Title: Relativity Lecture - 103023

Speakers: David Kubiznak

Collection: Relativity 2023/24

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Applications, <u>Gravitational Waves</u>

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \qquad |h_{\mu\nu}| \ll 1$$

$$x^\mu \to x^\mu - \xi^\mu \qquad \delta g_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu$$

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$$

$$h \to h + 2\,\partial^\mu \xi_\mu$$

$$\partial^\mu h_{\mu\nu} - \ldots$$

$$\underbrace{\partial^\mu h_{\mu\nu} - \tfrac{1}{2}\partial_\nu h} + \partial^\mu\partial_\mu \xi_\nu + \cancel{\partial^\mu\partial_\nu \xi_\mu} - \cancel{\partial_\nu \partial^\mu \xi_\nu}$$

$$\Box\left(h_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu} h\right) = -16\pi G T_{\mu\nu}$$

$$\underbrace{\partial^\mu h_{\mu\nu} - \frac{1}{2}\partial_\nu h} + \partial^\mu \partial_\mu \xi_\nu + \cancel{\partial^\mu \partial_\nu \xi_\mu} - \cancel{\partial_\nu \partial^\mu \xi_\nu} = 0$$

$$\Box\left(h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h\right) = -16\pi G T_{\mu\nu}$$

$$\underbrace{\partial^\mu h_{\mu\nu} - \tfrac{1}{2}\partial_\nu h} + \partial^\mu\partial_\mu \xi_\nu + \partial^\mu \cancel{\partial_\nu \xi_\mu} - \cancel{\partial_\nu \partial^\mu \xi_\mu} = 0$$

$$\Box \xi_\mu = -(\partial^\mu h^{old}_{\mu\nu} - \tfrac{1}{2}\partial_\nu h^{old})$$

$$\Box(\underbrace{h_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu} h}_{\bar{h}_{\mu\nu}}) = -16\pi G T_{\mu\nu}$$

$$\bar{h} = -h$$

$$\Box \bar{h}_{\mu\nu} = -16\pi G$$

$$\partial^\mu h^{new}_{\mu\nu} - \frac{1}{2}\partial_\nu h^{new} = 0$$

$$\partial^\mu \bar{h}_{\mu\nu} = 0$$

Harmonic / de Donder gauge

(Transverse)

$$\Updownarrow$$

$$\Gamma^\mu_{\alpha\beta}\, g^{\alpha\beta} = 0 \iff \nabla_\alpha \nabla^\alpha x^\mu = 0$$

Vacuum: $T_{\mu\nu} = 0$

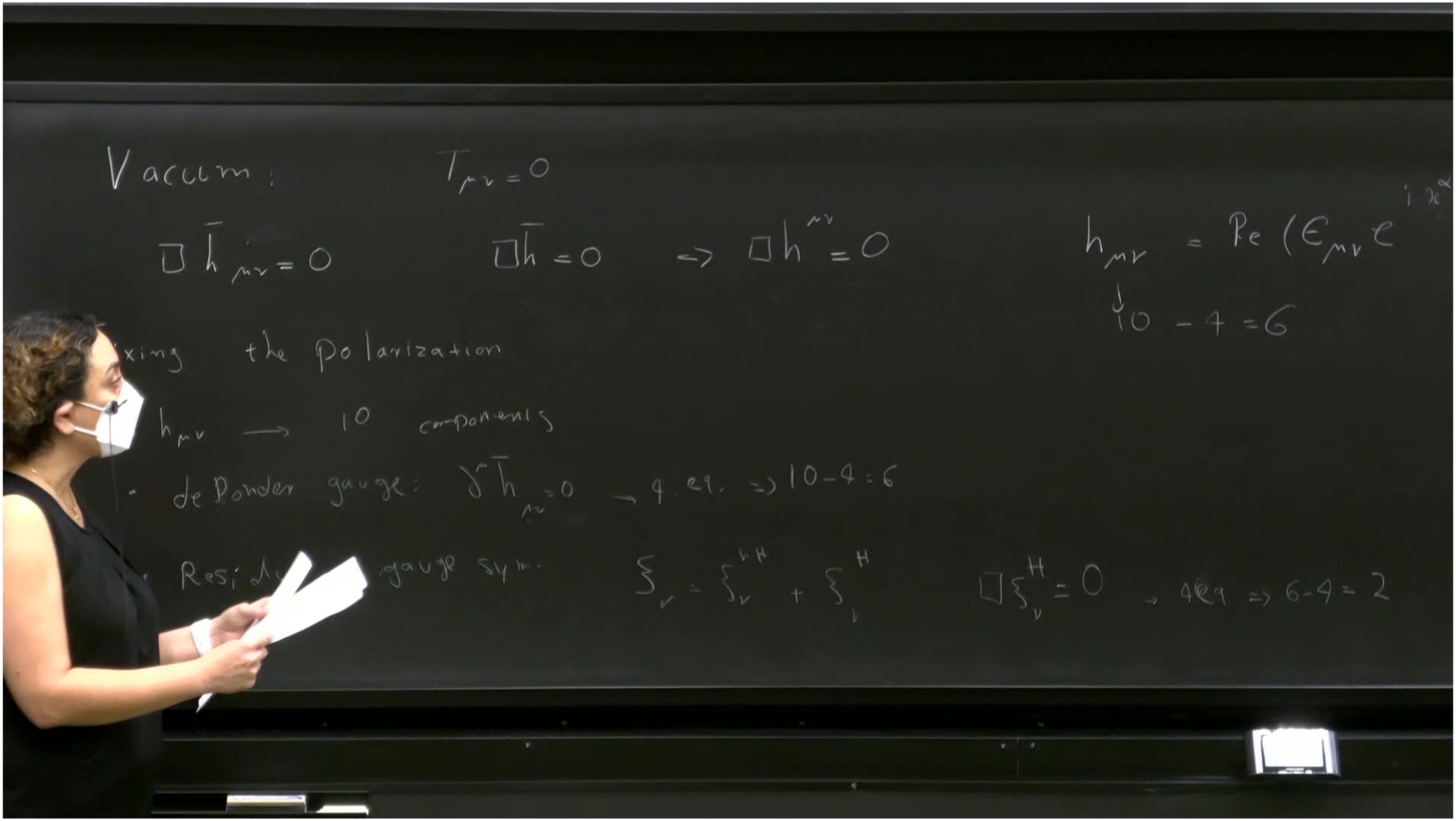
$\Box \bar{h}_{\mu\nu} = 0$ $\quad \Box \bar{h} = 0 \quad \Rightarrow \quad \Box h^{\mu\nu} = 0$

