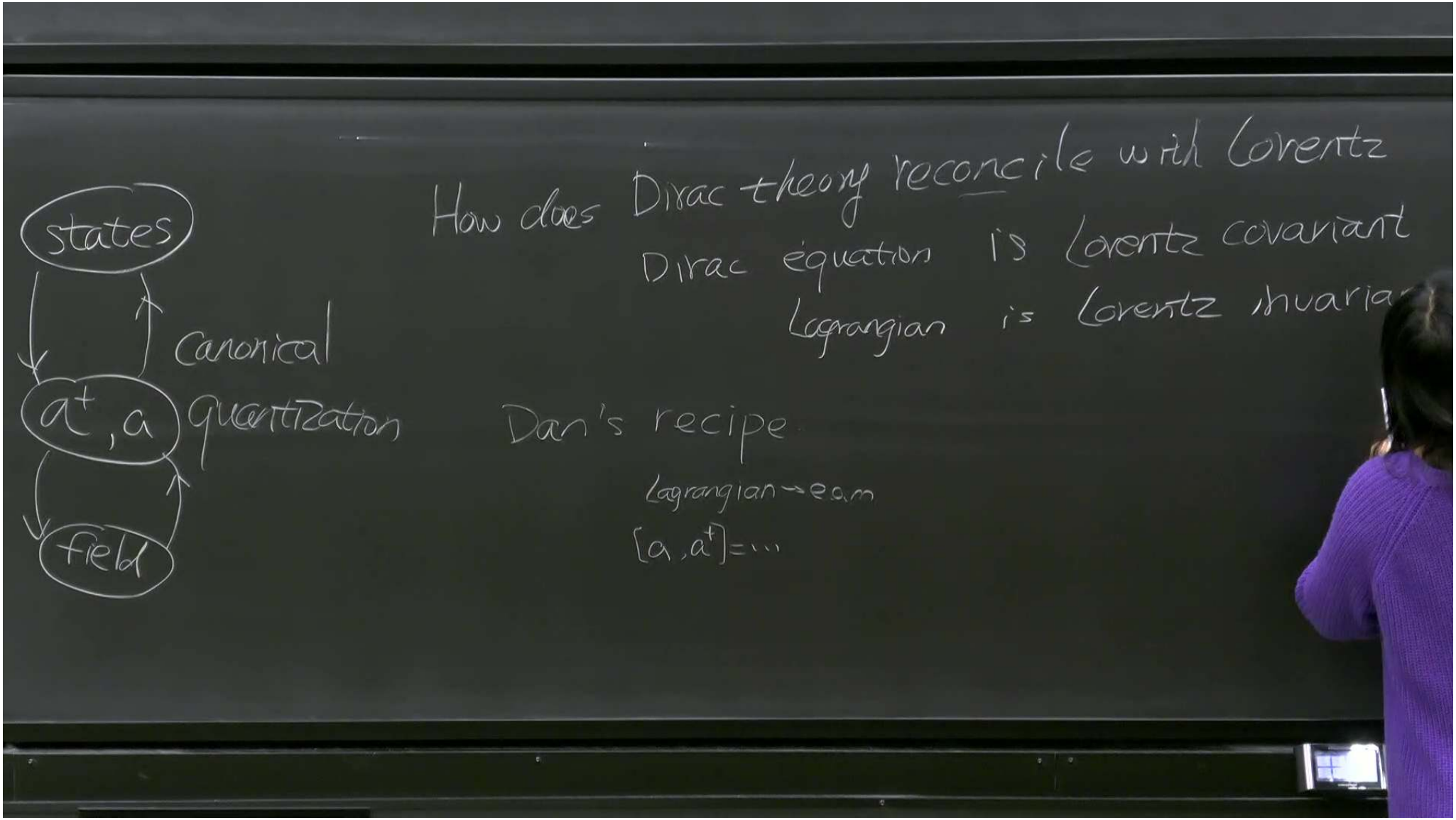
Title: QFT1 Lecture - 102323

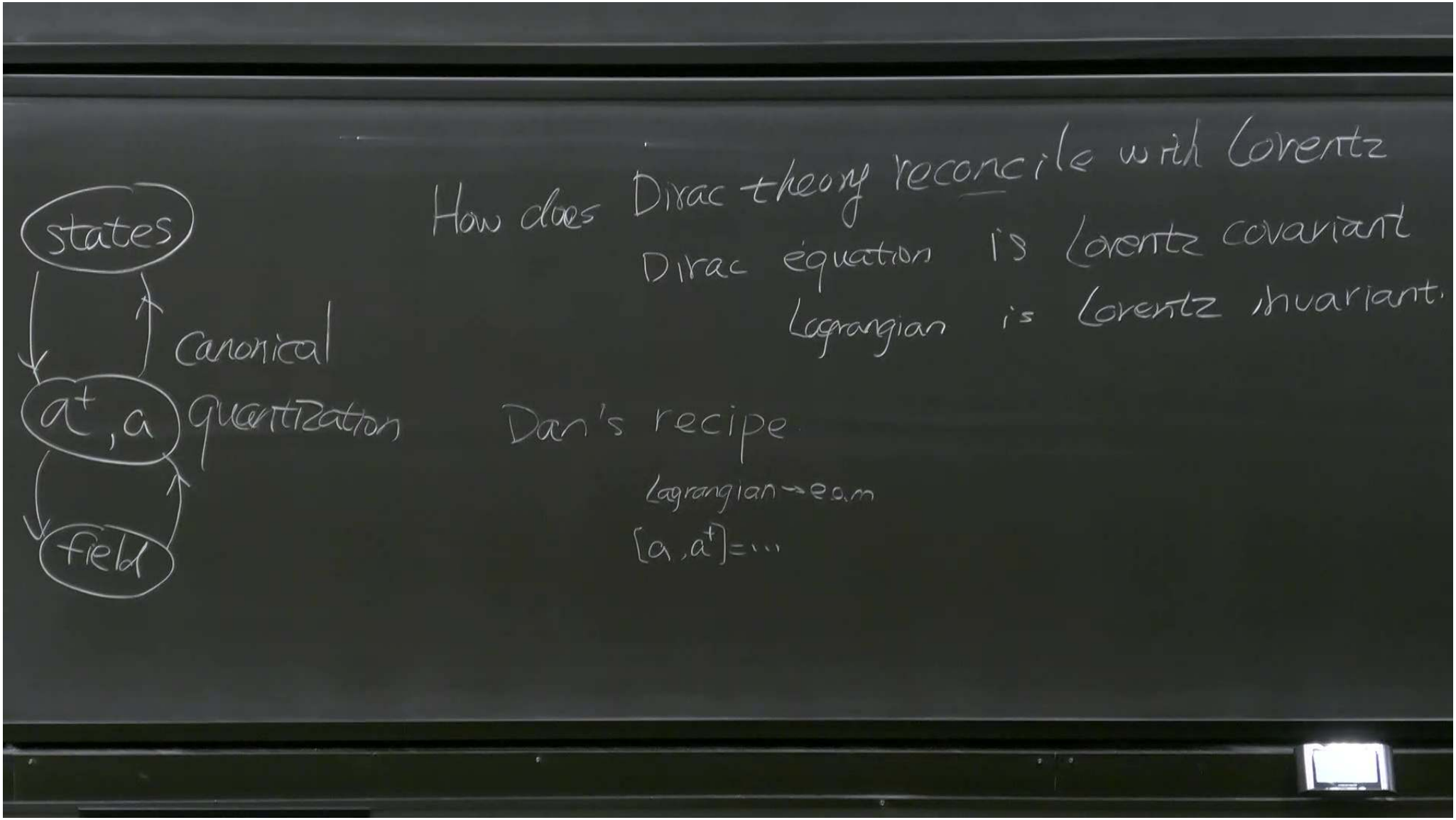
Speakers: Gang Xu

Collection: Quantum Field Theory 1 2023/24

Date: October 23, 2023 - 10:45 AM

URL: https://pirsa.org/23100051


How does Dirac theory reconcile with Lorentz
Dirac equation is Lorentz covariant
Lagrangian is Lorentz invaria
states
canonical
quantization
$a^\dagger, a$
field
Dan's recipe
Lagrangian → e.o.m
$[a, a^\dagger] = \cdots$


How does Dirac theory reconcile with Lorentz
Dirac equation is Lorentz covariant
Lagrangian is Lorentz invariant.
states
canonical
quantization
$a^\dagger, a$
field
Dan's recipe
Lagrangian → e.o.m
$[a, a^\dagger] = \cdots$

Lorentz

covariant

invariant.

• Derive Dirac equation.

popular KG

$$(\partial_\mu \partial^\mu + m^2)\Phi = 0$$

$$\rho_{KG} = i\,(\partial_t \Phi^*)\Phi - \Phi^*(\partial_t \Phi)$$

not positive definite

Dirac's criteria

① 1 time derivative

② 1 spatial derivative

③ $p^2 = m^2$

field still satisfy KG equation

