{6}} + \cdots\right),$$

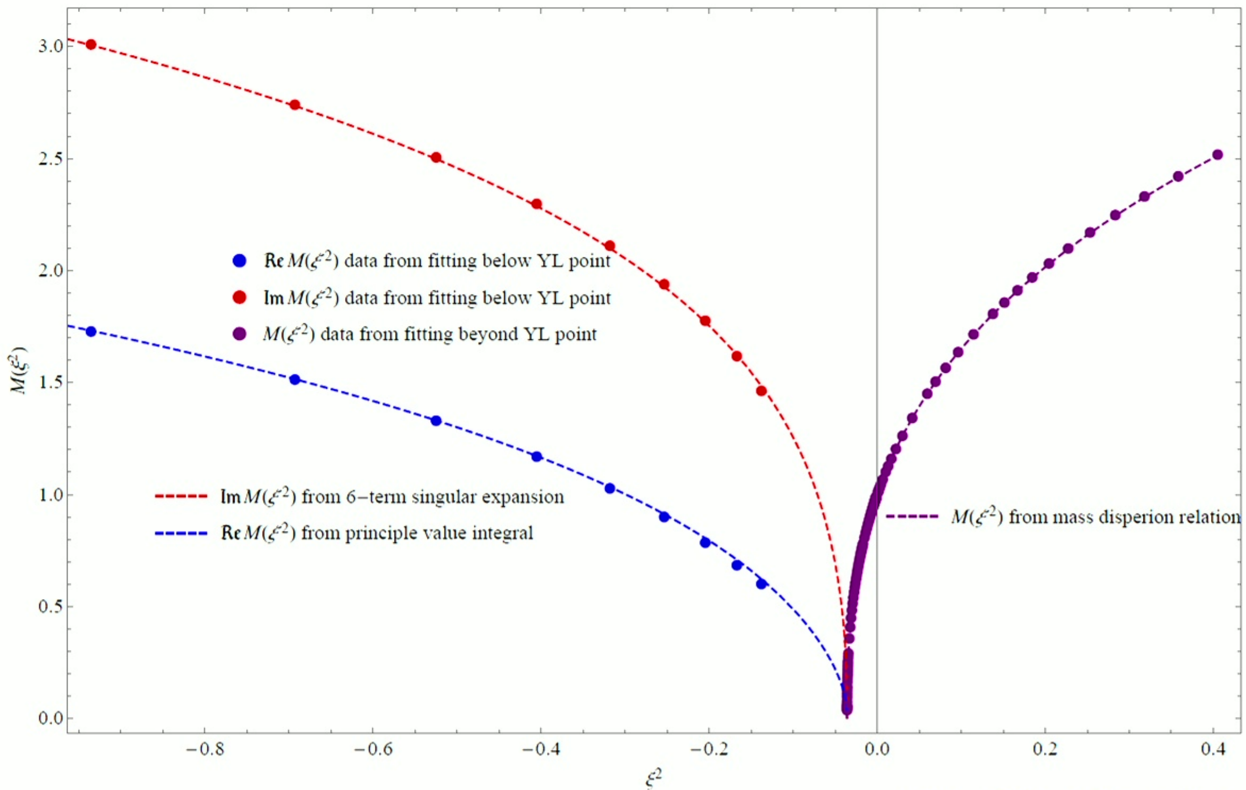
The amplitudes are measurable and give effective couplings.

- Measurements: $b_0 = 4.228(5)$, $b_1 = 21.9(9)$ and $c_0 = -14.4(6)$.



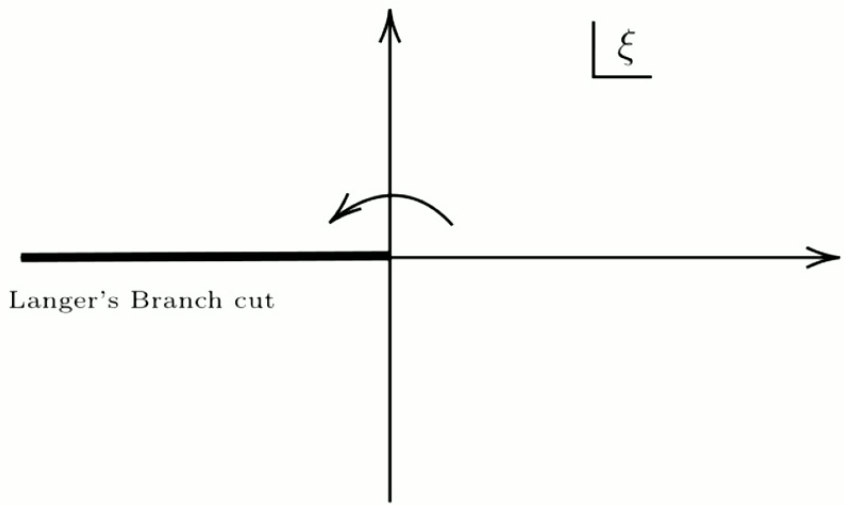
Numerical check of high-T dispersion relation of M_1.

- Near $\xi^2 = -\infty$: regular expansion in m or η since $\varepsilon(x)$ is relevant.
- Able to build discontinuities and check the dispersion relation in high-T.



Analyticity properties at low-T (I).

- By choosing $\langle\sigma\rangle = +\bar{\sigma}$ as vacuum: scaling functions are single valued functions for real positive ξ.
- Analyticity conjecture for low-T: both $\hat{M}_1(\xi)$[13] and $\mathcal{G}(\xi)$[14] can be analytically continued to the full complex ξ-plane from positive real axis, leaving discontinuities along the Fisher-Langer's branch cut: $-\infty < \xi < 0$.



[13] Hao-Lan Xu. "Ising Field Theory in a Magnetic Field: Extended analyticity properties of M_1". In: *In preparation* (2023).

[14] P Fonseca and A Zamolodchikov. "Ising field theory in a magnetic field: analytic properties of the free energy". In: *Journal of statistical physics* 110.3-6 (2003), pp. 527–590.

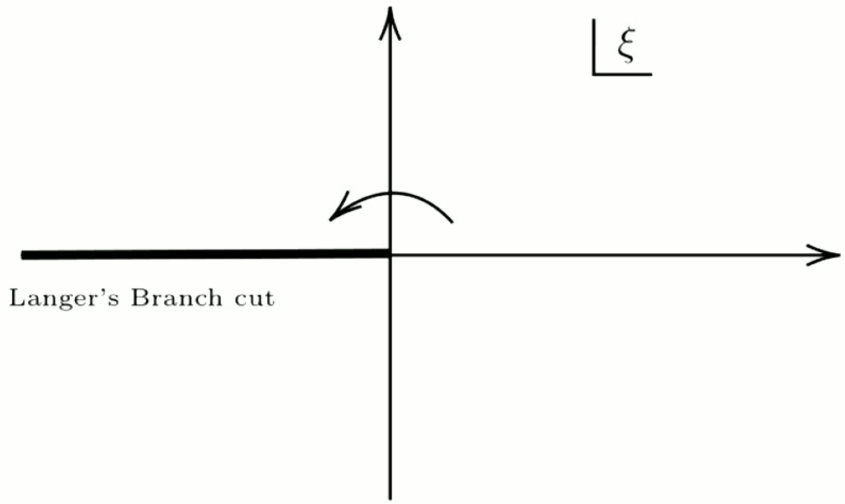
Analyticity properties at low-T (II).