$h_{\mu\nu} = \mathrm{Re}\,(\epsilon$

$\downarrow$
10

fixing the polarization

- $h_{\mu\nu} \longrightarrow 10$ components
- de Donder gauge: $\partial^\mu \bar{h}_{\mu\nu} = 0 \rightarrow 4$ eq. $\Rightarrow 10 - 4 = 6$
- Residual gauge sym.


Vacuum: $T_{\mu\nu}=0$
$\Box \bar{h}_{\mu\nu}=0$
$\Box \bar{h}=0 \Rightarrow \Box h^{\mu\nu}=0$
$h_{\mu\nu} = Re(\epsilon_{\mu\nu} e^{i x^\alpha}$
$10-4=6$
Fixing the polarization
$h_{\mu\nu} \longrightarrow 10$ components
de Donder gauge: $\partial^\mu \bar{h}_{\mu\nu}=0 \to 4$ eq. $\Rightarrow 10-4=6$
Residual gauge sym. $\xi_\nu = \xi_\nu^{nH} + \xi_\nu^{H}$
$\Box \xi_\nu^{H}=0 \to 4$ eq $\Rightarrow 6-4=2$

$$h_{\mu\nu} = \mathrm{Re}\left(\epsilon_{\mu\nu}\, e^{i x^\alpha k_\alpha}\right) \quad ; \quad k^\mu k_\mu = 0 \qquad k^\mu = (\omega, \vec{k})$$

$$-\omega^2 + \vec{k}\cdot\vec{k} = 0 \qquad \omega = \pm |\vec{k}|$$

$\downarrow$

$10 - 4 = 6$

$0 \to 4\,\mathrm{eq} \Rightarrow 6 - 4 = 2 \to \xi^H_\mu = \tilde{\xi}_\mu e^{i x^\alpha k_\alpha}$

$$h_{\mu\nu} = \mathrm{Re}\left(\epsilon_{\mu\nu}\, e^{i x^\alpha k_\alpha}\right) \quad ; \quad k^\mu k_\mu = 0 \qquad k^\mu = (\omega, \vec{k})$$

$$\downarrow$$

$$10 - 4 = 6 \qquad -\omega^2 + \vec{k}\cdot\vec{k} = 0 \qquad \omega = \pm |\vec{k}|$$

$$= 0 \;\rightarrow\; 4\,\text{eq} \Rightarrow 6 - 4 = 2 \;\rightarrow\; \xi_\mu = \tilde{\xi}_\mu\, e^{i x^\alpha k_\alpha} \;\rightarrow \begin{cases} \epsilon \rightarrow \epsilon + 2ik^\mu \xi_\mu \\ \epsilon_{\mu\nu} \rightarrow \epsilon_{\mu\nu} + ik_\mu \tilde{\xi}_\nu + ik_\nu \tilde{\xi}_\mu \end{cases} \qquad \epsilon = \epsilon^\mu{}_\mu$$

- de Donder gauge: $\partial^\mu h_{\mu\nu} = 0 \rightarrow 4$ eq. $\Rightarrow 10-4=6$

