Title: Relativity Lecture - 102723

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Collection: Relativity 2023/24

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c) VARIATIONAL PRINCIPLE FOR GRAVITY

EINSTEIN-HILBERT ACTION

$$\boxed{S_{EH}[g] = \frac{1}{16\pi G}\int d^4x \sqrt{-g}\, R(g, \partial g, \partial^2 g)}$$

$$\delta S = \frac{1}{16\pi G}\int d^dx\, \delta\left(\sqrt{-g}\, g^{\mu\nu} R_{\mu\nu}\right) = \frac{1}{16\pi G}\int d^dx \left(\underbrace{\delta\left(\sqrt{-g}\, g^{\mu\nu}\right)}_{\downarrow\; G_{\mu\nu}\delta g^{\mu\nu}} R_{\mu\nu} + \underbrace{\sqrt{-g}\, g^{\mu\nu}\delta R_{\mu\nu}}_{\substack{\text{BOUNDARY}\\ \text{TERM}\\ \text{"WEIRD"}}}\right)$$

$\Rightarrow$ VACUUM EE: $\boxed{G_{\mu\nu} = 0} \overset{d\neq 2}{\Leftrightarrow} \boxed{R_{\mu\nu} = 0}$

$D\,\delta g^{\mu\nu}$

• DIRICHLET BOUNDARY VALUE PROBLEM:

$$S = S_{EH} + \frac{1}{8\pi G}\int d^3x \sqrt{h}\, K \quad + \text{COUNTER TERMS}$$

$$\delta S = EE - \frac{1}{2}\int d^3x \sqrt{h}\, \tau_{ab}\, \delta h^{ab} \quad \ldots \text{IMPOSING } \delta h^{ab}/_{\partial M} = 0,$$

HOLOGRAPHIC ENERGY MOM. TENSOR

REMARKS:

1) $S_{EH}[g]$ "2ND-ORDER FORMALISM"

"$\partial^2 g$ ARE PRESENT"

$D\delta g^{\mu\nu}$

INSTEAD ONE CAN CONSIDER 1ST-ORDER FORMALISM

$$S_{\text{PALATINI}}[g, \Gamma] = \frac{1}{16\pi G}\int d^4x\, \underbrace{\sqrt{-g}\, g^{\mu\nu}}_{\substack{\text{DEP ON } g \\ \text{"0TH-ORDER"}}}\, \underbrace{R_{\mu\nu}(\Gamma)}_{\substack{\text{DEP ON } \Gamma, \partial\Gamma \\ \text{1ST-ORDER IN } \Gamma}}$$

$\underline{\delta g^{\mu\nu}}$: "IMMEDIATELY GIVES EE" $\boxed{G_{\mu\nu} = 0}$

$\underline{\delta \Gamma^{\alpha}_{\mu\nu}}$: FIXES CONNECTION TO BE CHRISTOFFEL:

$$\Gamma = (\partial g + \partial g - \partial g)$$

(Γ)

ON $\Gamma, \partial\Gamma$

RDER IN Γ

2) UNIQUENESS:

LOVELOCK THEOREM (1971): IN 4D

$$\boxed{S = \frac{1}{16\pi G}\int \sqrt{g}\, d^4x \, (R - 2\Lambda) \text{ IS THE ONLY ACTION (UP TO TOTAL DERIVATIVES) THAT YIELDS 2ND-ORDER EOM FOR THE METRIC}}$$

SPEC: R^2, $R_{\mu\nu}R^{\mu\nu}$, $D_\alpha R D^\alpha R$, ... $\Rightarrow$ HIGHER-ORDER EOM.

IN HIGHER d WE CAN HAVE

LOVELOCK GRAVITIES

EX: 5D: $R + \alpha \left(R^2 - 4 R_{\mu\nu} R^{\mu\nu} + R_{\mu\nu\rho\lambda} R^{\mu\nu\rho\lambda} \right)$

!! 2ND-ORDER EQS. (SEE DESER REALL)

NOTE:

$$\sqrt{-g}\, R(g, \partial g, \partial^2 g) = \boxed{\underbrace{\sqrt{-g}\, \tilde{R}(g, \partial g)}_{\text{NOT TENSOR}}} + \underbrace{\partial_\mu \hat{R}^\mu (g, \partial g)}_{\text{BOUNDARY TERM}}$$