$$(\partial_\mu \partial^\mu + m^2)\Phi = 0 \quad - \quad ③ \quad \begin{cases} \Phi \\ \psi \\ V \end{cases}$$

$$a^2 + b^2 = (-ia-b)(ia-b)$$

$$(i\partial_\mu \gamma^\mu - m)\psi = 0 \quad \leftarrow \text{linear coeffients}$$

$$(-i\partial_\nu \gamma^\nu - m)(i\partial_\mu \gamma^\mu - m)\psi = 0$$

$$(\partial_\nu \partial_\mu \gamma^\nu \gamma^\mu + m^2)\psi = 0$$

$$\partial_\mu \partial^\mu = \partial_\mu \partial_\nu \eta^{\mu\nu}$$

$$\left(\partial_\nu \partial_\mu \frac{\gamma^\nu \gamma^\mu + \gamma^\mu \gamma^\nu}{2} + m^2\right)\psi = 0$$

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} 1$$

$$(\partial_\mu \partial^\mu + m^2)\Phi = 0 \quad (3) \qquad \Phi,\ \psi,\ V$$

$$a^2 + b^2 = (-ia-b)(ia-b)$$

$$(i\partial_\mu \gamma^\mu - m)\psi = 0 \quad \leftarrow \text{linear coeffients}$$

$$(-i\partial_\nu \gamma^\nu - m)(i\partial_\mu \gamma^\mu - m)\psi = 0$$

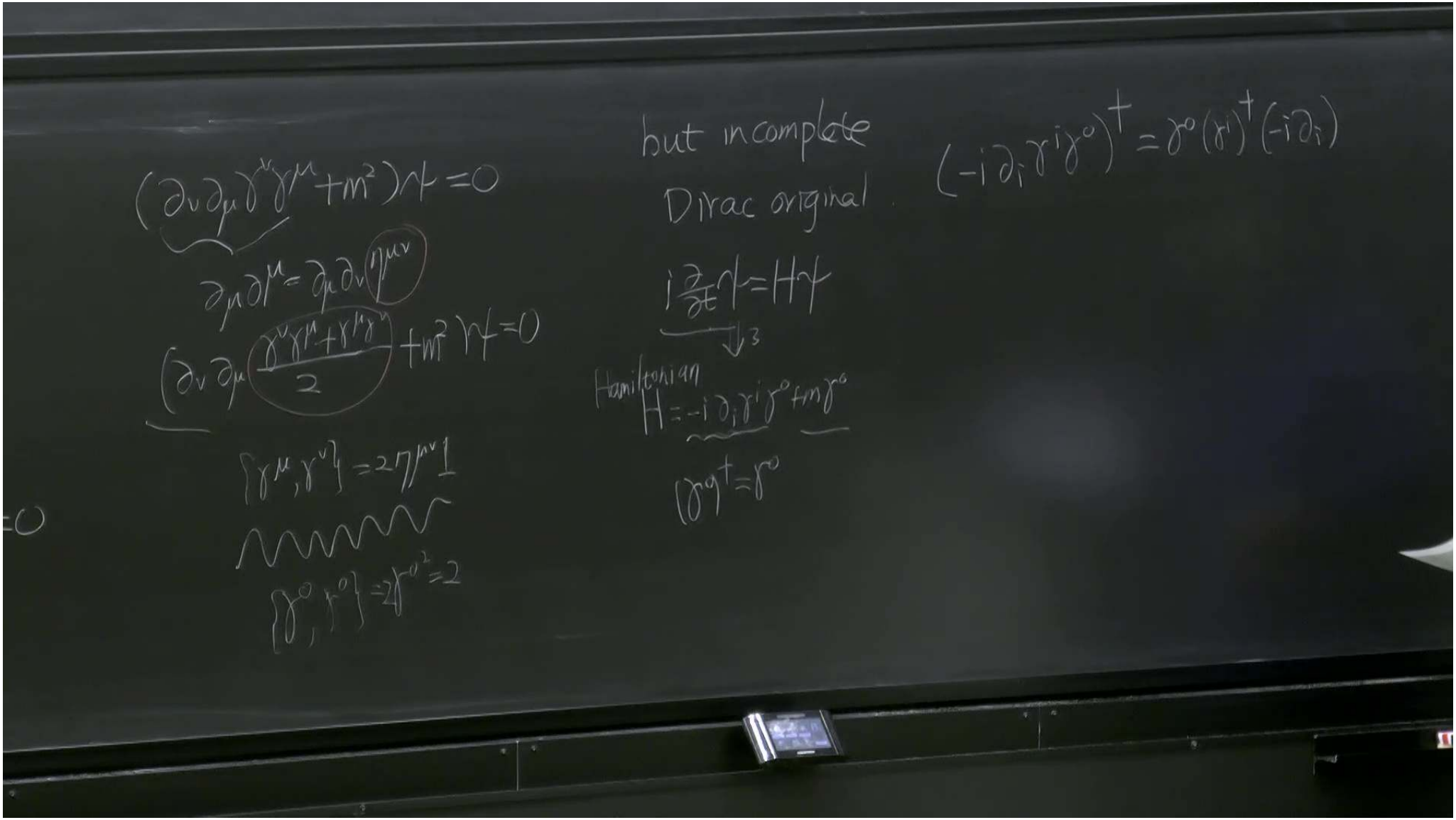
$$(\partial_\nu \partial_\mu \gamma^\nu \gamma^\mu + m^2)\psi = 0$$