- Residual gauge sym. $\xi_\nu = \xi_\nu^{nH} + \xi_\nu^{H}$ $\quad \Box \xi_\nu^H = 0$, 4 eq $\Rightarrow 6-4=2 \longrightarrow \xi^H_\mu = \tilde{\xi}_\mu e^{i x^\alpha k_\alpha}$

$$\begin{bmatrix} \frac{1}{2}\epsilon \\ \epsilon_{01} \\ \epsilon_{02} \\ \epsilon_{03} \end{bmatrix} + i \begin{bmatrix} \omega & k_1 & k_2 & k_3 \\ k_1 & -\omega & 0 & 0 \\ k_2 & 0 & -\omega & 0 \\ k_3 & 0 & 0 & -\omega \end{bmatrix} \begin{bmatrix} \tilde{\xi}_0 \\ \tilde{\xi}_1 \\ \tilde{\xi}_2 \\ \tilde{\xi}_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \epsilon_{\mu\nu} = \begin{bmatrix} \epsilon_{00} & 0 & 0 & 0 \\ 0 & & & \\ 0 & & & \\ 0 & & & \end{bmatrix}$$

$\epsilon = 0$

Transverse-Traceless

$$k^0 = \omega$$

$$i k^\mu \epsilon_{\mu\nu} - \frac{1}{2} i k^\nu \epsilon = 0$$

$$\frac{1}{2}\epsilon \rightarrow \frac{1}{2}\epsilon + \underbrace{i\left(k^0 \xi_0 + k^1 \xi_1 + k^2 \xi_2 + k^3 \xi_3\right)}_{i \begin{bmatrix} \omega & k^1 & k^2 & k^3 \end{bmatrix} \begin{bmatrix} \xi_0 \\ \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix}}$$

$k^0 = \omega$

$$i k^\mu \epsilon_{\mu\nu} - \frac{1}{2} \cancel{i k^\nu \epsilon} = 0$$

$$\nu = 0 \Rightarrow i k^\mu \epsilon_{\mu 0} = 0 \Rightarrow \underbrace{k^0}_{\omega \neq 0} \epsilon_{00} = 0 \Rightarrow \epsilon_{00} = 0$$

$$k^0 = \omega$$

$$i k^\mu \epsilon_{\mu\nu} - \frac{1}{2} \cancel{i k^\nu \epsilon} = 0$$

$$\nu = 0 \Rightarrow i k^\mu \epsilon_{\mu 0} = 0 \Rightarrow \underset{\omega \neq 0}{k^0} \epsilon_{00} = 0 \Rightarrow \epsilon_{00} = 0$$

$$\epsilon_{ii} = 0 \qquad k^\mu = (k, k, 0, 0)$$

$$k^\mu \epsilon_{\mu i} = 0 \Rightarrow k^0 \cancel{\epsilon_{0i}} + k^1 \epsilon_{1i} = 0 \Rightarrow \epsilon_{1i} = 0$$

Applications, Gravitational waves

$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \qquad |h_{\mu\nu}| \ll 1$

$x^\mu \to x^\mu - \xi^\mu \qquad \delta g_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu$

$h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$

$h \to h + 2\,\partial^\mu \xi_\mu$

$x'^\mu = x^\mu - \xi^\mu \qquad g'_{\mu\nu}(x'^\alpha) = \eta_{\mu\nu}(x^\alpha) + h'_{\mu\nu}(x')$

$$\underbrace{\partial^\mu h_{\mu\nu} - \tfrac{1}{2}\partial_\nu h}_{} + \partial^\mu\partial_\mu \xi_\nu + \cancel{\partial^\mu \partial_\nu \xi_\mu} - \cancel{\partial_\nu \partial^\mu \xi_\mu} = 0$$

$$\Box(\underbrace{h_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu} h}_{\bar h_{\mu\nu}}) = -16\pi G T_{\mu\nu} \qquad \bar h = -h$$

$$\Box \bar h_{\mu\nu} = -16\pi G$$

$$\Box \xi_\mu = -(\partial^\mu h^{old}_{\mu\nu} - \tfrac{1}{2}\partial_\nu h^{old}$$

$$\boxed{\partial^\mu h^{new}_{\mu\nu} - \tfrac{1}{2}\partial_\nu h^{new} = 0}$$

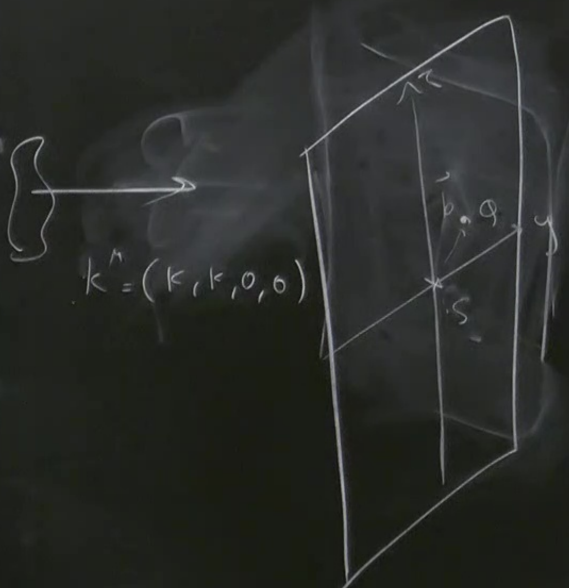
$$\boxed{\partial^\mu \bar h_{\mu\nu} = 0}$$

Harmonic / de Do
gauge
(Transverse)

$$\Updownarrow$$

$$\Gamma^\mu_{\alpha\beta}\, g^{\alpha\beta} = 0 \iff \nabla_\alpha \nabla^\alpha x^\mu = 0$$

