Title: QFT1 Lecture - 101323

Speakers: Gang Xu

Collection: Quantum Field Theory 1 2023/24

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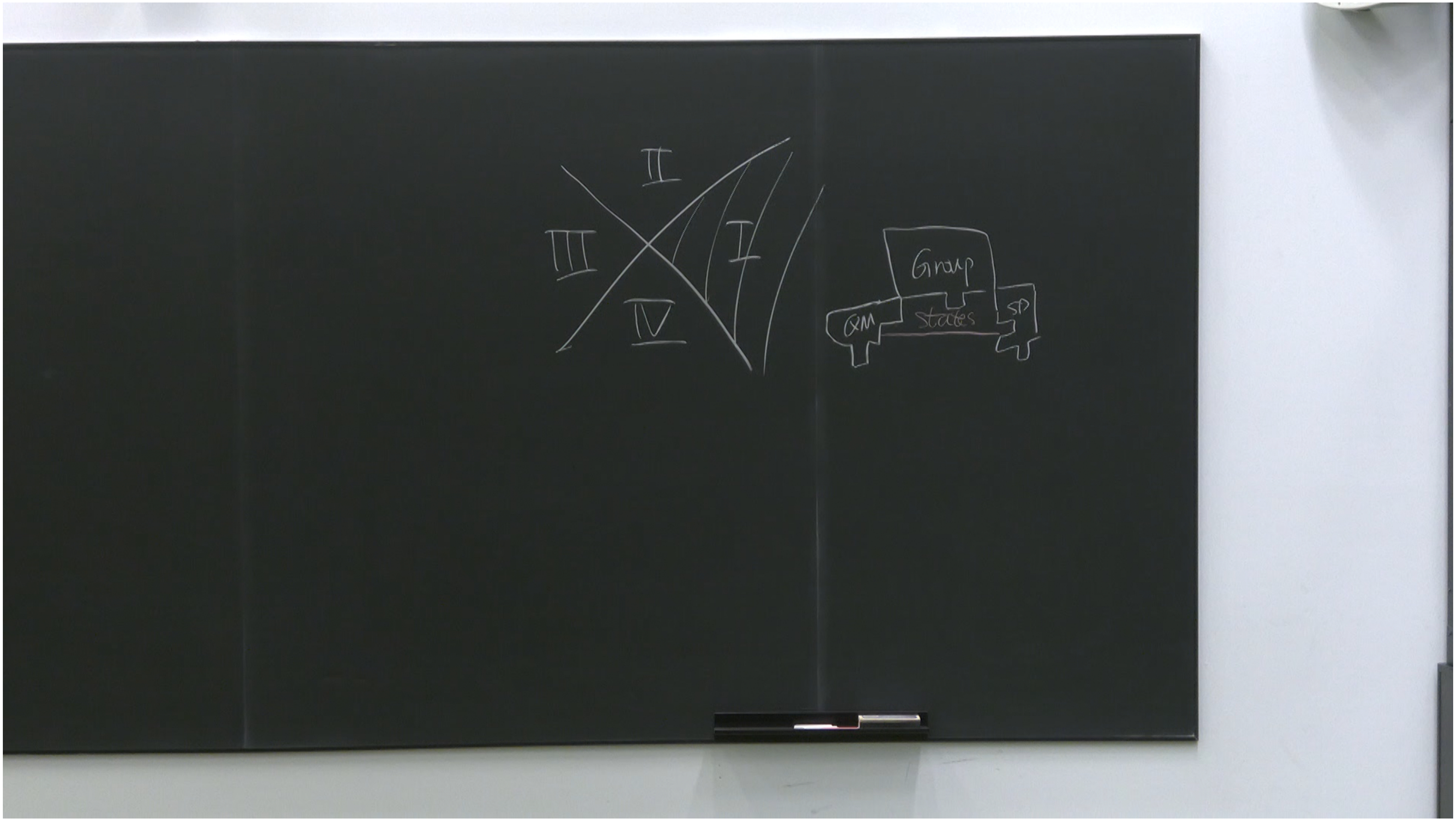


II
III
I
IV
Group
QM
states
SP

$O(B_1)$ is our only hope. Lie

$R^2 \otimes D_2$ is our only hope

$SU(2)$ is our only hope.

