Title: QFT1 Lecture - 102323

Speakers: Gang Xu

Collection: Quantum Field Theory 1 2023/24

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states

canonical quantization

$a^\dagger, a$

field

How does Dirac theory reconcile with Lorentz

Dirac equation is Lorentz covariant

Lagrangian is Lorentz invariant

Dan's recipe

Lagrangian $\to$ e.o.m

$[a, a^\dagger] = \cdots$

$$
\begin{tikzcd}
\text{states} \arrow[d, shift right] \\
a^{+}, a \arrow[u, shift right, "\text{canonical quantization}"'] \arrow[d, shift right] \\
\text{field} \arrow[u, shift right]
\end{tikzcd}
$$

How does Dirac theory reconcile with Lorentz

Dirac equation is Lorentz covariant

Lagrangian is Lorentz invariant.

Dan's recipe

Lagrangian → e.o.m

$[a, a^{+}] = \cdots$

Lorentz

covariant

invariant.

· Derive Dirac equation

KG

popular

$$(\partial_\mu \partial^\mu + m^2)\Phi = 0$$

$$\rho_{KG} = i\left((\partial_t \Phi^*)\Phi - \Phi^*(\partial_t \Phi)\right)$$

not positive definite

Dirac's criteria

① 1 time derivative

② 1 spatial derivative

③ $\underbrace{p^2 = m^2}_{\text{field still satisfy KG equation}}$

$$(\partial_\mu \partial^\mu + m^2)\,\Phi = 0 \quad \textcircled{3} \quad \Phi,\ \psi,\ V$$

$$a^2 + b^2 = (-ia-b)(ia-b)$$

$$(i\partial_\mu \gamma^\mu - m)\psi = 0 \quad \leftarrow \text{linear coeffients}$$

$$(-i\partial_\nu\gamma^\nu - m)(i\partial_\mu \gamma^\mu - m)\psi = 0$$

$$(\partial_\nu \partial_\mu \gamma^\nu \gamma^\mu + m^2)\psi = 0$$

$$\partial_\mu \partial^\mu = \partial_\mu \partial_\nu \eta^{\mu\nu}$$

$$\left(\partial_\nu \partial_\mu \frac{\gamma^\nu\gamma^\mu + \gamma^\mu\gamma^\nu}{2} + m^2\right)\psi = 0$$

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbb{1}$$

but incom

$$(\partial_\mu \partial^\mu + m^2)\,\Phi = 0 \quad ③ \quad \Phi,\ \psi,\ V$$

$$a^2 + b^2 = (-ia-b)(ia-b)$$

$$(i\partial_\mu \gamma^\mu - m)\psi = 0$$

$\gamma^\mu \leftarrow$ linear coeffients

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$$(\partial_\nu \partial_\mu \gamma^\nu \gamma^\mu + m^2)\psi = 0$$

$$\partial_\mu \partial^\mu = \partial_\mu \partial_\nu \eta^{\mu\nu}$$

$$\left(\partial_\nu \partial_\mu \frac{\gamma^\nu\gamma^\mu + \gamma^\mu\gamma^\nu}{2} + m^2\right)\psi = 0$$

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$$\left(\partial_\nu \partial_\mu \frac{\gamma^\nu\gamma^\mu + \gamma^\mu\gamma^\nu}{2} + m^2\right)\psi = 0$$