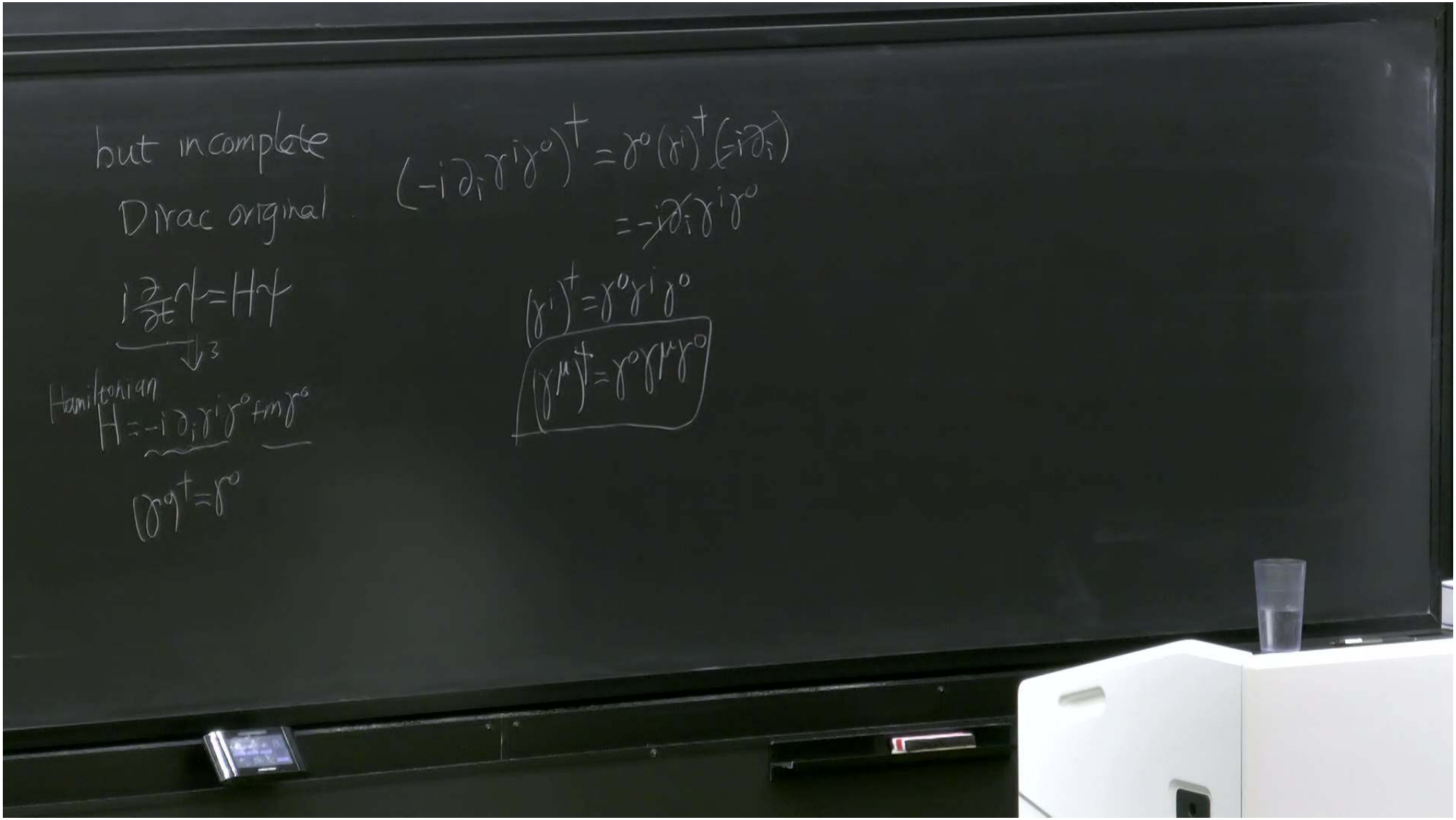
$$\partial_\mu \partial^\mu = \partial_\mu \partial_\nu \eta^{\mu\nu}$$

$$\left(\partial_\nu \partial_\mu \frac{\gamma^\nu \gamma^\mu + \gamma^\mu \gamma^\nu}{2} + m^2\right)\psi = 0$$