Lie

$SU(2) \sim SO(3) \subsetneq SO(1,3) \subset SO(1,3) \ltimes R^{1,3}$ semi product

$U(\Lambda, a) = U(1,a)U(\Lambda$

$[T_a, T_b] = i f_{abc} T_c$

SU(2) - unitary
compact
simple

real
completely
anti-symmetric

ε_{abc}

$m=2$ defining rep

$m=3$

Lie

$SU(2) \xrightarrow{2} SO(3) \subsetneq SO(1,3) \subset SO(1,3) \underset{\text{product}}{\overset{\text{semi}}{}} R^{1,3}$

$U(\Lambda, a) = U(1,a)\,U(\Lambda$

$[T_a, T_b] = i f_{abc} T_c$

SU(2) - unitary
compact
simple

real
completely
anti-symmetric
ε_{abc}

$m=2$ defining rep

$m=3$ rotation

$T_1 = \begin{pmatrix} & & \\ & & -i \\ & i & \end{pmatrix}$

$(T_1)_{23} = -i\varepsilon_{123}$

Recipe

$SU(2)$

1 maximally commuting $\qquad J_3$ special

2 eigenvector of special $\qquad [J_3, \bullet] = \text{const} \times \bullet$

$$\bullet ? = J_1 + a J_2$$

3 Highest weight method

$$J_\pm = \frac{1}{\sqrt{2}}(J_1 \pm i J_2)$$

eigenvalue

$$[J_3, J_\pm] = \pm J_\pm$$

$$O(3) \subsetneq SO(1,3) \subset SO(1,3) \ltimes_{\text{semi product}} \mathbb{R}^{1,3}$$