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbb{1}$$

$$\{\gamma^0, \gamma^0\} = 2\gamma^{0^2} = 2$$

$=0$

but incomplete

Dirac original

$$i\frac{\partial}{\partial t}\psi = H\psi$$

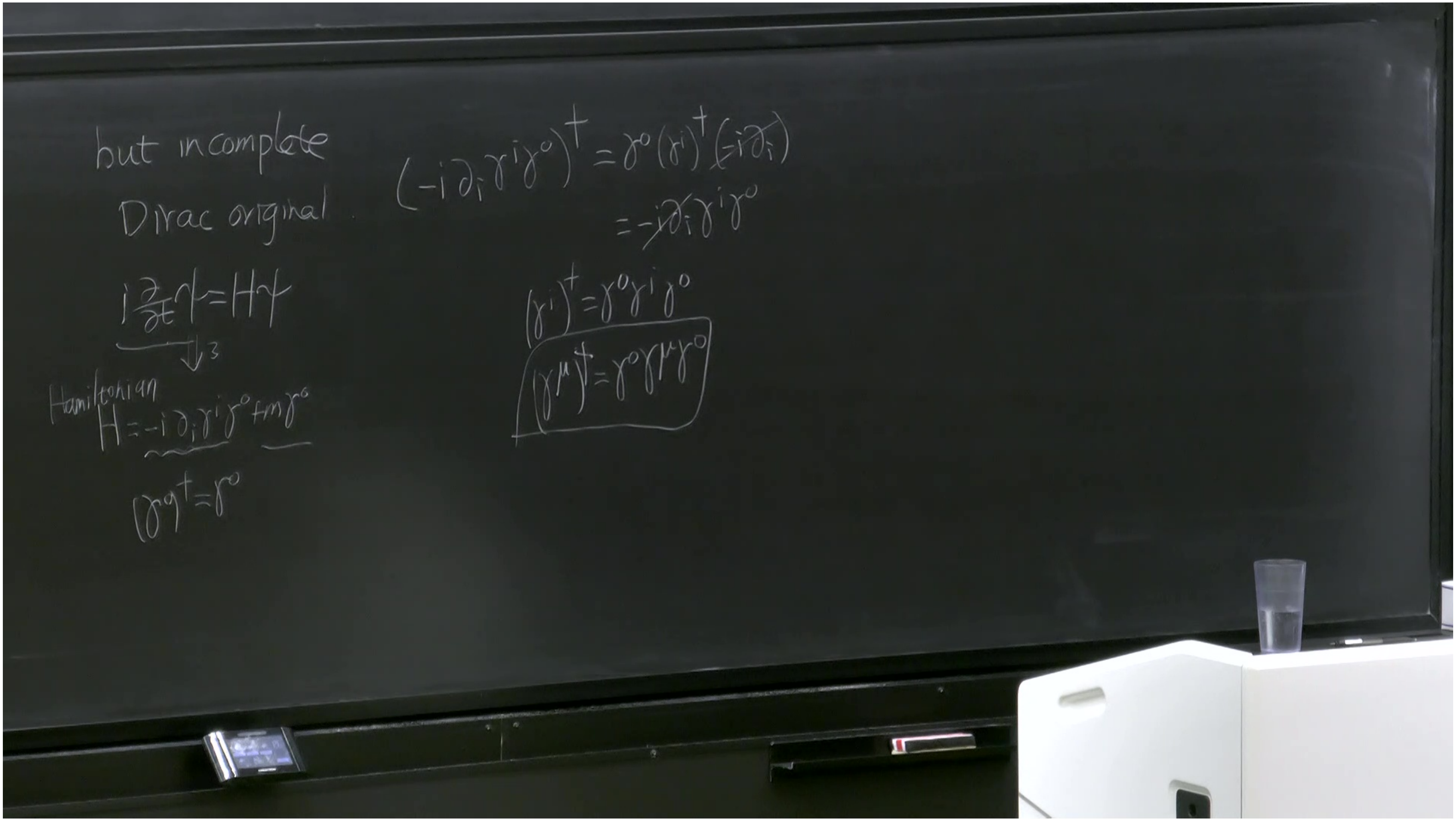
$\Downarrow 3$

Hamiltonian

$$H = -i\partial_i \gamma^i \gamma^0 + m\gamma^0$$

$$(\gamma^0)^\dagger = \gamma^0$$

$$(-i\partial_i \gamma^i \gamma^0)^\dagger = \gamma^0 (\gamma^i)^\dagger (-i\partial_i)$$


but incomplete
Dirac original
$i\frac{\partial}{\partial t}\psi = H\psi$
$\Downarrow 3$
Hamiltonian
$H = -i\partial_i\gamma^i\gamma^0 + m\gamma^0$
$(\gamma^0)^\dagger = \gamma^0$
$(-i\partial_i\gamma^i\gamma^0)^\dagger = \gamma^0(\gamma^i)^\dagger(-i\partial_i)$
$= -i\partial_i\gamma^i\gamma^0$
$(\gamma^i)^\dagger = \gamma^0\gamma^i\gamma^0$
$\boxed{(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0}$