- Low-T dispersion relation of $\hat{M}_1(\xi)$: (and similarly for $\mathcal{G}(\xi)$)

$$\hat{M}_1(\xi) = 2 + \xi \int_0^{+\infty} \frac{dt}{\pi} \frac{\Im m\, \hat{M}_1(-\xi + i0)}{t(t+\xi)} \,,$$

as an integral on FL branch cut $-\infty < \xi < 0$, with no other singularity.

- However, at the condensation point $\xi \to 0^-$ there exist non-analytic contributions to the discontinuities, leads to an essential singularity here.



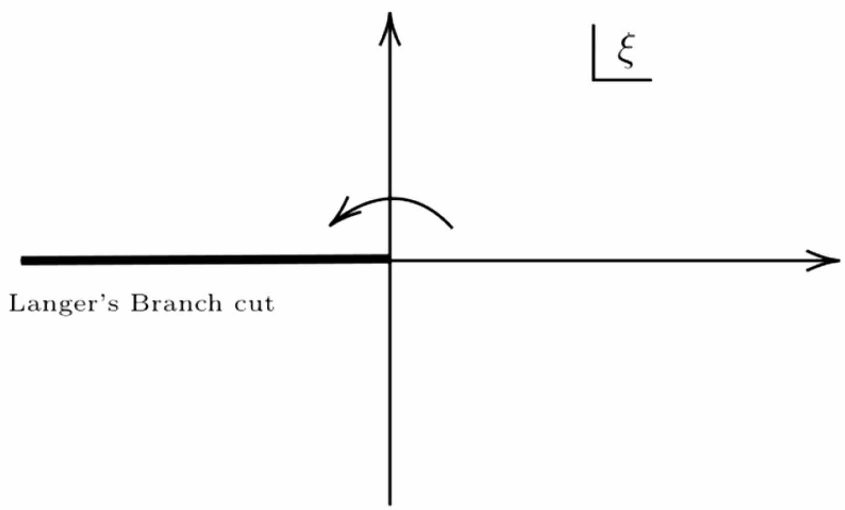
Fisher-Langer's branch cut and essential singularities (I).

- Scaling functions are expandable at $\xi = 0$ and $\xi = \infty$, i.e.:

$$\hat{M}_1(\xi) = 2 + a_1 \xi^{2/3} + a_2 \xi^{4/3} + \cdots, \quad \text{at} \quad \xi \to 0,$$

$$\hat{M}_1(\xi) = m_1^{(0)}/\eta + m_1^{(1)} + m_1^{(2)}\eta + \cdots = m_1^{(0)} \xi^{8/15} + \cdots, \quad \text{at} \quad \xi \to \infty.$$

- Small positive $\xi \Longrightarrow$ choosing the stable vacuum.
  Analytically continuing $\xi \to e^{\pm \pi i} \xi \Longrightarrow$ exchange the roles of both vacuums.
- Thus, $\xi = -\epsilon \pm i0$ would correspond to IFT sitting in the metastable vacuum, and the tunneling effects would contribute to the scaling functions.

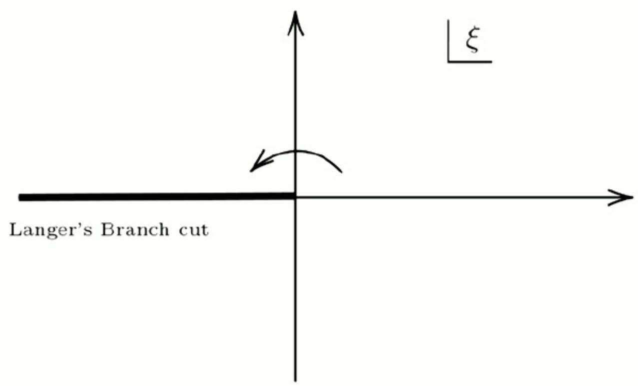


Fisher-Langer's branch cut and essential singularities (II).

- At finite T, in the metastable vacuum thermal fluctuation would generate "bubbles" of stable vacuum. Big bubbles would condensate and cause the vacuum decay.
- Computations of bubbles in metastable states would give at $\xi \to 0^-$, i.e.[15,16,17]:

$$\Im m\, \mathcal{G} \to \frac{\lambda}{4\pi} e^{-\frac{\pi}{\lambda}}, \quad \Im m\, \hat{M}_1 \to (\text{Analytic terms}) + \frac{1}{\pi} e^{-\frac{\pi}{\lambda}} + \cdots .$$

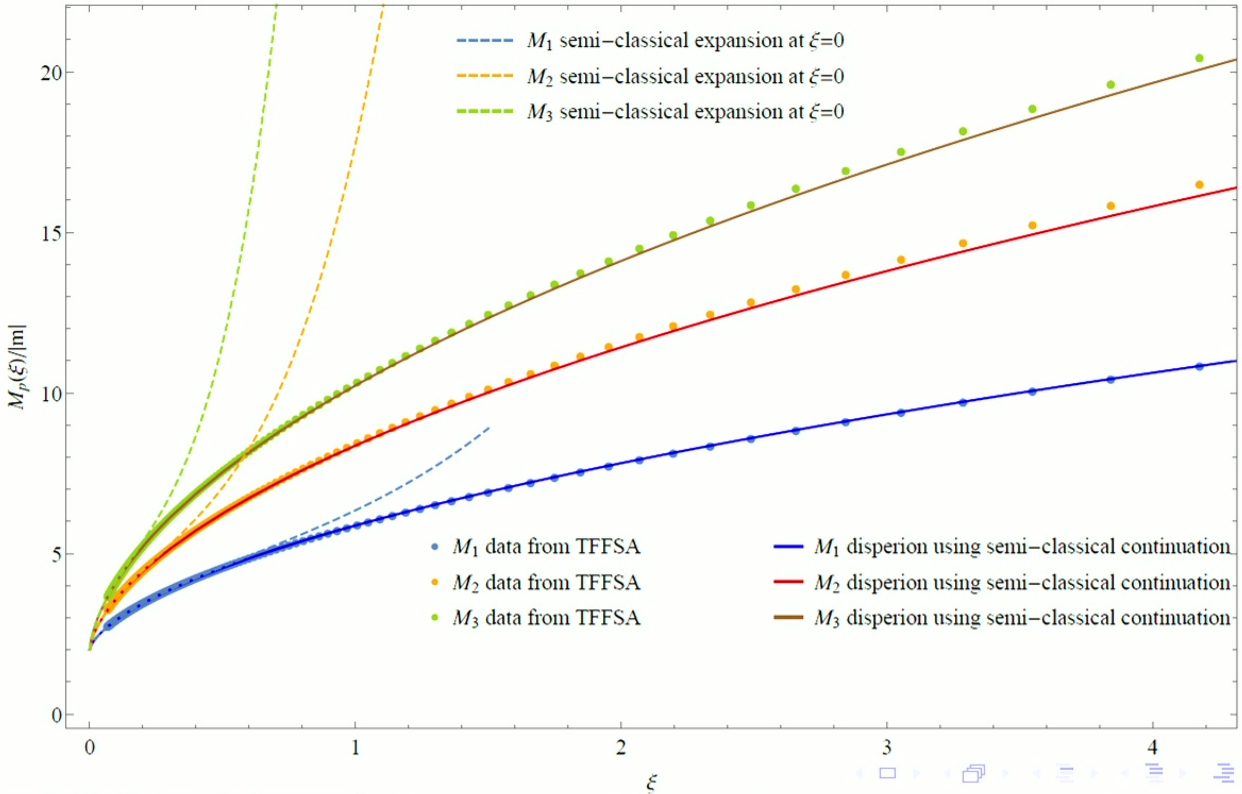
  where $\lambda = -2\bar{s}\xi > 0$.
- The metastable $F(m,h)$ is related to the vacuum decay rate: $\Im m\, F_{\text{meta}} \sim \Gamma$.