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \mathbb{1}$$

but incom


$(\partial_\nu \partial_\mu \gamma^\nu \gamma^\mu + m^2)\psi = 0$
$\partial_\mu \partial^\mu = \partial_\mu \partial_\nu \eta^{\mu\nu}$
$(\partial_\nu \partial_\mu \frac{\gamma^\nu \gamma^\mu + \gamma^\mu \gamma^\nu}{2} + m^2)\psi = 0$
$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} 1$
$\{\gamma^0, \gamma^0\} = 2\gamma^{0^2} = 2$
but incomplete
Dirac original
$i\frac{\partial}{\partial t}\psi = H\psi$
Hamiltonian
$H = -i\partial_i \gamma^i \gamma^0 + m\gamma^0$
$(\gamma^0)^\dagger = \gamma^0$
$(-i\partial_i \gamma^i \gamma^0)^\dagger = \gamma^0 (\gamma^i)^\dagger (-i\partial_i)$


but incomplete
Dirac original
$i\frac{\partial}{\partial t}\psi = H\psi$
Hamiltonian
$H = -i\partial_i \gamma^i \gamma^0 + m\gamma^0$
$\gamma^0\gamma^{0\dagger} = \gamma^0$
$(-i\partial_i \gamma^i \gamma^0)^\dagger = \gamma^0 (\gamma^i)^\dagger (-i\partial_i)$
$= -i\partial_i \gamma^i \gamma^0$
$(\gamma^i)^\dagger = \gamma^0 \gamma^i \gamma^0$
$(\gamma^\mu)^\dagger = \gamma^0 \gamma^\mu \gamma^0$

Lorentz covariance

$x' = \Lambda x$

$(i\gamma^\mu \partial_\mu - m)\psi = 0$

$\partial_\mu = \frac{\partial}{\partial x^\mu} = \frac{\partial}{\partial x'^\nu}\frac{\partial x'^\nu}{\partial x^\mu} = \partial'_\nu \Lambda^\nu_\mu$

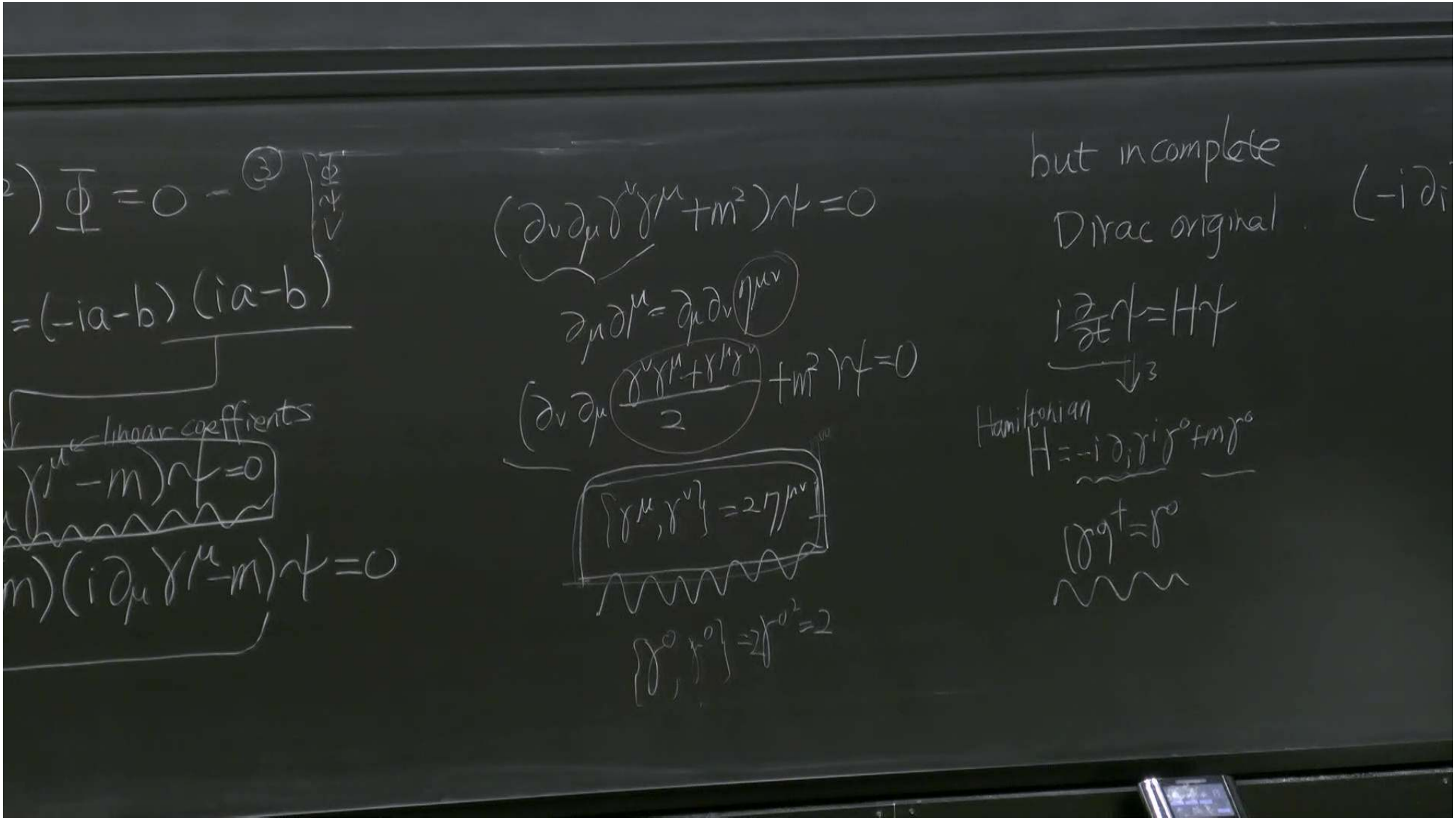
$\psi = D^{-1}(\Lambda)\psi'$

wish $\psi' = D(\Lambda)\psi$

$D(\Lambda)(i\gamma^\mu \partial'_\nu \Lambda^\nu_\mu - m) D^{-1}(\Lambda)\psi' = 0$

$(i\gamma^\mu \partial'_\mu - m)\psi' = 0$

$\gamma^\mu = D(\Lambda)\gamma^\nu \Lambda^\mu_\nu D^{-1}(\Lambda)$



$$\ldots)\,\Phi = 0 \quad - \quad ③ \quad \left\{ \begin{matrix} \Phi \\ \psi \\ V \end{matrix} \right.$$

$$= (-ia-b)(ia-b)$$

linear coeffients

$$\ldots\gamma^\mu - m)\psi = 0$$

$$\ldots m)(i\partial_\mu\gamma^\mu - m)\psi = 0$$

$$(\partial_\nu\partial_\mu\gamma^\nu\gamma^\mu + m^2)\psi = 0$$

$$\partial_\mu\partial^\mu = \partial_\mu\partial_\nu\eta^{\mu\nu}$$

$$\left(\partial_\nu\partial_\mu\frac{\gamma^\nu\gamma^\mu + \gamma^\mu\gamma^\nu}{2} + m^2\right)\psi = 0$$