$$\boxed{U(\Lambda,a)\,|\psi\rangle = |\psi'\rangle}$$

$$U(\Lambda,a) = U(1,a)\,U(\Lambda,0)$$

$$] = i f_{abc} T_c$$

real

completely

anti-symmetric

$$\varepsilon_{abc}$$

$n=2$ defining rep

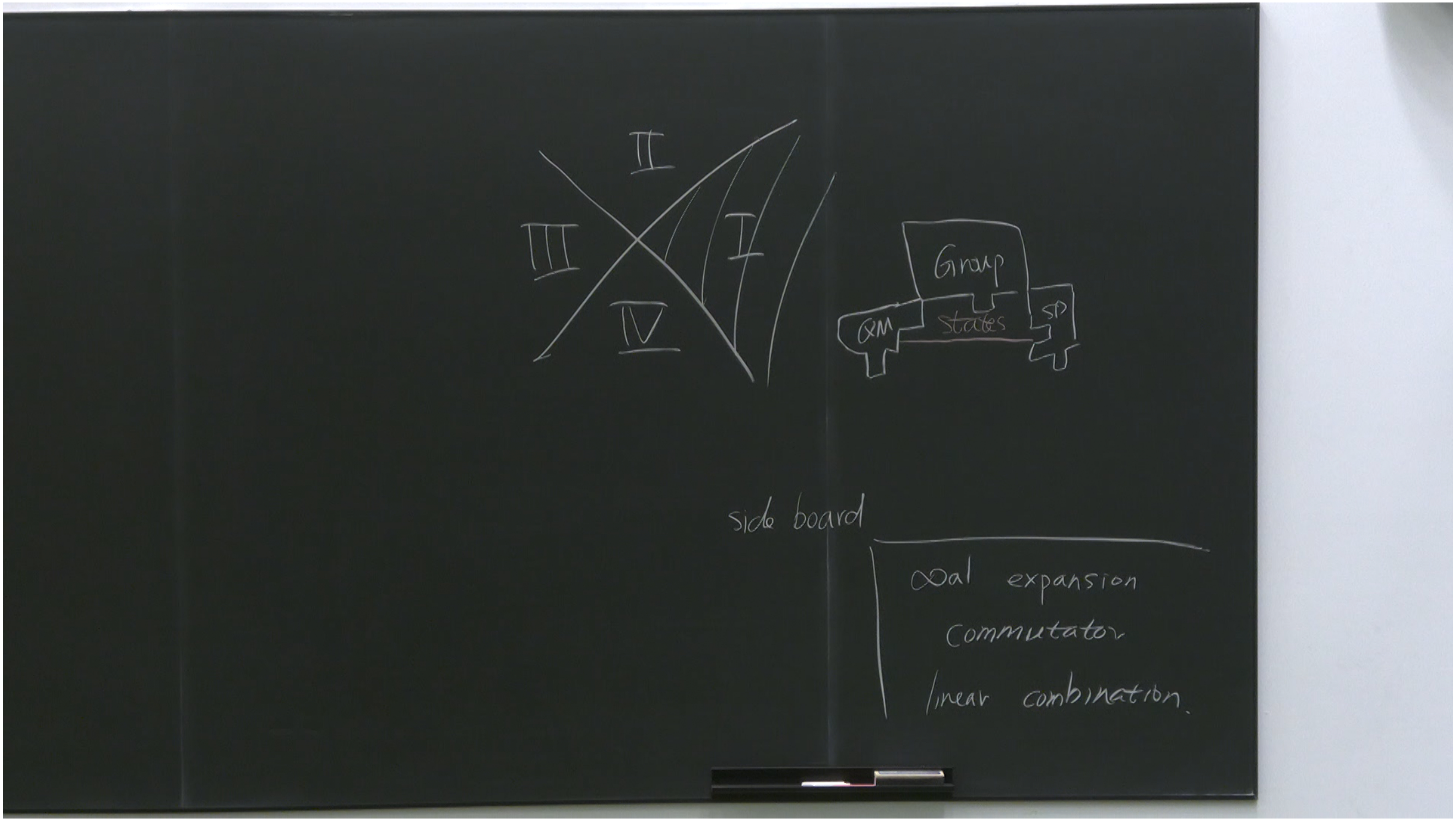
$n=3$ rotation

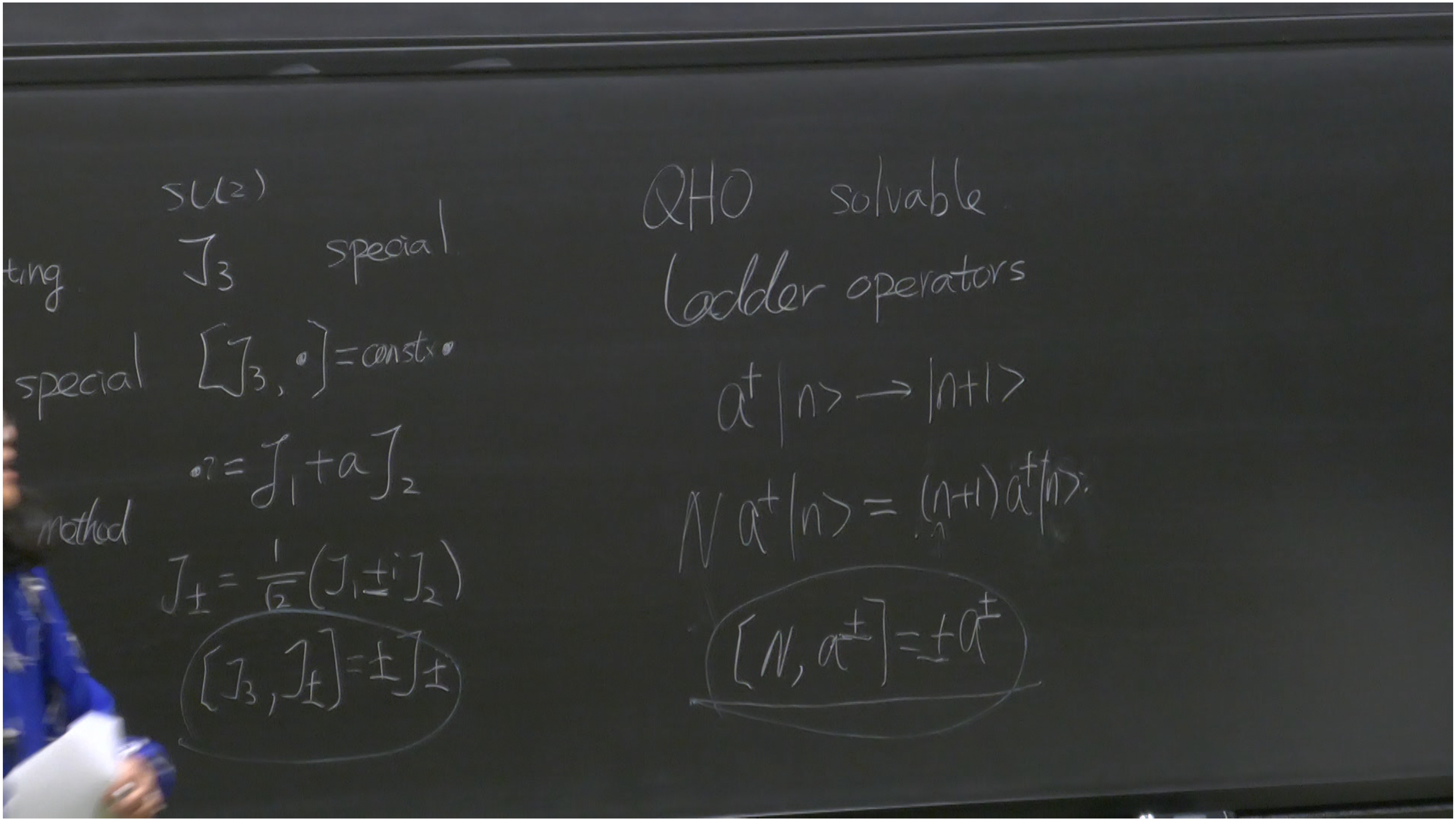
$$T_1 = \begin{pmatrix} & & \\ & & -i \\ & i & \end{pmatrix}$$