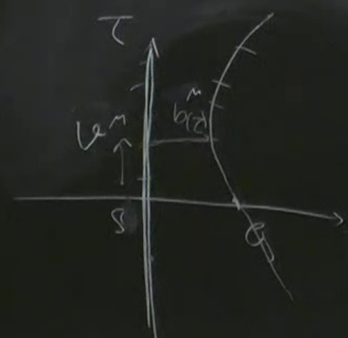
$$ds^2 = -dt^2 + dx^2 + (1 + \underbrace{\epsilon_+ e^{ik^\alpha x_\alpha}}_{\mathrm{Re}})\, dy^2 + (1 - \epsilon_+ e^{ik^\alpha x_\alpha})\, dz^2 + 2\epsilon_\times e^{ik^\alpha x_\alpha}\, dy\, dz$$



$k^\mu = (k, k, 0, 0)$

$$h_{22} \to \epsilon_+ \cos(\omega t - kx)$$

$$h_{33} \to \epsilon_+ \cos(\omega t - kx + \pi)$$



$$b^\mu = \delta x^\mu$$

$$v^\mu = \frac{dx^\mu}{d\tau} = (1, 0, 0, 0)$$

$$\frac{dx^i}{d\tau} = 0 \qquad \frac{dt}{d\tau} = 1$$

geodesic deviation Eq:

$$\frac{D^2 b^\mu}{D\tau^2} = -R^\mu{}_{\underset{\substack{\uparrow \\ 0}}{\nu}\rho\underset{\substack{\uparrow \\ 0}}{\sigma}}\, v^\nu v^\sigma b^\rho = -R^\mu{}_{0\rho\sigma}\, b^\rho$$

geodesic deviation Eq:

$$\frac{D^2 b^\mu}{D\tau^2} = -R^\mu{}_{\underset{\uparrow \atop 0}{\nu}\rho\underset{\uparrow \atop 0}{\sigma}}\, v^\nu v^\sigma b^\rho = -R^\mu{}_{0\rho\sigma}\, b^\rho$$

$$R^\mu{}_{0\rho 0} = \frac{1}{2}\eta^{\mu\lambda}\left(\partial_\rho \partial_0 \cancel{h_{0\lambda}} - \partial_\rho \partial_\lambda \cancel{h_{00}} \boxed{- \partial_0^2 h_{\rho\lambda}} + \partial_0 \partial_\lambda \cancel{h_{0\rho}}\right)$$

$$\frac{d^2 b^\mu}{dt^2} = +\frac{1}{2}\frac{\partial^2 h^\mu{}_\rho}{\partial t^2}\, b^\rho$$

geodesic deviation Eq:

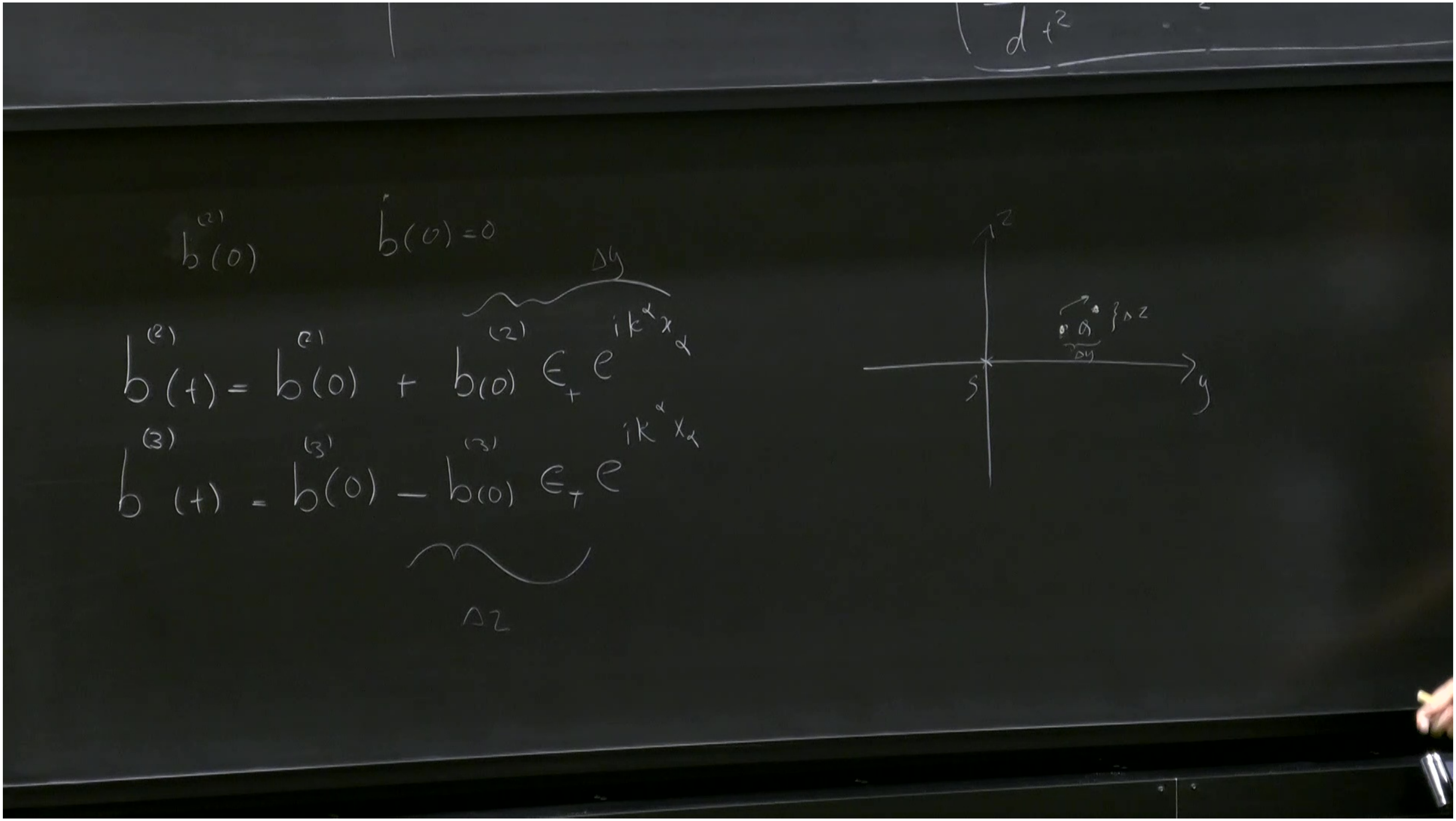
$$\frac{D^2 b^\mu}{D\tau^2} = -R^\mu{}_{\underset{0}{\nu}\rho\underset{0}{\sigma}} v^\nu v^\sigma b^\rho = -R^\mu{}_{0\rho\sigma} b^\rho$$