$$\boxed{\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}}$$

$$\{\gamma^0, \gamma^0\} = 2\gamma^{0\,2} = 2$$

but incomplete

Dirac original

$$i\frac{\partial}{\partial t}\psi = H\psi$$

$$\Downarrow 3$$

Hamiltonian

$$H = -i\partial_i\gamma^i\gamma^0 + m\gamma^0$$

$$(\gamma^0)^\dagger = \gamma^0$$

$$(-i\partial_i \ldots$$

KG equation

$$D(\Lambda)_{ab} = \left(1 + \frac{i}{2}\omega_{\mu\nu}(J^{\mu\nu})\right)_{ab}$$

$$\Lambda^{\nu}_{\ \mu} = 1 + \omega^{\nu}_{\ \mu}$$

$\Downarrow$ 3 lines

$$[J_{\alpha\beta}, \gamma_\mu] = -i(\gamma_\alpha \eta_{\beta\mu} - \gamma_\beta \eta_{\alpha\mu})$$

$$J_{\alpha\beta} = -\frac{i}{4}[\gamma_\alpha, \gamma_\beta]$$

Dirac field is a spinor.

• irreducible rep Weyl spinor

$$D(\Lambda_{rot})_{4\times4} = \begin{pmatrix} e^{\frac{i}{2}\vec{\theta}\cdot\vec{\sigma}} & \\ & e^{\frac{i}{2}\vec{\theta}\cdot\vec{\sigma}} \end{pmatrix}$$

$$D(\Lambda_{boost})_{4\times4} = \begin{pmatrix} e^{\frac{1}{2}\vec{\chi}\cdot\vec{\sigma}} & \\ & e^{-\frac{1}{2}\vec{\chi}\cdot\vec{\sigma}} \end{pmatrix}$$

block diagonal.

separate $\psi = \begin{pmatrix} W_+ \\ W_- \end{pmatrix}$

$W_\pm \xrightarrow{\text{rot}} e^{\pm \frac{1}{2}\vec{\theta}\cdot\vec{\sigma}} W_\pm$

$W_\pm \xrightarrow{\text{boost}} e^{\pm \frac{1}{2}\vec{\chi}\cdot\vec{\sigma}} W_\pm$

Look for projection operator

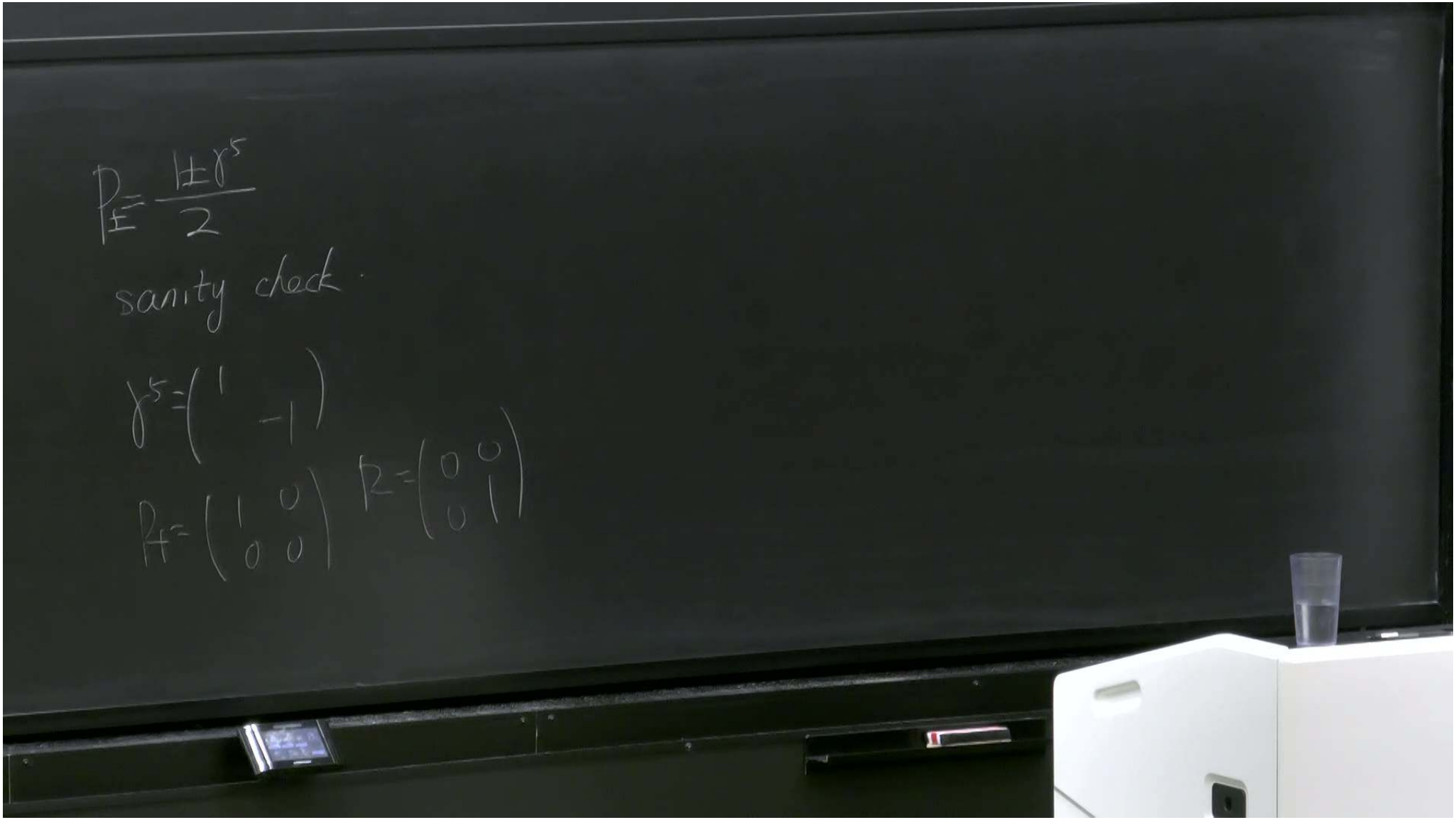
$P_1 + P_2 = 1 \quad P^2 = P \quad P_1 P_2 = 0$