$$(T_1)_{23} = -i\varepsilon_{123}$$

$$[J_a, J_b] = i\epsilon_{abc} J_c$$

$$J_1 \quad J_2 \quad J_3$$


II
III
I
IV
Group
QM
states
SP
side board
∞al expansion
commutator
linear combination


SU(2)
ting
J_3 special
special $[J_3, \cdot] = \text{const} \times \cdot$
$\cdot = J_1 + a J_2$
method
$J_\pm = \frac{1}{\sqrt{2}}(J_1 \pm i J_2)$
$[J_3, J_\pm] = \pm J_\pm$
QHO solvable
ladder operators
$a^\dagger |n\rangle \to |n+1\rangle$
$N a^\dagger |n\rangle = (n+1) a^\dagger |n\rangle$
$[N, a^\pm] = \pm a^\pm$

$SU(2)$

ting. J_3 special

special $[J_3, \cdot\,] = \text{const} \times \cdot$

$\cdot\,? = J_1 + a J_2$

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$[J_3, J_\pm] = \pm J_\pm$

QHO solvable.

ladder operators

$a^\dagger |n\rangle \longrightarrow |n+1\rangle$

$N a^\dagger |n\rangle = (n+1) a^\dagger |n\rangle$

$[N, a^\pm] = \pm a^\pm$

Recipe

1 maximally commuting

2 eigenvector of special

3 Highest weight method

eigenvalue

$SU(2)$

J_3 special

$[J_3, \cdot] = \text{const} \cdot \cdot$

$\cdot ? = J_1 + a J_2$

$J_\pm = \frac{1}{\sqrt{2}}(J_1 \pm i J_2)$

$[J_3, J_\pm] = \pm J_\pm$

QHO solvable

ladder operators

$a^\dagger |n\rangle \longrightarrow |n+1\rangle$

$N a^\dagger |n\rangle = (n+1) a^\dagger |n\rangle$

$[N, a^\pm] = \pm a^\pm$

$\delta_{nn'}$

① $\hat{N}|n\rangle = n|n\rangle$

② $a^{\pm}|n\rangle \propto |n\pm 1\rangle$

③ $a|0\rangle = 0$ (bc)

SU(2)

①' $J_3|m\rangle = m|m\rangle$ $\quad \langle m'|m\rangle = \delta_{m'm}$

②' $J_\pm|m\rangle \longrightarrow |m\pm 1\rangle$

③ $J_+|j\rangle = 0$ (bc)