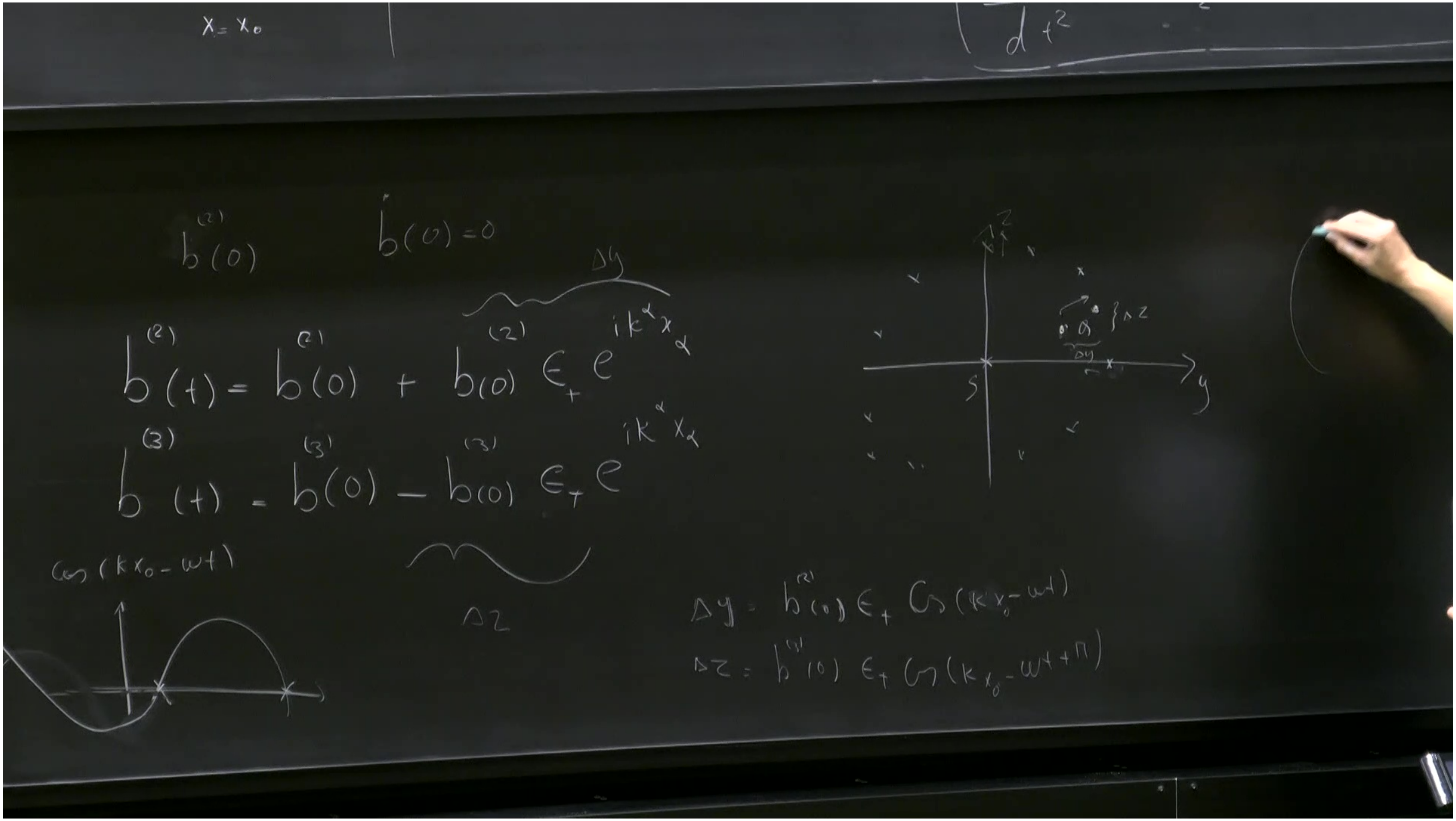
$$R^\mu{}_{0\rho 0} = \frac{1}{2}\eta^{\mu\lambda}\left(\partial_\rho \partial_0 \cancel{h_{0\lambda}} - \partial_\rho \partial_\lambda \cancel{h_{00}} \boxed{- \partial_0^2 h_{\rho\lambda}} + \partial_0 \partial_\lambda \cancel{h_{0\rho}}\right)$$