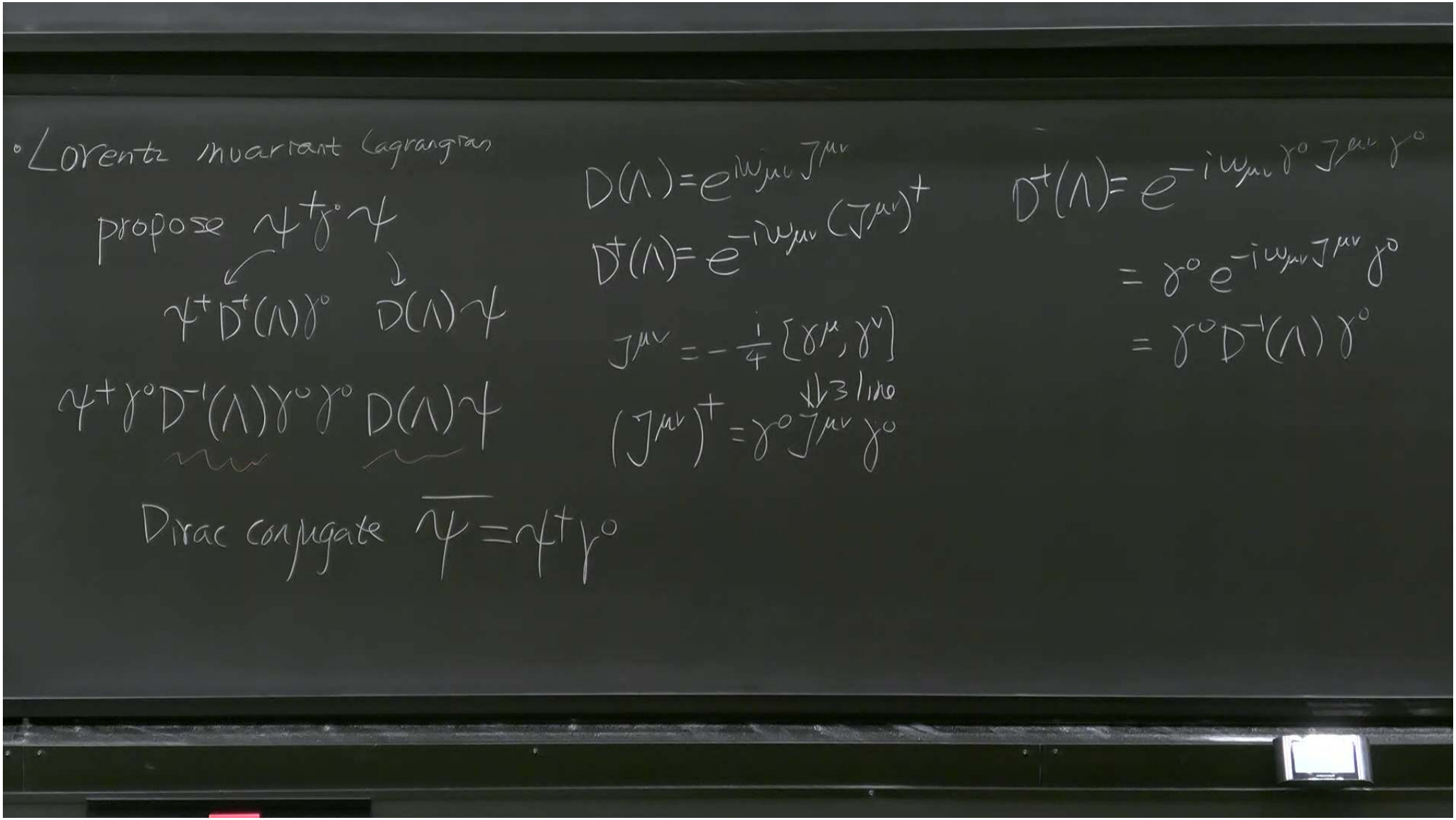
$[P, J^{\mu\nu}] = 0$

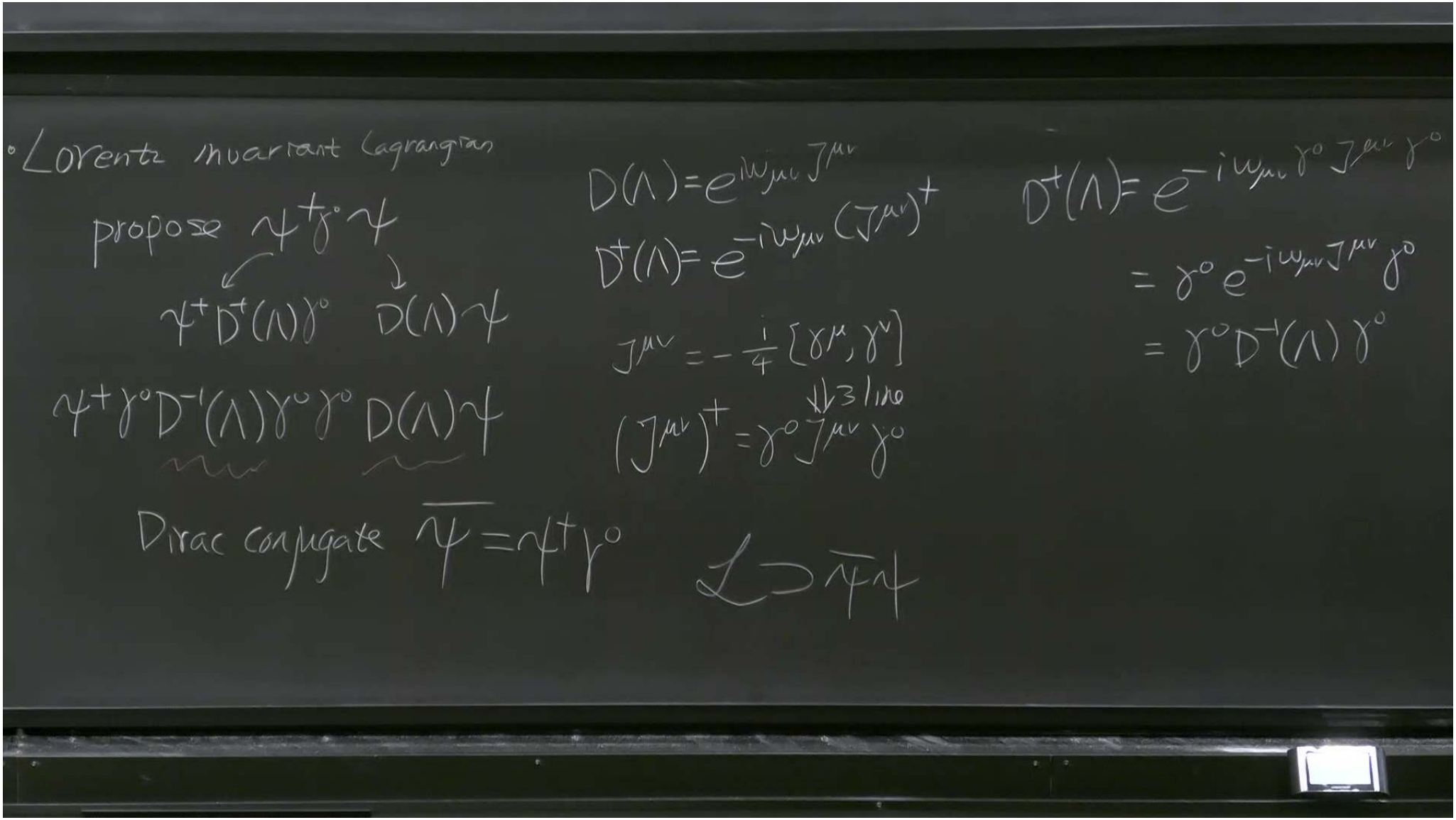
all $[P, \gamma^\mu] = 0$ or $\{P, \gamma^\mu\} = 0$