$J_-|m\rangle = N(m)\,|m-1\rangle$

$\langle m-1|J_-|m\rangle = N(m)$

$J_+|m\rangle = N(m+1)\,|m+1\rangle$

$\delta_{nn'}$

① $\hat{N}|n\rangle = n|n\rangle$

② $a^{\pm}|n\rangle \propto |n\pm 1\rangle$

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③' $J_+|j\rangle = 0$ (b.c)

$J_-|m\rangle = N(m)\,|m-1\rangle$

$\langle m-1|J_-|m\rangle = N(m)$

$J_+|m\rangle = N(m+1)|m+1\rangle$

$$[J_+, J_-] = J_3$$

$$\langle m | [J_+, J_-] | m \rangle = \langle m | J_3 | m \rangle$$

$$J_+ J_- - J_- J_+$$

$$| J_- |m\rangle |^2 - | J_+ |m\rangle |^2 = m$$

$$\begin{cases} N^2(m) - N^2(m+1) = m \\ N(j+1) = 0 \end{cases}$$

$$N(m) = \sqrt{(j+m)(j-m+1)}$$

$$[J_+, J_-] = J_3$$

$$\langle m | [J_+, J_-] | m \rangle = \langle m | J_3 | m \rangle$$

$$J_+ J_- - J_- J_+$$

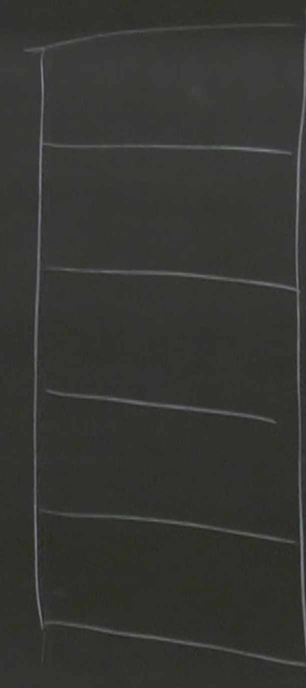
$$\left| J_- |m\rangle \right|^2 - \left| J_+ |m\rangle \right|^2 = m$$