Lorentz covariance

$$x' = \Lambda x$$

$$(i\gamma^\mu \partial_\mu - m)\psi = 0$$

wish $\psi' = D(\Lambda)\psi$

$$(i\gamma^\mu \partial'_\mu - m)\psi' = 0$$

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = \frac{\partial}{\partial x'^\nu}\frac{\partial x'^\nu}{\partial x^\mu} = \partial'_\nu \Lambda^\nu{}_\mu$$

$$\psi = D^{-1}(\Lambda)\psi'$$

$$D(\Lambda)(i\gamma^\mu \partial'_\nu \Lambda^\nu{}_\mu - m)D^{-1}(\Lambda)\psi' = 0$$

$$\gamma^\mu = D(\Lambda)\gamma^\nu \Lambda_\nu{}^\mu D^{-1}(\Lambda)$$

$) \Phi = 0$ — ③

$= (-ia-b)(ia-b)$

linear coeffients

$\boxed{\gamma^\mu - m)\psi = 0}$

$m)(i\partial_\mu \gamma^\mu - m)\psi = 0$

$(\partial_\nu \partial_\mu \gamma^\nu \gamma^\mu + m^2)\psi = 0$

$\partial_\mu \partial^\mu = \partial_\mu \partial_\nu \eta^{\mu\nu}$

$\left(\partial_\nu \partial_\mu \frac{\gamma^\nu \gamma^\mu + \gamma^\mu \gamma^\nu}{2} + m^2\right)\psi = 0$

$\boxed{\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}}$

$\{\gamma^0, \gamma^0\} = 2{\gamma^0}^2 = 2$

but incomplete

Dirac original.

$i\frac{\partial}{\partial t}\psi = H\psi$

$\Downarrow 3$

Hamiltonian

$H = -i\partial_i \gamma^i \gamma^0 + m\gamma^0$

$\gamma^{0\dagger} = \gamma^0$

$(-i\partial_i$

KG equation

$$\left(D(\Lambda)_{ab} = (1)_{ab} + i\omega_{\mu\nu}(J^{\mu\nu})_{ab} \right)$$

$$\Lambda^{\nu}{}_{\mu} = 1 + \omega^{\nu}{}_{\mu}$$

$\Downarrow$ 3 lines

$$[J_{\alpha\beta}, \gamma_\mu] = -i(\gamma_\alpha \eta_{\beta\mu} - \gamma_\beta \eta_{\alpha\mu})$$

$$J_{\alpha\beta} = -\frac{i}{4}[\gamma_\alpha, \gamma_\beta]$$

Dirac field is a spinor.

• irreducible rep Weyl spinor

$$D(\Lambda_{rot})_{4\times4} = \begin{pmatrix} e^{\frac{i}{2}\vec{\theta}\cdot\vec{\sigma}} & \\ & e^{\frac{i}{2}\vec{\theta}\cdot\vec{\sigma}} \end{pmatrix}$$

$$D(\Lambda_{boost})_{4\times4} = \begin{pmatrix} e^{\frac{1}{2}\vec{\chi}\cdot\vec{\sigma}} & \\ & e^{-\frac{1}{2}\vec{\chi}\cdot\vec{\sigma}} \end{pmatrix}$$

block diagonal.

separate $\psi = \begin{pmatrix} W_+ \\ W_- \end{pmatrix}$

$$W_\pm \xrightarrow{\text{rot}} e^{\frac{i}{2}\vec{\theta}\cdot\vec{\sigma}}\, W_\pm$$

$$W_\pm \xrightarrow{\text{boost}} e^{\pm\frac{1}{2}\vec{\chi}\cdot\vec{\sigma}}\, W_\pm$$

Look for projection operator

$$P_1+P_2=\mathbb{1} \quad P_1^2=P_1 \quad P_1\cdot P_2=0$$

$$[P, J^{\mu\nu}]=0$$

all $[P, \gamma^\mu]=0$ or $\{P, \gamma^\mu\}=0$

$\mathbb{1}$ $\quad \gamma^5=-i\gamma^0\gamma^1\gamma^2\gamma^3$

$$(\gamma^5)^2=1$$

$$P_\pm = \frac{1 \pm \gamma^5}{2}$$

sanity check.