[15] James S Langer. "Theory of the condensation point". In: *Annals of Physics* 281.1-2 (2000), pp. 941–990.
[16] M. B. Voloshin. "DECAY OF FALSE VACUUM IN (1+1)-DIMENSIONS". In: *Yad. Fiz.* 42 (1985), pp. 1017–1026.
[17] Hao-Lan Xu. "Ising Field Theory in a Magnetic Field: Extended analyticity properties of M_1". In: *In preparation* (2023).

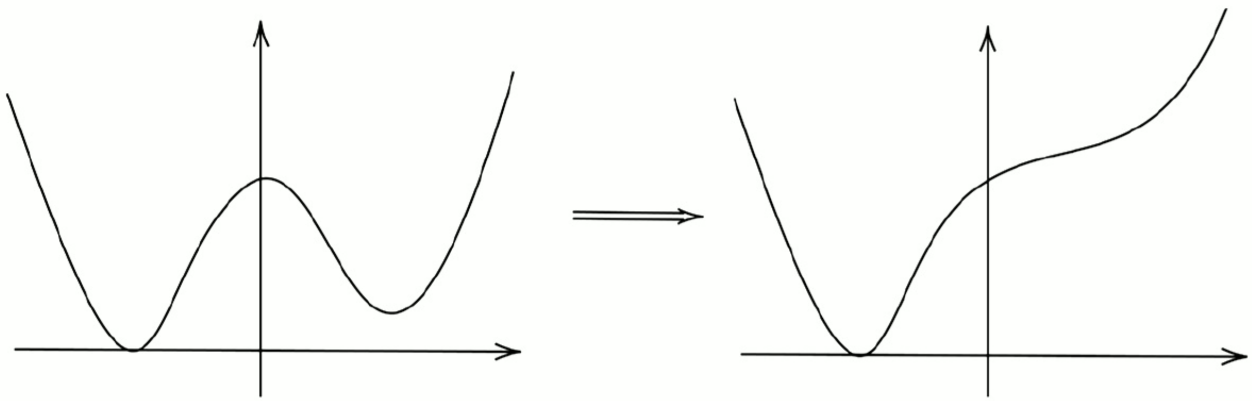
Low-T dispersion relations with Langer's branch cut.

- By approximating discontinuities, the low-T dispersion relation can be verified.
- For $\hat{M}_1(\xi)$, computation shows the non-analytic term is negligible. Also for $\hat{M}_2(\xi)$ and $\hat{M}_3(\xi)$ similar dispersion relations exist, and can be checked.



Spinodal point on Fisher-Langer's branch cut?

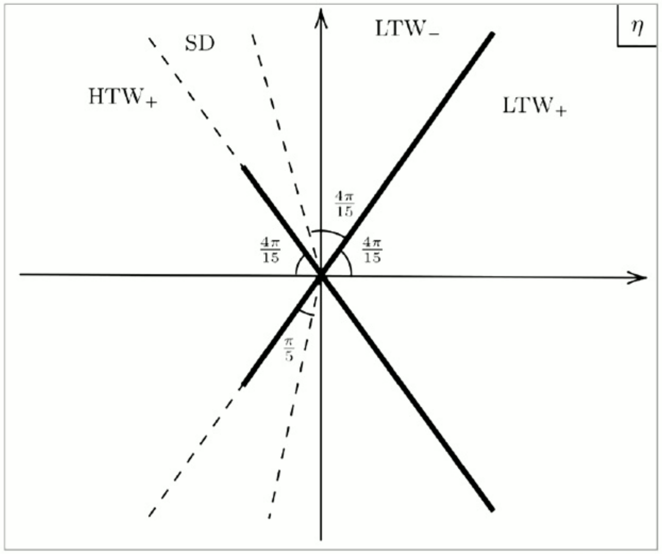
- Behaviours at $\xi \to 0^-$ is given by the decay of metastable vacuum, which is due to thermal fluctuations and tunneling effects.
- However, at $\xi \to -\infty$ there would be only one true vacuum, and the phase transition should happened classically.
- From mean field theory point of view, at negative ξ there exist a point where the metastable vacuum becomes classically unstable, and the picture changes.
- The point is known as spinodal point. However, no other singularity found on the discontinuities[18]. Where is and what happened to the spinodal point?



[18] V Privman and LS Schulman. "Analytic continuation at first-order phase transitions". In: *Journal of Statistical Physics* 29 (1982), pp. 205–229.