$$\begin{cases} N^2(m) - N^2(m+1) = m \\ N(j+1) = 0 \end{cases}$$

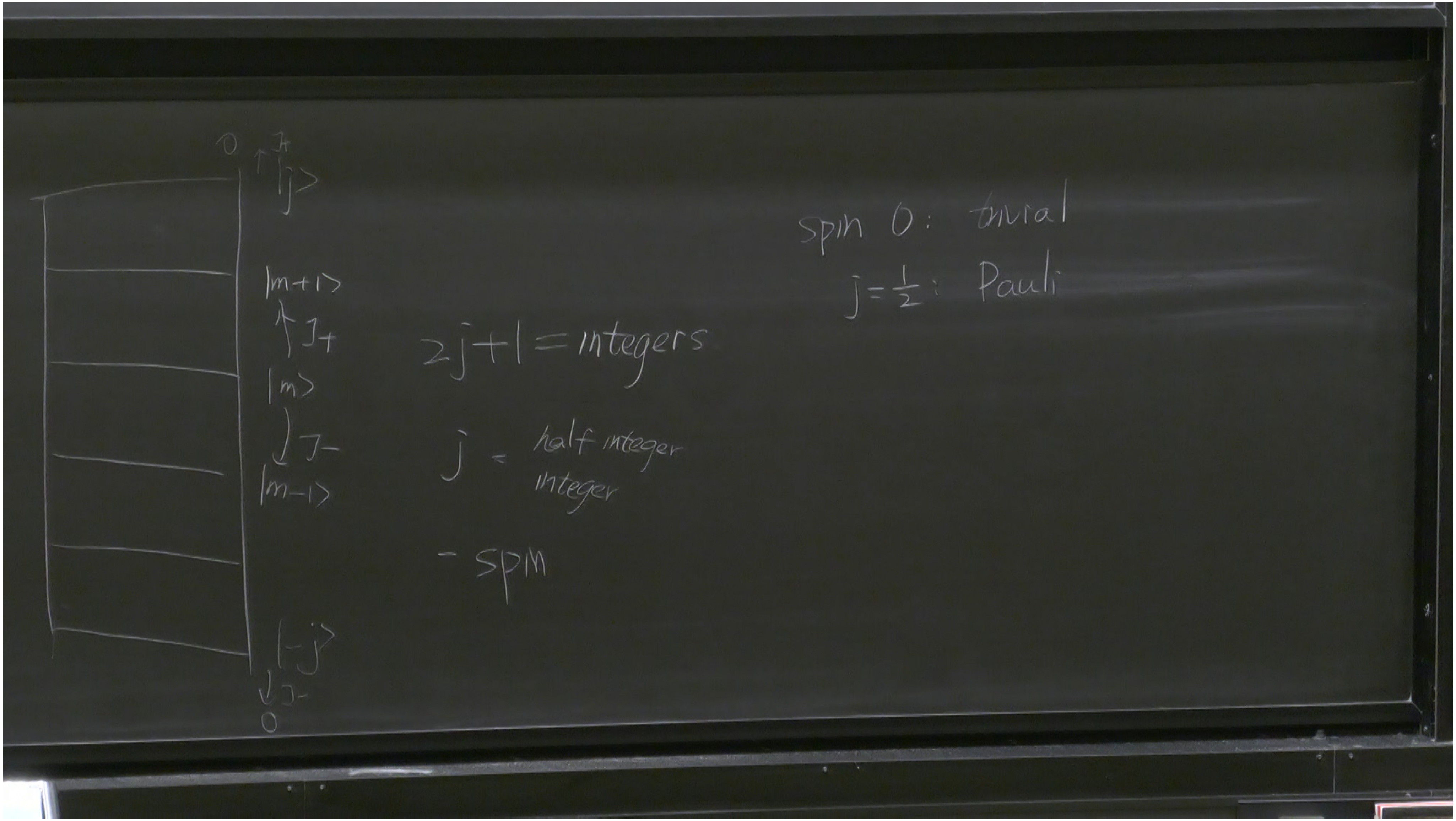
$$N(m) = \sqrt{(j+m)\,\underbrace{(j-m+1)}_{m=j+1}}$$