$$\gamma^5 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$$

$$P_+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad P_- = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

- Lorentz invariant Lagrangian

propose $\psi^\dagger \gamma^0 \psi$

$\psi^\dagger D^\dagger(\Lambda)\gamma^0 \quad D(\Lambda)\psi$

$\psi^\dagger \gamma^0 D^{-1}(\Lambda)\gamma^0\gamma^0 D(\Lambda)\psi$

Dirac conjugate $\bar{\psi} = \psi^\dagger \gamma^0$

$D(\Lambda) = e^{i\omega_{\mu\nu} J^{\mu\nu}}$

$D^\dagger(\Lambda) = e^{-i\omega_{\mu\nu}(J^{\mu\nu})^\dagger}$

$J^{\mu\nu} = -\frac{i}{4}[\gamma^\mu, \gamma^\nu]$

$\Downarrow$ 3 line

$(J^{\mu\nu})^\dagger = \gamma^0 J^{\mu\nu} \gamma^0$

$D^\dagger(\Lambda) = e^{-i\omega_{\mu\nu}\gamma^0 J^{\mu\nu}\gamma^0}$

$= \gamma^0 e^{-i\omega_{\mu\nu}J^{\mu\nu}}\gamma^0$

$= \gamma^0 D^{-1}(\Lambda)\gamma^0$

• Lorentz invariant Lagrangian

propose $\psi^\dagger \gamma^0 \psi$

$\psi^\dagger D^\dagger(\Lambda)\gamma^0 \quad D(\Lambda)\psi$

$\psi^\dagger \gamma^0 D^{-1}(\Lambda)\gamma^0 \gamma^0 D(\Lambda)\psi$

Dirac conjugate $\overline{\psi} = \psi^\dagger \gamma^0$

$D(\Lambda) = e^{i\omega_{\mu\nu} J^{\mu\nu}}$

$D^\dagger(\Lambda) = e^{-i\omega_{\mu\nu}(J^{\mu\nu})^\dagger}$

$J^{\mu\nu} = -\frac{i}{4}[\gamma^\mu, \gamma^\nu]$

$\Downarrow$ 3 line

$(J^{\mu\nu})^\dagger = \gamma^0 J^{\mu\nu} \gamma^0$

$\mathcal{L} \supset \overline{\psi}\psi$

$D^\dagger(\Lambda) = e^{-i\omega_{\mu\nu}\gamma^0 J^{\mu\nu}\gamma^0}$

$= \gamma^0 e^{-i\omega_{\mu\nu}J^{\mu\nu}}\gamma^0$

$= \gamma^0 D^{-1}(\Lambda)\gamma^0$

$$[illegible]) = e^{-i\omega_{\mu\nu}\gamma^0 J^{\mu\nu}\gamma^0}$$

$$= \gamma^0 e^{-i\omega_{\mu\nu}J^{\mu\nu}}\gamma^0$$

$$= \gamma^0 D^{-1}(\Lambda)\gamma^0$$

$$\underline{\bar\psi\gamma^\mu\psi} \to \underline{\Lambda^\mu{}_\nu\bar\psi\gamma^\nu\psi}$$

$$\underline{D^{-1}(\Lambda)\gamma^\mu D(\Lambda) = \Lambda^\mu{}_\nu\gamma^\nu}$$

$$\bar\psi\psi \qquad \underset{[-4]}{\mathcal{L}} = \bar\psi\gamma^\mu\partial_\mu\psi - m\bar\psi\psi = \bar\psi(\gamma^\mu\partial_\mu - m)\psi$$

$$\partial^\mu\phi$$

$$S = \int d^4x\,\mathcal{L} \qquad x \quad \dim[\psi]\cdot 2 - 1 = -4$$

$$[S] = 0 \qquad \psi = -\frac{3}{2}$$

$${}^2\phi^\dagger\phi$$

$\frac{1 \pm \gamma^5}{2}$

nity check.

$= \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad R = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$$\psi_a(x) = \int \sum_s dV_p \, u_a^s(p) \, a_{\vec{p}}^s \, e^{-ipx} + v_a^s(p) \, a_{\vec{p}}^{\dagger s} \, e^{+ipx}$$