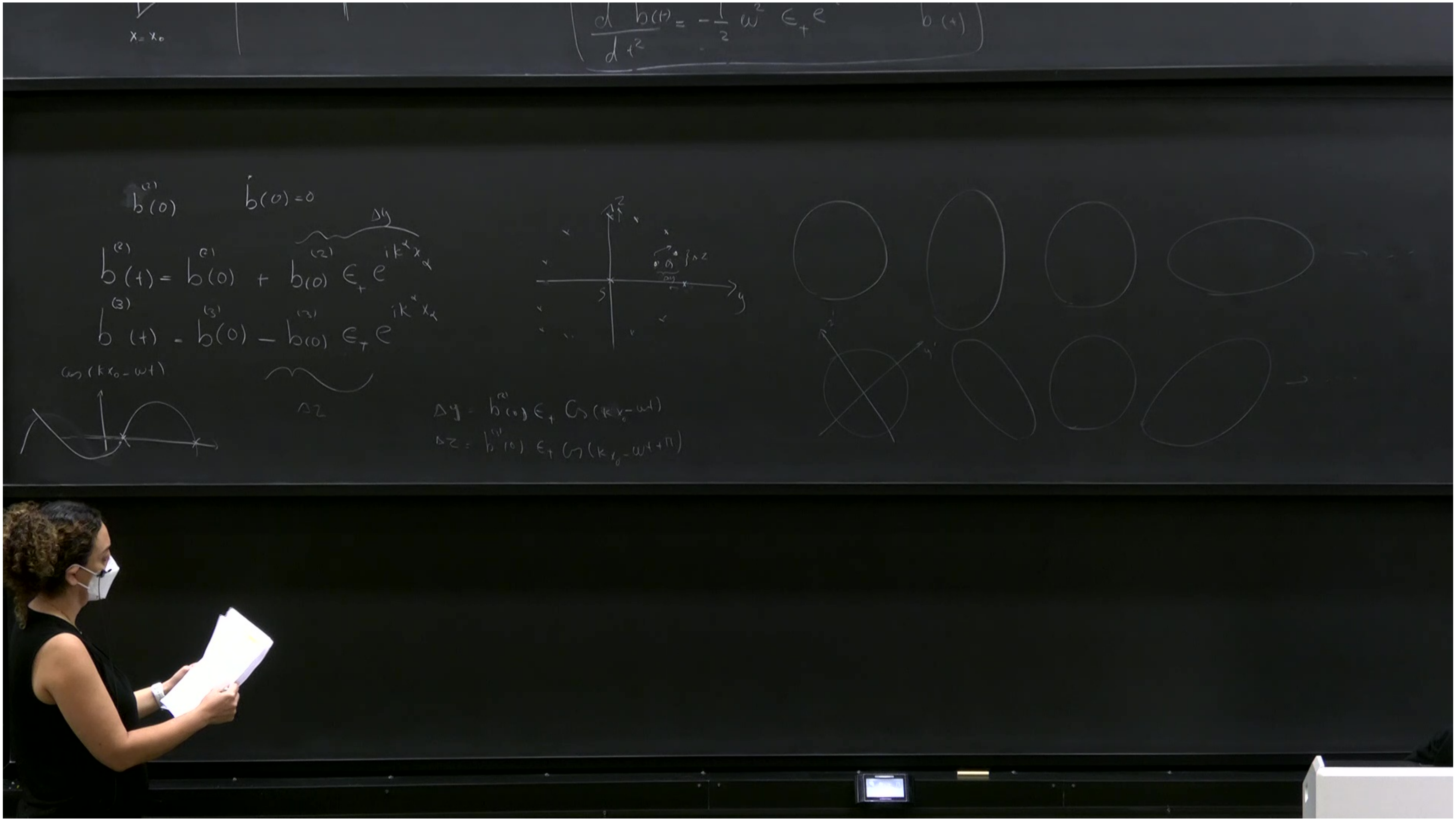
$$\frac{d^2 b^\mu}{dt^2} = +\frac{1}{2}\frac{\partial^2 h^\mu{}_\rho}{\partial t^2} b^\rho$$