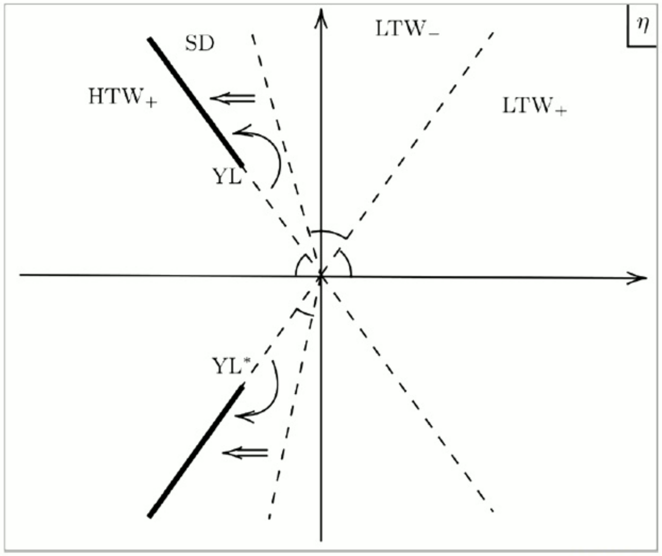
Extended analyticity conjectures: connecting all-T

- The analyticity conjectures on $\xi = h/|m|^{\frac{15}{8}}$ plane treat $T < T_c$ and $T > T_c$ differently. How about on the complex $\eta = m/h^{\frac{8}{15}} = \xi^{-8/15}$ plane?
- $-\frac{8\pi}{15} \leq \text{Arg}\,\eta \leq +\frac{8\pi}{15}$: Low-T wedge (LTW), represents the full ξ-plane with $m > 0$.
- $-\frac{4\pi}{15} \leq \text{Arg}\,(-\eta) \leq +\frac{4\pi}{15}$: High-T wedge$_+$ (HTW$_+$), represents $\Re e\,\xi > 0$ for $m < 0$.
- In between: shadow domain (SD), which is under the FL branch cut.
- YL branch cut: $\eta = -ye^{\pm\frac{4\pi i}{15}}$ with $y \leq Y_0$, $Y_0 \approx 2.4293$.



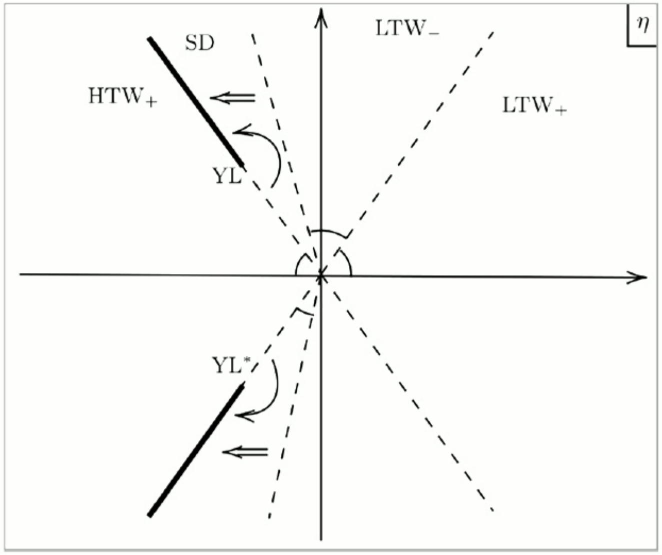
Extended analyticity conjectures: scaling functions on the η-plane.

- Scaling functions on the η-plane: to avoid pole at $\eta = 0$, use instead: For real η: $\mathcal{M}_1(\eta) = M_1/|h|^{\frac{8}{15}}$ and $\tilde{\Phi}(\eta) = F/|h|^{\frac{16}{15}} - \frac{\eta^2}{8\pi}\log|h|^{\frac{16}{15}}$.
- For complex η, possible to rotate YL branch cut by redefinitions in SD.
- Near YL point the singular expansion also continued, as $y \to -Y_0 + (y - Y_0)e^{\pi i}$.
- The discontinuities on the rotated branch cuts are now controlled by the behaviours near YL point and FF point.
- Question: what are the analytical structures within the SD?



Extended analyticity conjecture as minimal conjecture.

- Extended analyticity conjecture: the scaling function is analytical anywhere on the complex η-plane, except on the rotated YL branch cuts[19][20].
- No extra singularities within SD, YL point is the nearest one under FL branch cut.
- The extended analyticity conjecture is the most elegant conjecture. Meanwhile, if other singularities exist, then one must consider their physical interpretations.



[19] P Fonseca and A Zamolodchikov. "Ising field theory in a magnetic field: analytic properties of the free energy". In: *Journal of statistical physics* 110.3-6 (2003), pp. 527–590.

[20] Hao-Lan Xu. "Ising Field Theory in a Magnetic Field: Extended analyticity properties of M_1". In: *In preparation* (2023).

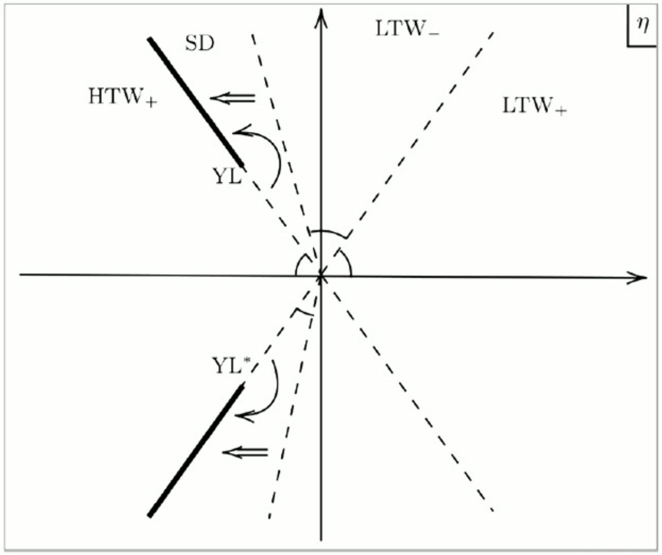
Extended dispersion relations.

- As a result, the extended dispersion relation can be formulated, for example:

$$\mathcal{M}_1(\eta) = M_1^{(0)} + M_1^{(1)}\eta + \frac{2\eta^2}{\pi}\int_{Y_0}^{\infty}\frac{dy}{y^2}\frac{y\,\Re e\left(e^{-\frac{11\pi i}{15}}\Delta_1(y)\right) - \eta\,\Re e\,\Delta_1(y)}{y^2 - 2\cos\left(\frac{11\pi}{15}\right)\eta y + \eta^2},$$