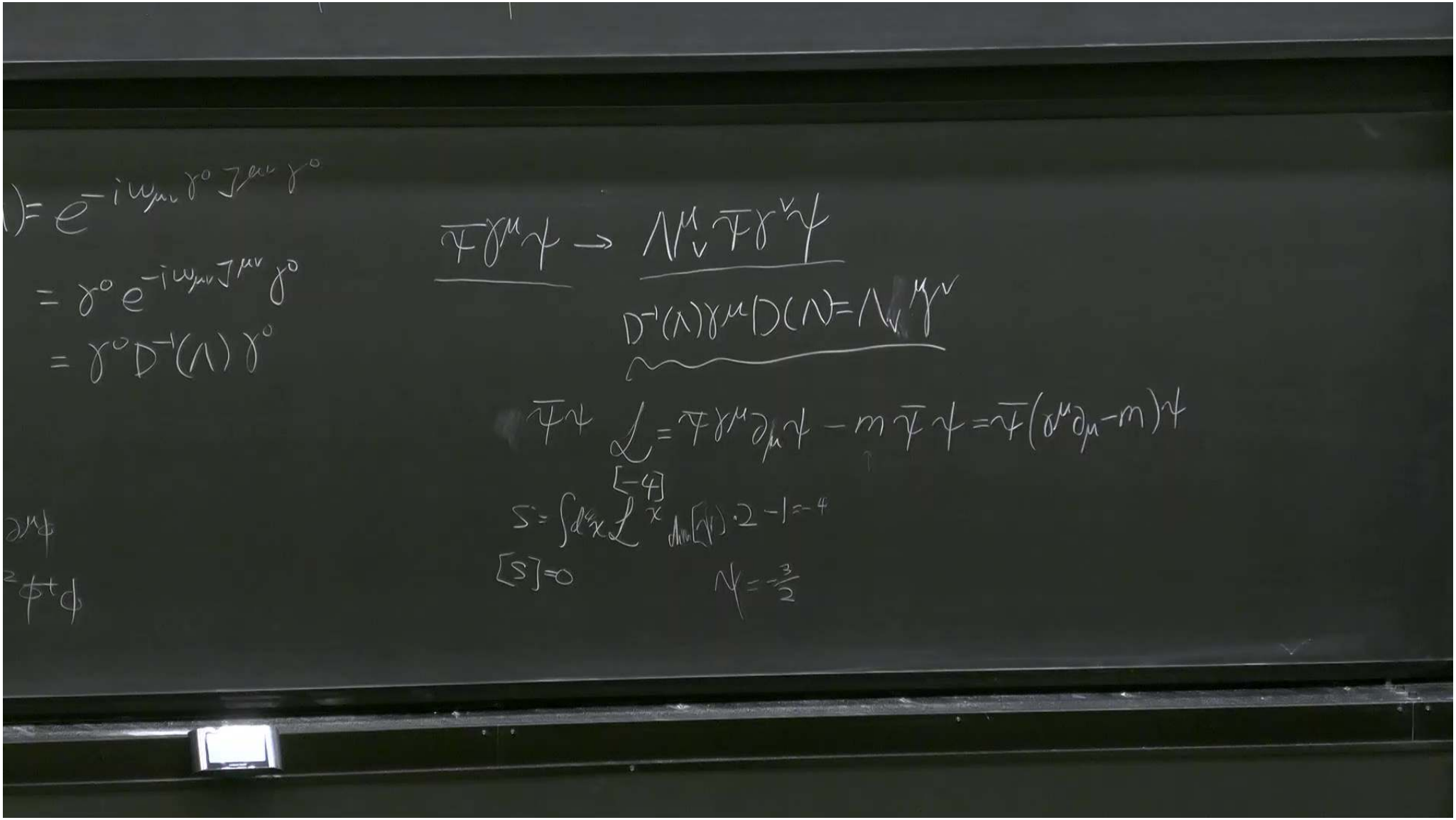
1 $\quad \gamma^5 = -i\gamma^0\gamma^1\gamma^2\gamma^3$