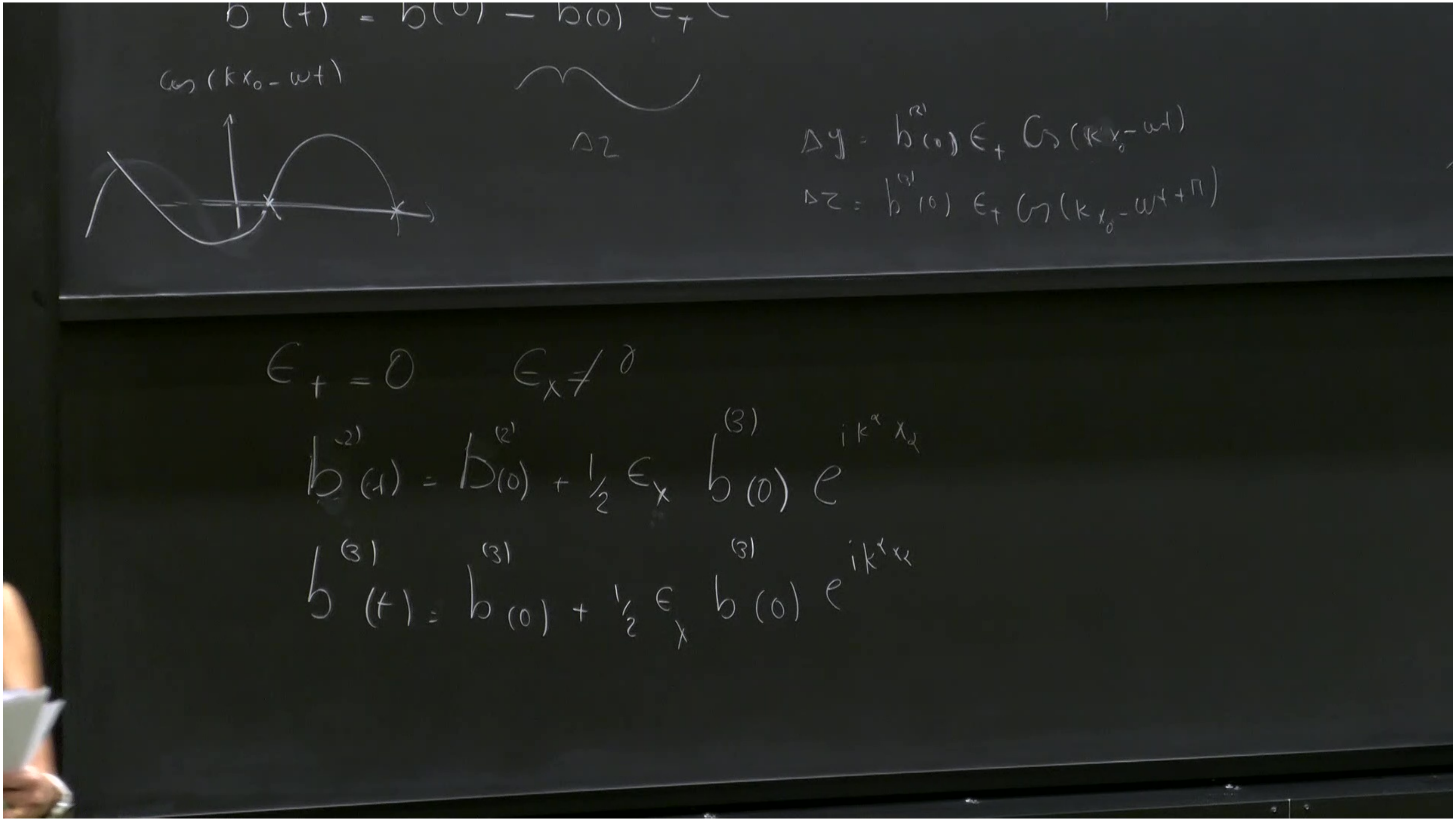
$$h^2{}_2 = \epsilon_+ e^{i(kx-\omega t)}$$