EE

d) EINSTE

$\nabla \delta g^{\mu\nu}$

d) EINSTEIN EQUATIONS

$$\boxed{S = S_{EH}[g] + \alpha_m \int d^4x \sqrt{-g}\,\mathcal{L}_m}$$

$$\delta_g S = \int d^4x \sqrt{-g}\left(\frac{1}{16\pi G} G_{\mu\nu}\,\delta g^{\mu\nu} - \frac{\alpha_m}{2} T_{\mu\nu}\,\delta g^{\mu\nu}\right)$$

$$G_{\mu\nu} = 8\pi G \underset{\substack{\parallel \\ 1}}{\alpha_m} T_{\mu\nu}$$

(NEWTONIAN LIMIT)

$$\nabla^2\phi = 4\pi G\rho$$

$$\boxed{G^{\mu\nu} = 8\pi G\, T^{\mu\nu}}$$

• BUT THIS IS AN EOM FOR THE MATTER:

EE PLAY A "DOUBLE AGENT ROLE"

"MATTER TELLS SPACETIME HOW
TO CURVE (EE...FIELD EQ FOR $g_{\mu\nu}$)

AT THE SAME TIME SPACETIME
TELLS MATTER HOW TO MOVE:

ors.

S IS AN EOM FOR THE MATTER!

A "DOUBLE AGENT ROLE"

TELLS SPACETIME HOW

E (EE.. FIELD EQ FOR $g_{\mu\nu}$)

SAME TIME SPACETIME

MATTER HOW TO MOVE!

• IS ANYTHING CONSERVED WHEN $\nabla_\mu T^{\mu\nu}=0$?

<u>NO</u>! WE NEED SYMMETRIES!

SYMMETRIES IN GR.. KILLING VECTORS

$$\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 2 \nabla_{(\mu} \xi_{\nu)} = 0$$

=> WE CAN HAVE CONSERVED Q..

BY NOETHER'S TH.

PROPOSAL:

$$\boxed{J^\mu = T^{\mu\nu}\xi_\nu}$$

$$D_\mu J^\mu = \underbrace{\left(D_\mu T^{\mu\nu}\right)}_{0}\xi_\nu + T^{\mu\nu}\underbrace{D_{(\mu}\xi_{\nu)}}_{0} = \underline{0}$$

$\rightarrow \underline{Q}$

• NOTE IN FLAT SPACE:

WE HAVE LOTS OF KVS.

$$\xi = \{\partial_t, \partial_x, \partial_y, \partial_z, \dots\}$$

$J^\mu_{\partial_t} = T^{\mu t}$ ..... E

$J^\mu_{\partial_x} = T^{\mu x}$ ..... P_x

$J^\mu_{(\nu)} = T^\mu{}_\nu$ ... 4 QUANTITIES .. $\underline{P_\nu}$

OLS.

BY NOETHER'S TH.

NOTE IN FLAT SPACE.

WE HAVE LOTS OF KVS.

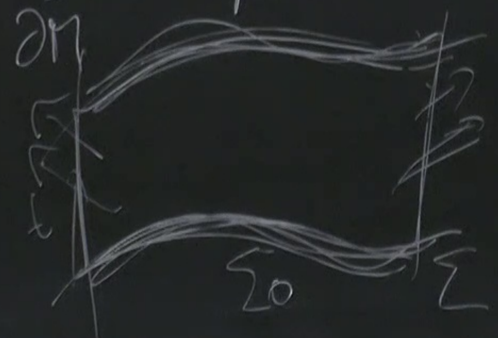
$$\xi = \{\partial_t, \partial_x, \partial_y, \partial_z, \ldots\}$$