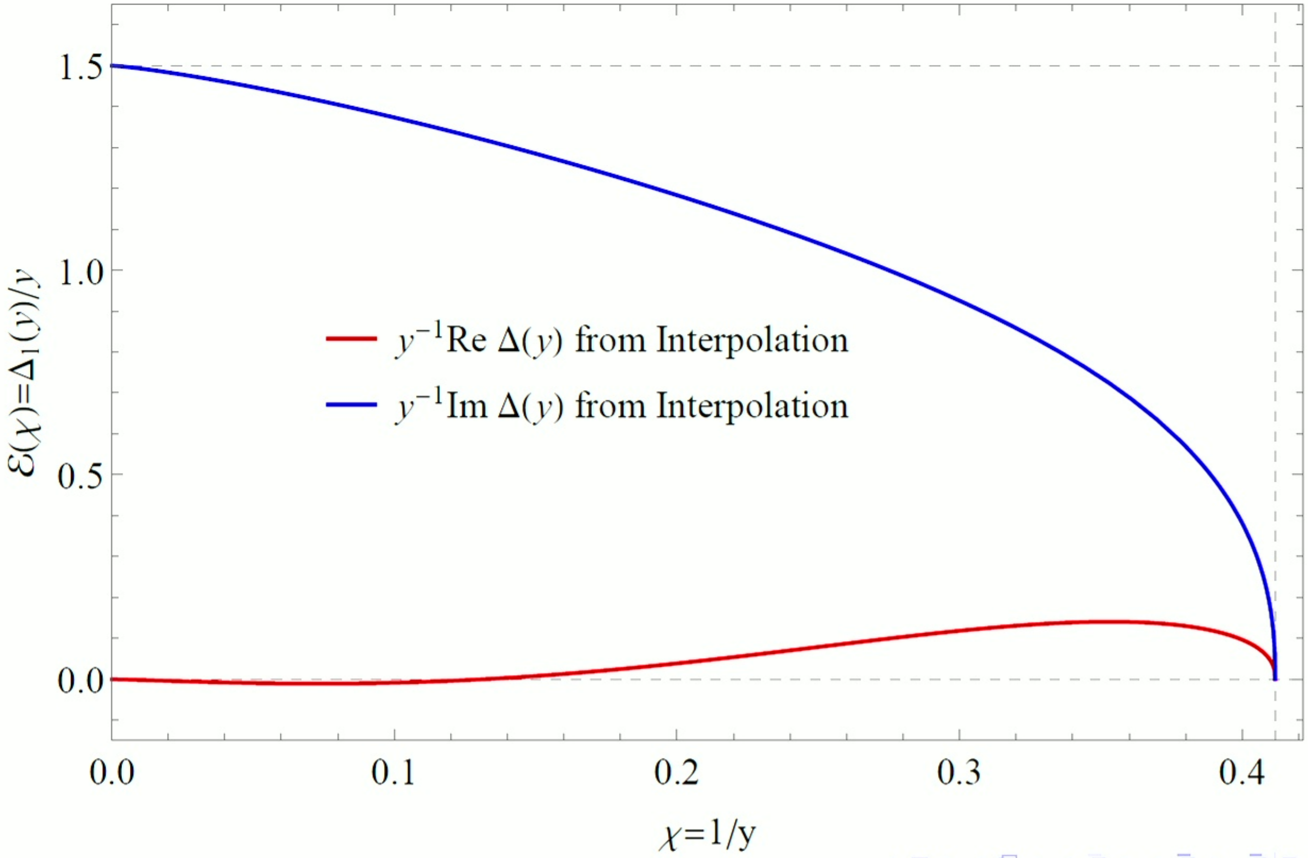
$$\text{where}\quad \Delta_1(y) = \frac{i}{2}e^{\frac{4\pi i}{15}}\left[\hat{\mathcal{M}}_1(ye^{\frac{11\pi i}{15}+i0}) - \hat{\mathcal{M}}_1(ye^{\frac{11\pi i}{15}-i0})\right].$$

- $\Delta(y)$ can be approximated using expressions near $\eta = \infty$ and $y = Y_0$.



Numerical verification of extended dispersion relations of M_1 (I)

- The discontinuities $\Delta_1(y)$ of $\mathcal{M}_1(\eta)$ is now complex on the rotated YL branch cut.



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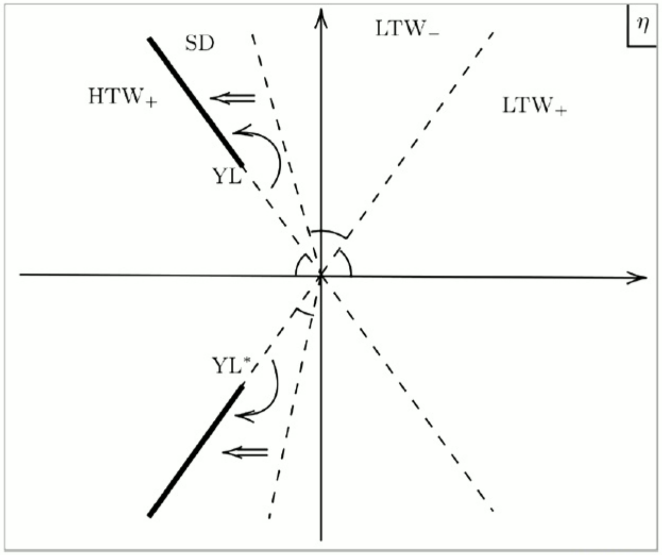
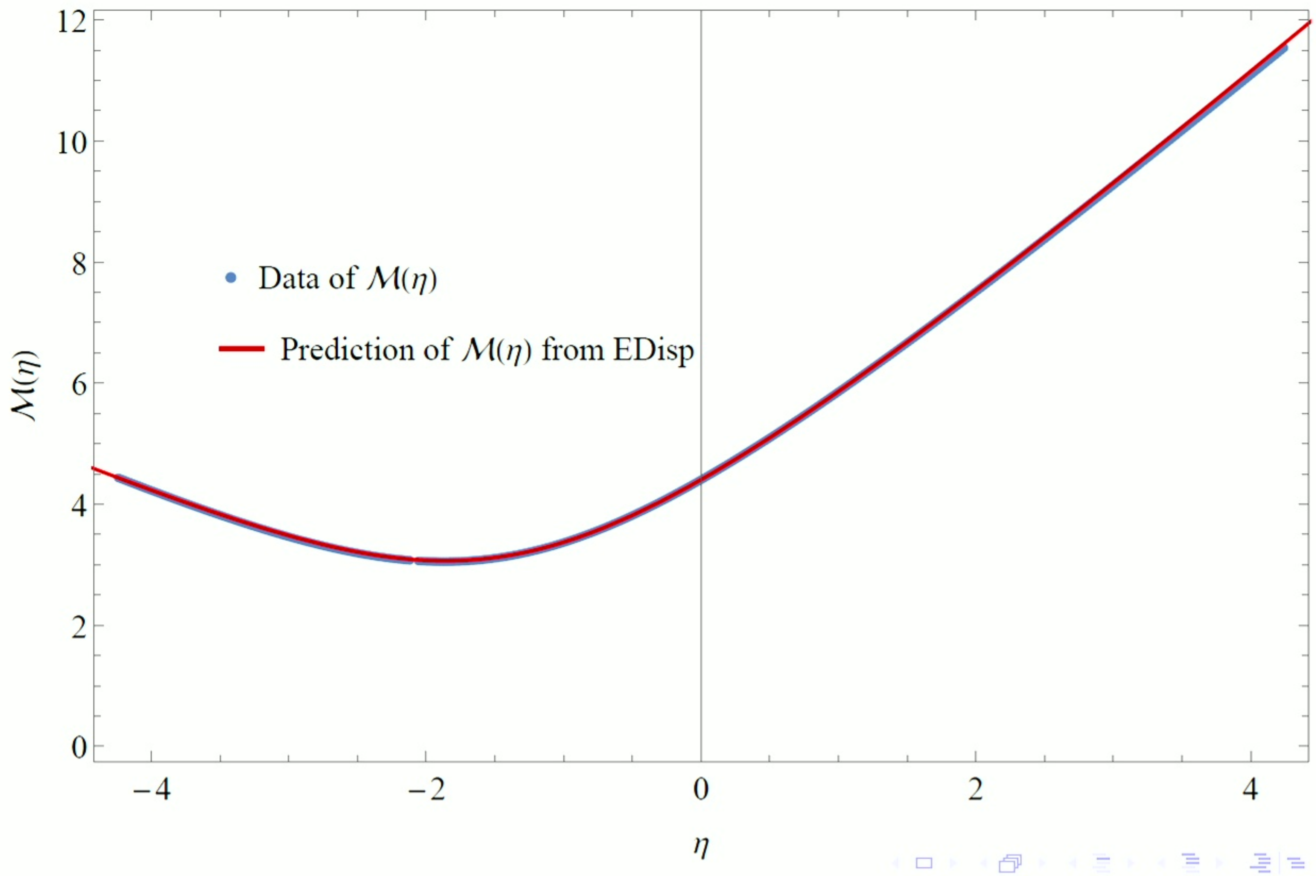