$m = -j$

$$J_- | -j \rangle = 0$$



$|m\rangle$



0
J+
|j>
|m+1>
J+
|m>
J-
|m-1>
|-j>
J-
0
2j+1 = integers
j = half integer
integer
- spin
spin 0: trivial
j = 1/2: Pauli

$$[J_+, J_-] = J_3$$

$$\langle m | [J_+, J_-] | m \rangle = \langle m | J_3 | m \rangle$$

$$J_+ J_- - J_- J_+$$

$$\big| J_- |m\rangle \big|^2 - \big| J_+ |m\rangle \big|^2 = m$$

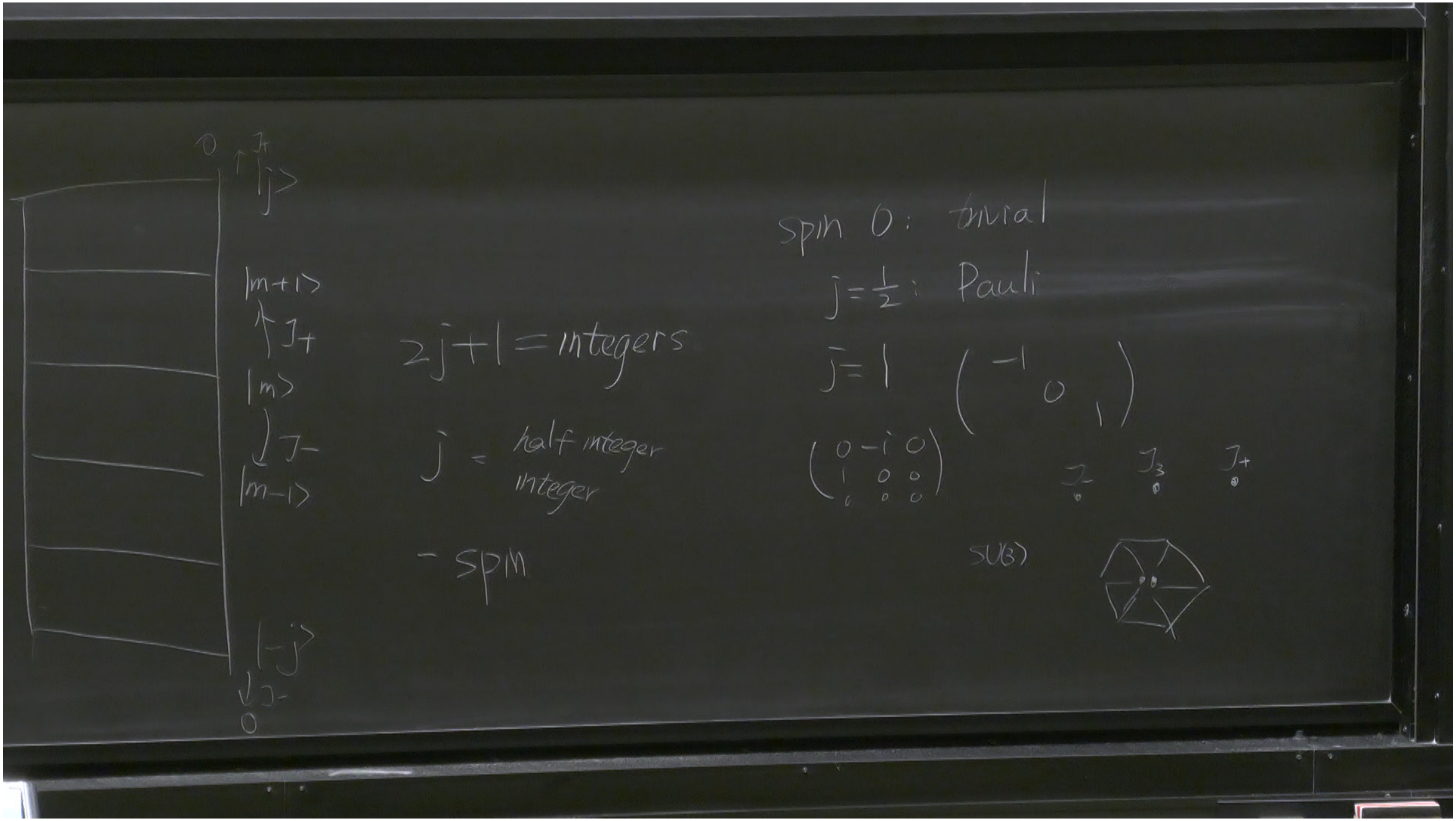
$$\begin{cases} N^2(m) - N^2(m+1) = m \\ N(j+1) = 0 \end{cases}$$

$$N(m) = \sqrt{\underset{m=-j}{(j+m)}\,\underbrace{(j-m+1)}_{m=j+1}}$$

$$J_- | -j \rangle = 0$$

$$(J_+)_{m'm} = \langle m' | J_+ | m \rangle$$

$$= N(m+1)\, \delta_{m', m+1}$$


|j>
|m+1>
|m>
|m-1>
|-j>
2j+1 = integers
j = half integer
integer
- spin
spin 0: trivial
j = 1/2: Pauli
j = 1
J-
J3
J+
SU(3)

$U(\varepsilon,\omega)|\psi\rangle = |\psi'\rangle$

$U(\varepsilon,\omega) = e^{i\epsilon_\mu P^\mu + \frac{1}{2}\omega_{\mu\nu}J^{\mu\nu}}$

non-abelian

non-compact

not-simple

not-semi-simple

oh crap!