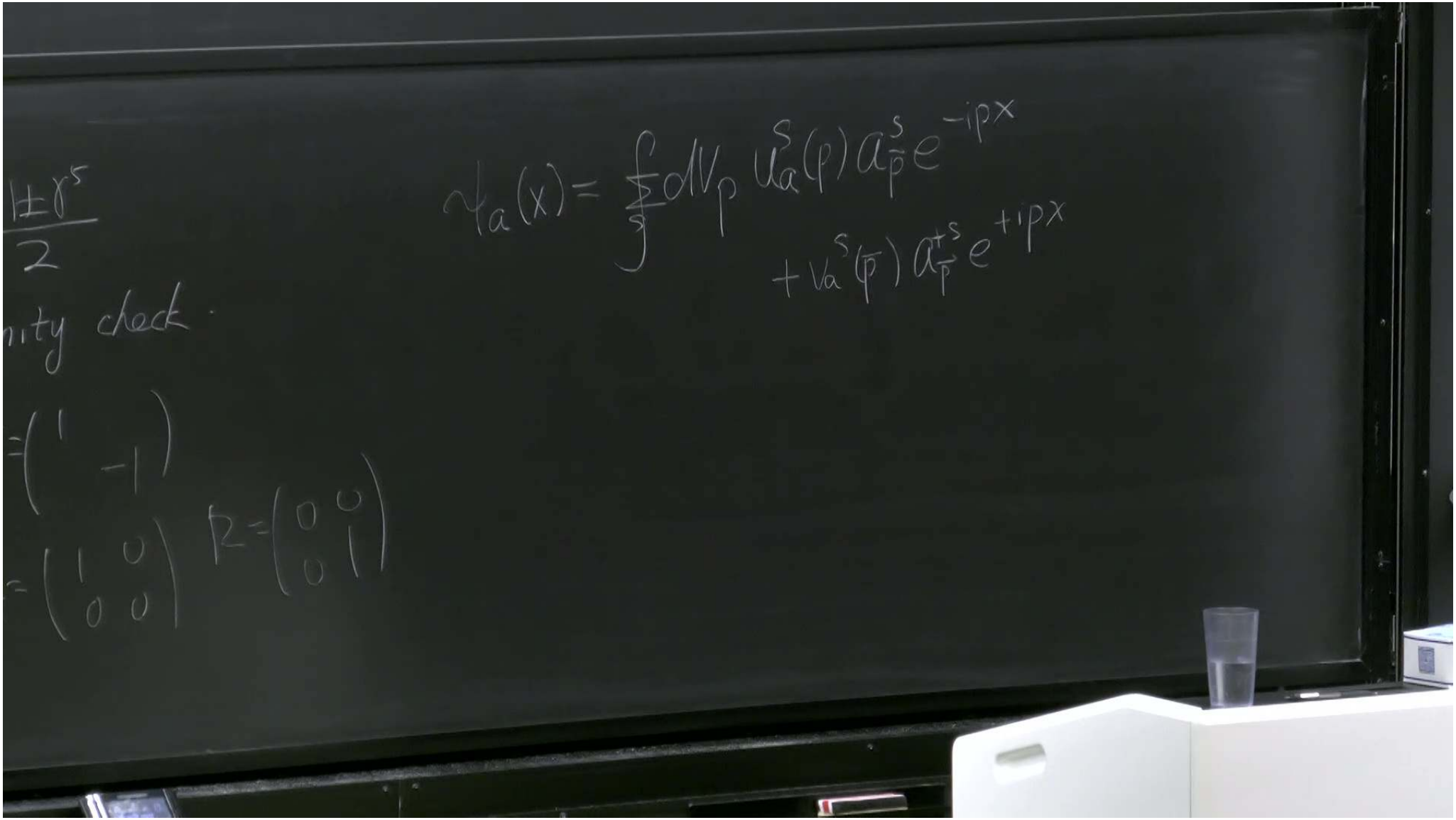
$(\gamma^5)^2 = 1$


$P_\pm = \frac{1 \pm \gamma^5}{2}$
sanity check.
$\gamma^5 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$
$P_L = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $P_R = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$


• Lorentz invariant Lagrangian
propose $\psi^\dagger \gamma^0 \psi$
$\psi^\dagger D^\dagger(\Lambda) \gamma^0$ $D(\Lambda)\psi$
$\psi^\dagger \gamma^0 D^{-1}(\Lambda) \gamma^0 \gamma^0 D(\Lambda) \psi$
$D(\Lambda) = e^{i\omega_{\mu\nu} J^{\mu\nu}}$
$D^\dagger(\Lambda) = e^{-i\omega_{\mu\nu} (J^{\mu\nu})^\dagger}$
$J^{\mu\nu} = -\frac{i}{4}[\gamma^\mu, \gamma^\nu]$
$(J^{\mu\nu})^\dagger = \gamma^0 J^{\mu\nu} \gamma^0$
$D^\dagger(\Lambda) = e^{-i\omega_{\mu\nu} \gamma^0 J^{\mu\nu} \gamma^0}$
$= \gamma^0 e^{-i\omega_{\mu\nu} J^{\mu\nu}} \gamma^0$
$= \gamma^0 D^{-1}(\Lambda) \gamma^0$
Dirac conjugate $\bar{\psi} = \psi^\dagger \gamma^0$


• Lorentz invariant Lagrangian
propose $\psi^\dagger \gamma^0 \psi$
$\psi^\dagger D^\dagger(\Lambda) \gamma^0$ $D(\Lambda)\psi$
$\psi^\dagger \gamma^0 D^{-1}(\Lambda) \gamma^0 \gamma^0 D(\Lambda)\psi$
$D(\Lambda) = e^{i\omega_{\mu\nu} J^{\mu\nu}}$
$D^\dagger(\Lambda) = e^{-i\omega_{\mu\nu} (J^{\mu\nu})^\dagger}$
$J^{\mu\nu} = -\frac{i}{4}[\gamma^\mu, \gamma^\nu]$
$(J^{\mu\nu})^\dagger = \gamma^0 J^{\mu\nu} \gamma^0$
$D^\dagger(\Lambda) = e^{-i\omega_{\mu\nu} \gamma^0 J^{\mu\nu} \gamma^0}$
$= \gamma^0 e^{-i\omega_{\mu\nu} J^{\mu\nu}} \gamma^0$
$= \gamma^0 D^{-1}(\Lambda) \gamma^0$
Dirac conjugate $\bar{\psi} = \psi^\dagger \gamma^0$
$\mathcal{L} \supset \bar{\psi}\psi$


$\bar{\psi}\gamma^\mu\psi \to \Lambda^\mu_\nu \bar{\psi}\gamma^\nu\psi$
$D^{-1}(\Lambda)\gamma^\mu D(\Lambda) = \Lambda^\mu_\nu \gamma^\nu$
$\mathcal{L} = \bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi = \bar{\psi}(\gamma^\mu\partial_\mu - m)\psi$
$[-4]$
$S = \int d^4x \mathcal{L}$
$[S] = 0$
$[\psi] = -\frac{3}{2}$


$\frac{1\pm\gamma^5}{2}$
nity clock.
$=\begin{pmatrix}1&\\&-1\end{pmatrix}$
$=\begin{pmatrix}1&0\\0&0\end{pmatrix}$ $R=\begin{pmatrix}0&0\\0&1\end{pmatrix}$
$\psi_a(x)=\sum_s\int dV_p\, u_a^s(p)\,a_p^s\,e^{-ipx}$
$+v_a^s(p)\,a_p^{\dagger s}\,e^{+ipx}$