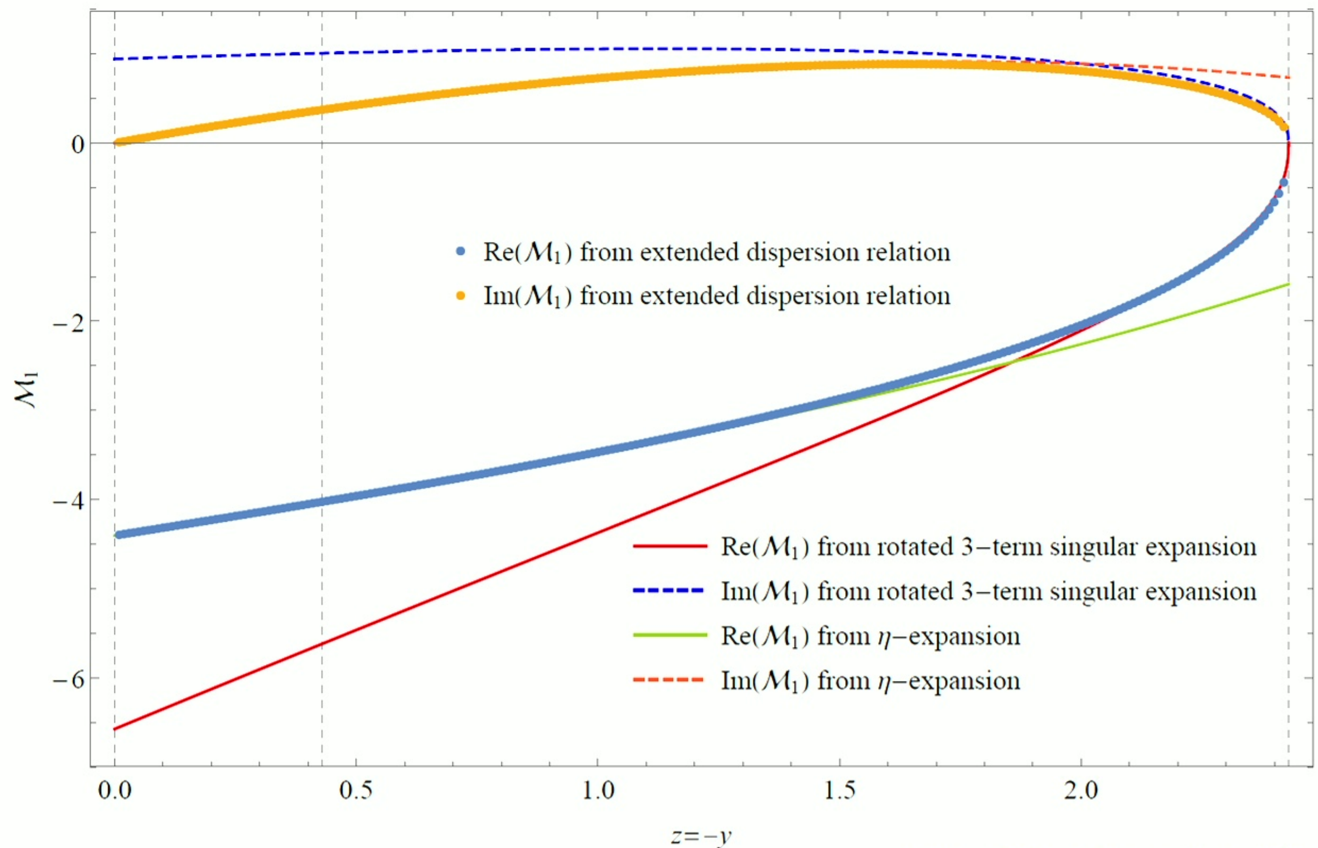
Numerical verification of extended dispersion relations of M_1 (II)

- For real η, numerical verification is straightforward.



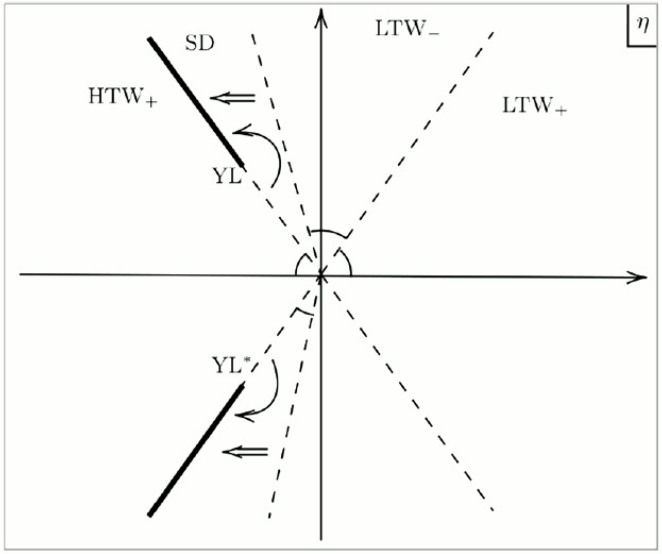
Numerical verification of extended dispersion relations of M_1 (III)

- For $\text{Arg}\,\eta = \frac{11\pi}{15}$, numerical verification is possible by comparing with expansions.



Extended analyticity conjecture and true spinodal point.