$[J_i, P_j]$

$= \epsilon_{ijk} P_k$

$[P^\mu, P^\nu] = 0$

quantum state

$|p, s_{\ldots}\rangle$

$U(1,a)|p,s\rangle = e^{i a_\mu p^\mu}|p,s\rangle$

$U(\Lambda, a) = U(1,a)U(\Lambda,0)$

$U(\Lambda)|p,s\rangle$

$$U(\Lambda)|p,s\rangle = \sum_{s'} C_{s's}(\Lambda p)\,|\Lambda p, s'\rangle$$

$$\Lambda p = p$$

new name

$$\underbrace{W}_{\text{little group}}{}^{\mu}{}_{\nu}\, k^{\nu} = k^{\mu}$$

$$U(W)|k,s\rangle = \underbrace{D_{s's}(W)}\,|k,s'\rangle$$

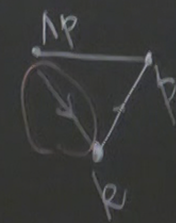
$$U(W_1)\,U(W_2) = U(W_1 W_2)$$

$\Rightarrow$ $D_{s's}(W)$ is rep of W

$$k^{\mu} = \begin{pmatrix} m \\ 0 \\ 0 \\ 0 \end{pmatrix} \longrightarrow SO(3) \to SU(2)$$

$|p,s\rangle$: $\quad p^\mu = L^\mu{}_\nu(p)\, k^\nu$

$$|p,s\rangle \equiv U(L(p))\,|k,s\rangle$$

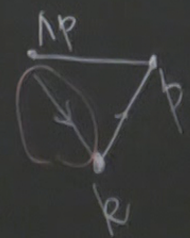


$$U(\Lambda)\,|p,s\rangle = U(\Lambda)\,U(L(p))\,|k,s\rangle$$

$$= U(L(\Lambda p))\,U(L^{-1}(\Lambda p))\,U(\Lambda)\,U(L(p))\,|k,s\rangle$$

$$= U(L(\Lambda p))\,U(W(\Lambda,p))\,|k,s\rangle$$

$$p^\mu = L^\mu{}_\nu(p)\, k^\nu$$



$$|p,s\rangle \equiv U(L(p))\,|k,s\rangle$$

$$|p,s\rangle = U(\Lambda)\,U(L(p))\,|k,s\rangle$$

$$= U(L(\Lambda p))\,U(L^{-1}(\Lambda p))\,U(\Lambda)\,U(L(p))\,|k,s\rangle$$

$$L^{-1}(\Lambda p)\,\Lambda\, L(p)$$

$$\Downarrow$$

$$= U(L(\Lambda p))\,U(W(\Lambda,p))\,|k,s\rangle$$

$$= \sum_{s'} D_{s's}(W(\Lambda,p))\,U(L(\Lambda p))\,|k,s'\rangle$$

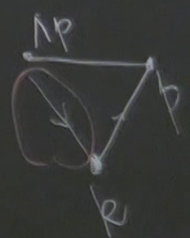
$$U(\Lambda)\,|p,s\rangle = \sum_{s'} D_{s's}(W(\Lambda,p))\,|\Lambda p, s'\rangle$$

$$L^{-1}(\Lambda p)\,\Lambda\, L(p)$$

$$\Downarrow$$

$$= U(L(\Lambda p))\; U(W(\Lambda,p))\,|k,s\rangle$$

$$)\,|k,s\rangle$$



$$= \sum_{s'} D_{s's}(W(\Lambda,p))\; U(L(\Lambda p))\,|k,s'\rangle$$

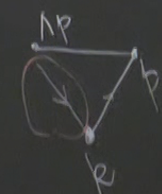
$$p))\,|k,s\rangle$$

$$U(\Lambda)\,U(L(p))\,|k,s\rangle \qquad U(\Lambda)\,|p,s\rangle = \sum_{s'} D_{s's}(W(\Lambda,p))\,|\Lambda p,s'\rangle$$

$$|\psi'\rangle = U|\psi\rangle$$

$|p,s\rangle$ $\quad p^\mu = L^\mu{}_\nu(p)\, k^\nu$

$$|p,s\rangle \equiv U(L(p))\,|k,s\rangle$$



$$U(\Lambda)\,|p,s\rangle = U(\Lambda)\,U(L(p))\,|k,s\rangle$$

$$= U(L(\Lambda p))\,U(L^{-1}(\Lambda p))\,U(\Lambda)\,U(L(p))\,|k,s\rangle$$

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$$= U(L(\Lambda p))\,U(W(\Lambda,p))\,|k,s\rangle$$

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$$|\psi'\rangle = U\,|\psi\rangle$$