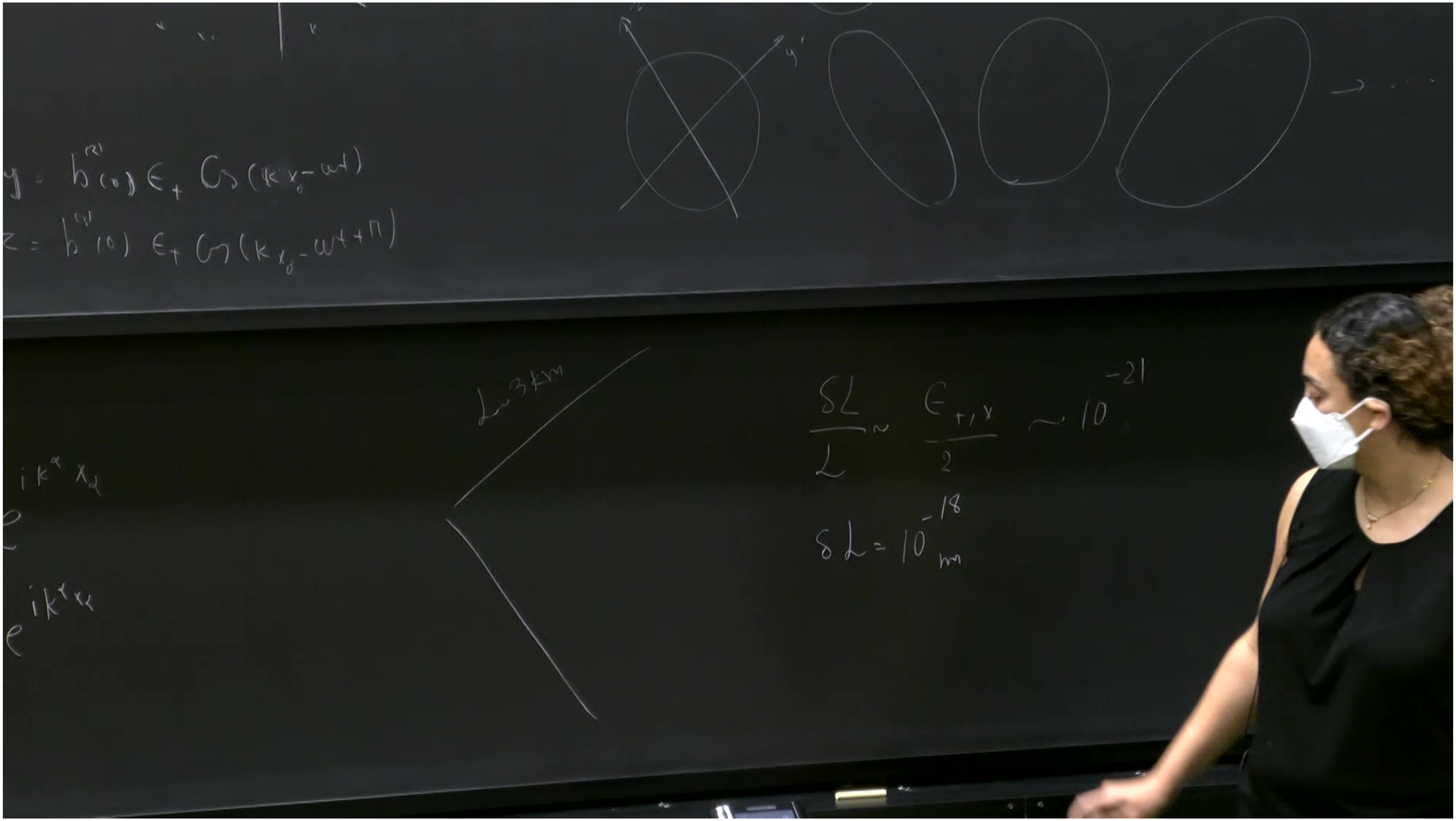
$$h^3{}_3 = -\epsilon_+ e^{i(kx-\omega t)}$$


$\overset{(2)}{b}(0)$
$\dot{b}(0)=0$
$\overset{(2)}{b}(t) = \overset{(2)}{b}(0) + \underbrace{\overset{(2)}{b}(0)\, \epsilon_+ e^{ik^\alpha x_\alpha}}_{\Delta y}$
$\overset{(3)}{b}(t) = \overset{(3)}{b}(0) - \underbrace{\overset{(3)}{b}(0)\, \epsilon_+ e^{ik^\alpha x_\alpha}}_{\Delta z}$
z
y
Δz
Δy


$x = x_0$
$\dot b^{(2)}(0)$
$\dot b(0) = 0$
Δy
$b^{(2)}(t) = b^{(2)}(0) + b^{(2)}(0)\,\epsilon_+\, e^{ik^\alpha x_\alpha}$
$b^{(3)}(t) = b^{(3)}(0) - b^{(3)}(0)\,\epsilon_+\, e^{ik^\alpha x_\alpha}$
$\cos(k x_0 - \omega t)$
Δz
z
y
Δz
Δy
$\Delta y = b^{(2)}(0)\,\epsilon_+ \cos(k x_0 - \omega t)$
$\Delta z = b^{(3)}(0)\,\epsilon_+ \cos(k x_0 - \omega t + \pi)$


x = x₀
b(0) = 0
Δy
Δz
cos(kx₀ − ωt)


$\cos(k x_0 - \omega t)$
Δz
$\Delta y = b^{(2)}(0)\, \epsilon_t \cos(k x_0 - \omega t)$
$\Delta z = b^{(3)}(0)\, \epsilon_t \cos(k x_0 - \omega t + \pi)$
$\epsilon_t = 0 \quad \epsilon_x \neq 0$
$b^{(2)}(t) = b^{(2)}(0) + \frac{1}{2} \epsilon_x\, b^{(3)}(0)\, e^{i k^\alpha x_\alpha}$
$b^{(3)}(t) = b^{(3)}(0) + \frac{1}{2} \epsilon_x\, b^{(3)}(0)\, e^{i k^\alpha x_\alpha}$


L~3 km
δL/L ~ ε+,× /2 ~ 10^-21
δL = 10^-18 m