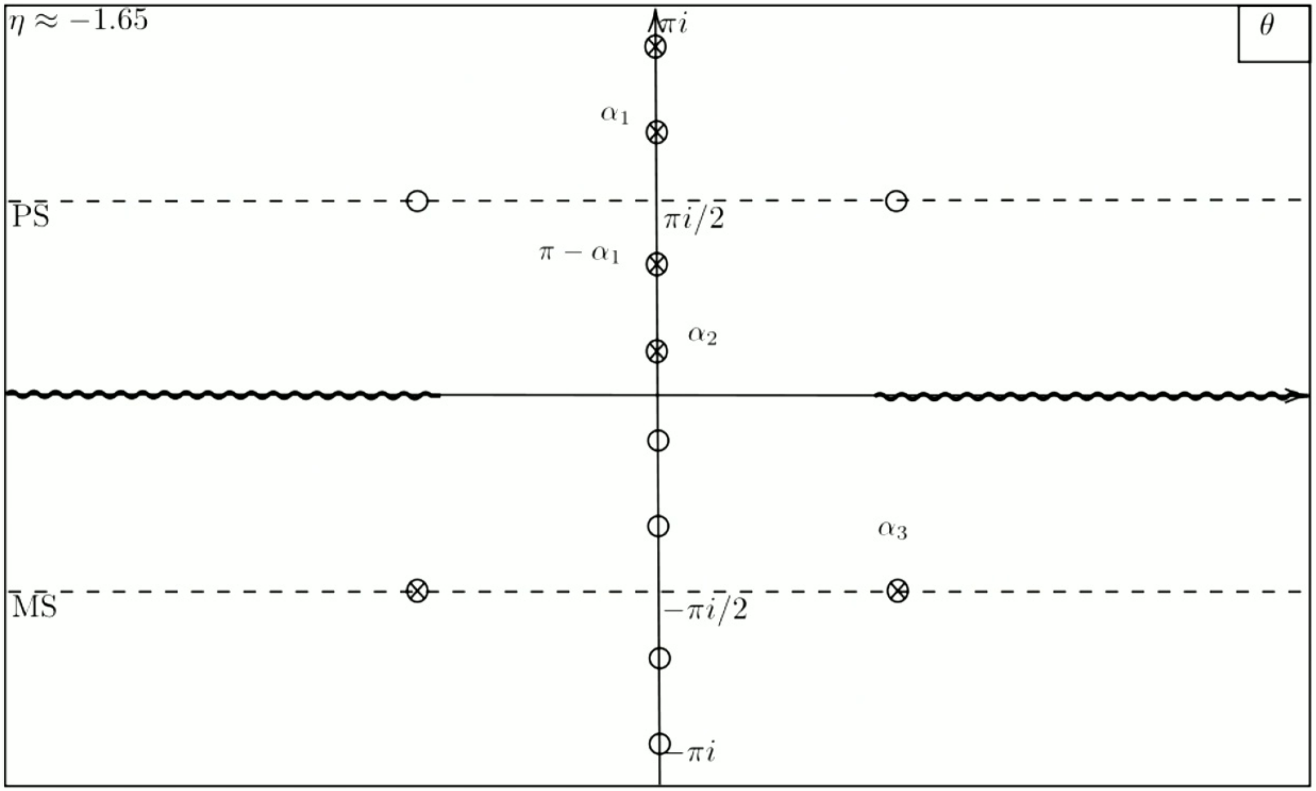
- How about the spinodal singularity? The low-T analyticity conjecture indicates that perturbations would "push down" the spinodal point inside the FL branch cut.
- Furthermore, by no other singularity within SD, we can recognize that the YL point is the non-perturbative spinodal point, and near which the scaling behaviours following the Yang-Lee universality class.
- It would be very interesting to see how this picture works for higher dimensions[21].



[21] Xin An, David Mesterhazy, and Mikhail A Stephanov. "On spinodal points and Lee-Yang edge singularities". In: *Journal of Statistical Mechanics: Theory and Experiment* 2018.3 (2018), p. 033207.

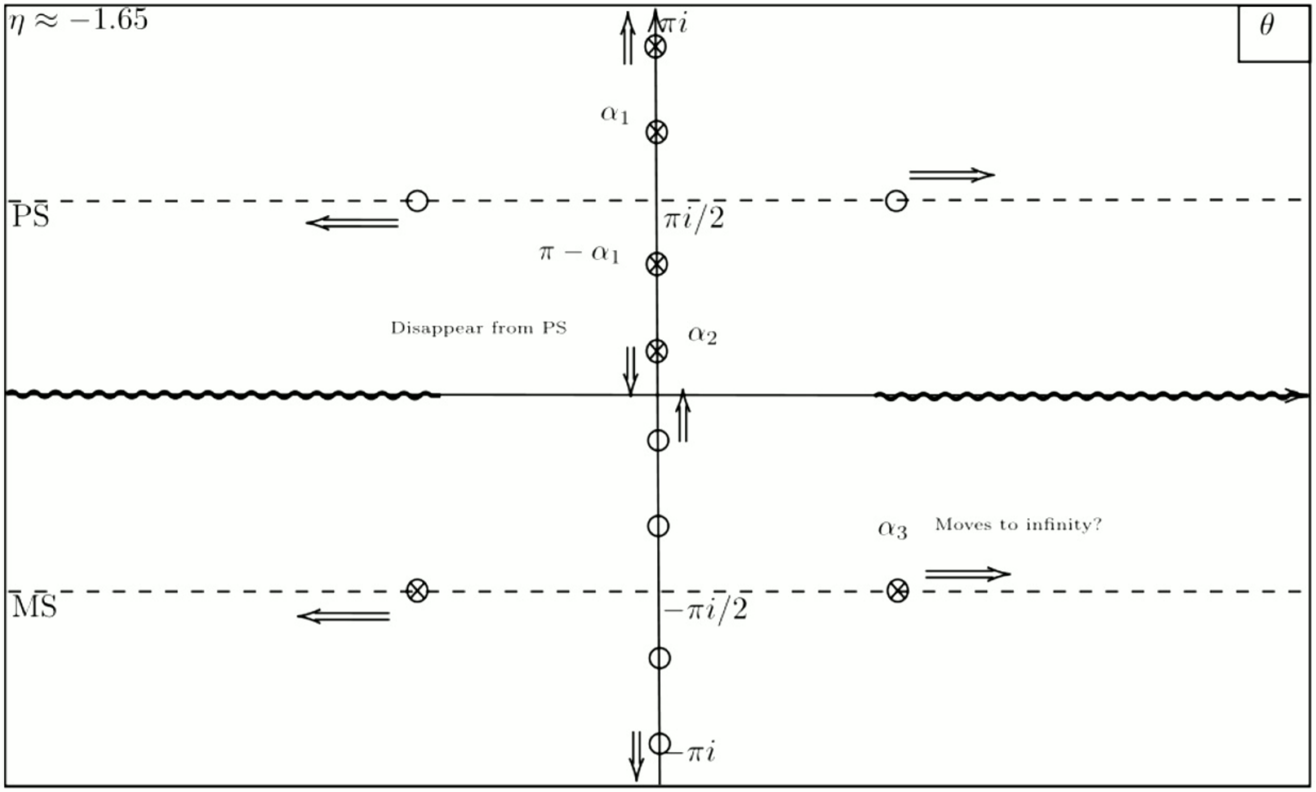
2d Ising scattering matrices.

- Away from integrable points: evolution of $S(\theta)$ analytical structure.



Evolution of 2d Ising scattering matrices.

- Decreasing ξ^2 from $+\infty$ to $-\xi_0^2$, or decreasing η from 0 to $-\infty$ then to $y = -Y_0$.



Evolution of 2d Ising scattering matrices.

- To describe the evolution quantitatively, perturbation theory can't help much.
- From $S(\theta)$ some scaling functions can be defined, and the corresponding dispersion relations can be established.
- Example (I)[24]:

$$\kappa = \kappa(\xi^2) = \frac{\sqrt{3}i}{2} \operatorname*{Res}_{\theta=\frac{2\pi i}{3}} S(\theta) = -\frac{\sqrt{3}}{2}\Gamma^2 \,,$$

which is proportional to the square effective 3-particle coupling of A_1, and also readable from the coefficient of leading exponential decay of $E_1(R)$.

- Example (II):

$$C_2 = C_2(\xi^2) = \frac{1}{B_2} = \frac{1}{\sin\alpha_2} \,,$$

describes the location of pole α_2, and related to the mass M_2 when α_2 still in PS.

- Both of these scaling functions have corresponding dispersion relations in high-T.

[24] Hao-Lan Xu and Alexander Zamolodchikov. "Ising Field Theory in a magnetic field: φ^3 coupling at $T > T_c$". In: (Apr. 2023). arXiv:2304.07886 [hep-th].

The dispersion relations.

- To establish (and verify) corresponding dispersion relations in high-T, the key is to find the discontinuity along the Yang-Lee branch cut.
- Near both Yang-Lee point and E_8 point, the form factor perturbation theory could help. For example, perturbing action with term $g\int \mathcal{O}(x)d^2x$ would leads to:

$$S(\theta) \to S(\theta)\left[1 + i\frac{g}{\sinh\theta}\left(f^{\mathcal{O}}_{\text{reg}}(\theta) - 2iS^{-1}(\theta)S'(\theta)F^{\mathcal{O}}_2(\pi i)\cosh\theta\right)\right],$$

where the components come from the 4-point form factor:

$$F^{\mathcal{O}}_4(\theta'_1, \theta'_2|\theta_1, \theta_2) = -i\left(\frac{\epsilon_1}{\epsilon_2} + \frac{\epsilon_2}{\epsilon_1}\right)S^{-1}(\theta_{12})S'(\theta_{12})F^{\mathcal{O}}_2(\pi i) + f^{\mathcal{O}}_{\text{reg}}(\theta_{12}) + O(\epsilon_1, \epsilon_2).$$

- Near Yang-Lee point: use effective action with irrelevant form factors, sometimes as perturbing the phase $\Delta(\theta)$.
- Near E_8 point, use form factors of energy operator $\varepsilon(x)$ (quite complicated).
- The dispersion relation of $\kappa(\xi^2)$ has been verified, with others ongoing.

Summary and Outlooks.

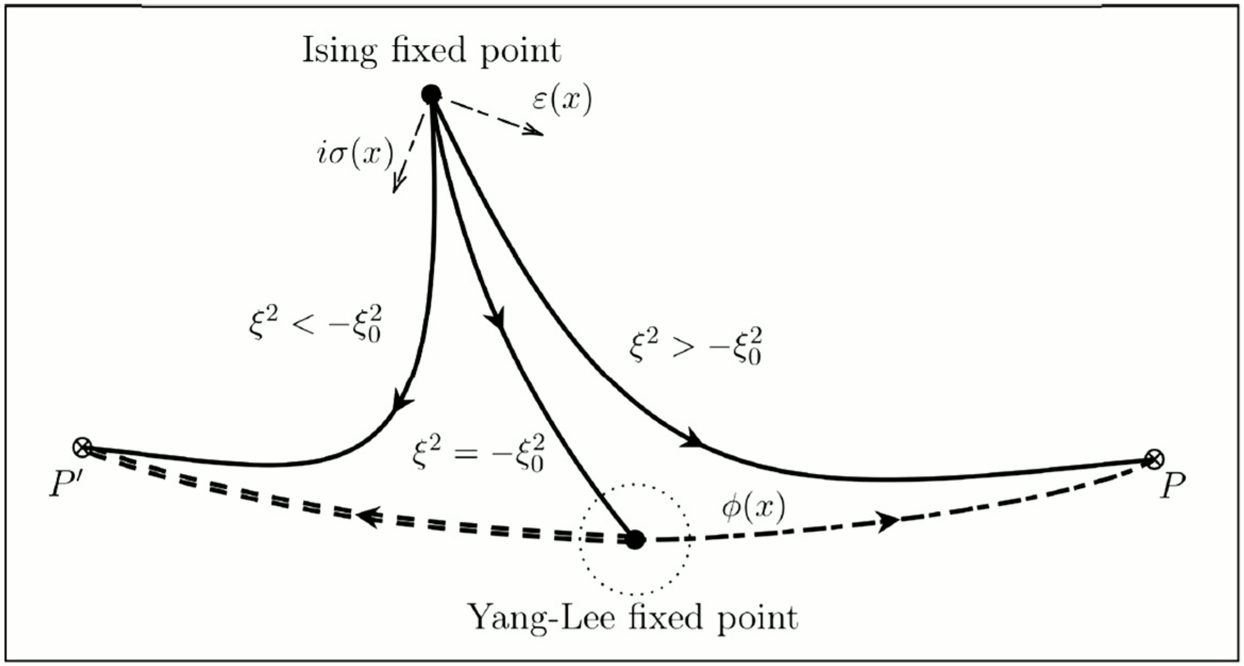
Summary: analyticity structure of 2d Ising field theories:

- Basics of 2d Ising field theory, theory space and scaling functions;
- Disordered/Ordered phases and different scenarios, Yang-Lee & Fisher-Langer;
- Analyticity properties of scaling functions and their dispersion relations;
- Extended analyticity and extended dispersion relation on η-plane;
- Analytical structures of S-matrices and their evolution in parameter space.

Outlook: unsolve questions and future directions:

- Goal: understand the structure of theory space of 2d IFTs non-perturbatively.
- Difficulties: limitations of both numerical methods and perturbative calculations, conceptual difficulties when consider some complicated phenomenons.
- Unsolved questions: (On going)
  (i): Evolution of $S(\theta)$ in low-T: McCoy-Wu scenario of meson, classical scattering with confining interaction, Bethe-Salpeter equation, etc.
  (ii): UV behaviours of scattering, inelastic scattering, and parities of poles.

Thank You for Listening!



Special thanks: prof. Zamolodchikov for his guidance.