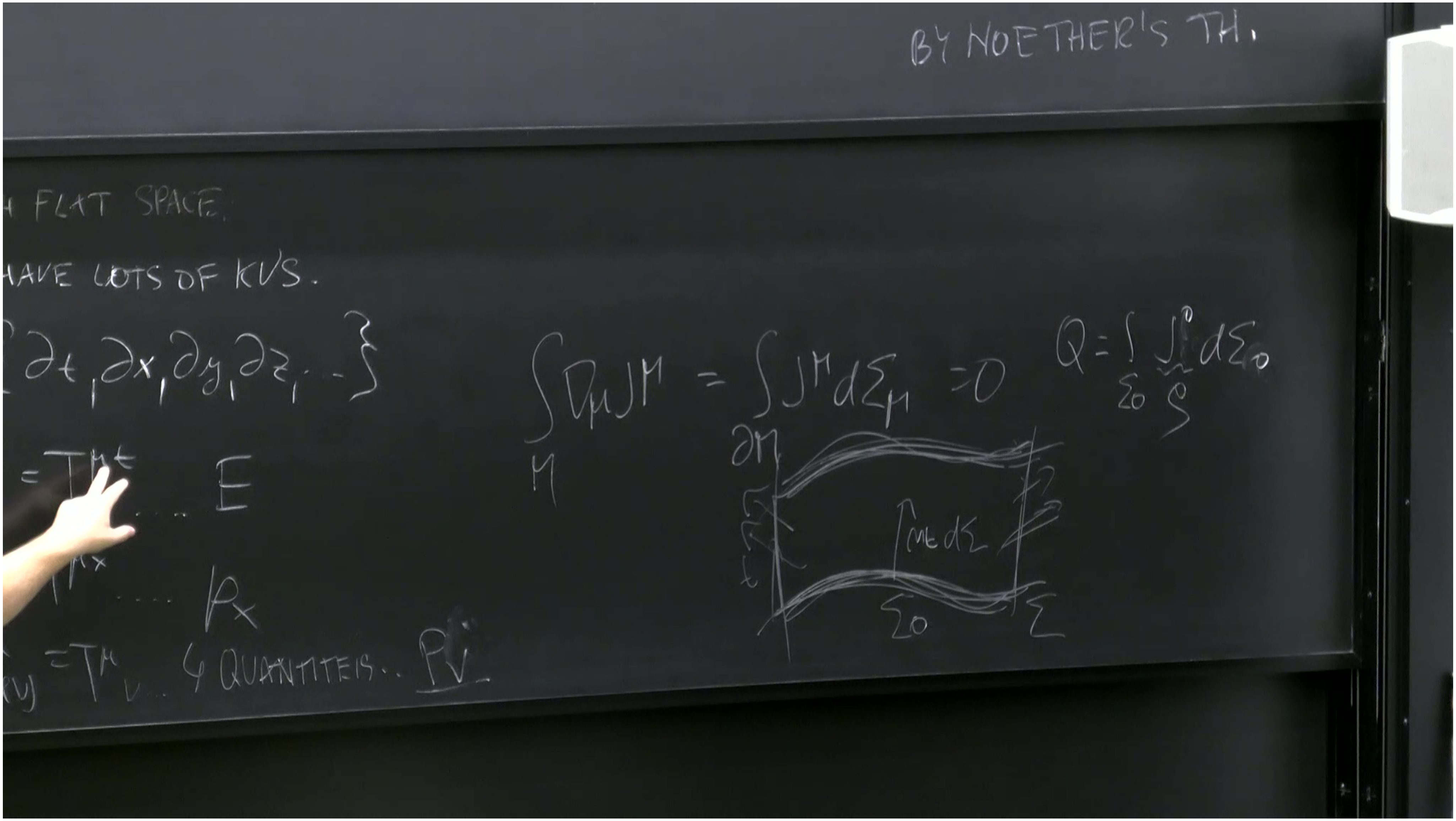
$$J^{\mu}_{\partial_t} = T^{\mu t} \ldots\ldots E$$

$$J^{\mu}_{\partial_x} = T^{\mu x} \ldots\ldots P_x$$

$$J^{\mu}_{(\nu)} = T^{\mu}{}_{\nu} \ldots \text{ 4 QUANTITEIS} \ldots \underline{P_\nu}$$

$$\int_M D_\mu J^\mu = \int J^\mu d\Sigma_\mu = 0 \qquad Q = \int_{\Sigma_0} J^\mu d\Sigma_\mu$$




BY NOETHER'S TH.
FLAT SPACE.
HAVE LOTS OF KVS.
$\{\partial_t, \partial_x, \partial_y, \partial_z, ..\}$
$= T^{\mu t}$ ..... E
$T^{\mu x}$ ..... P_x
$= T^{\mu}{}_{\nu}$ .. 4 QUANTITIES .. P_ν
$\int_M \nabla_\mu J^\mu = \int J^\mu d\Sigma_\mu = 0$
$Q = \int_{\Sigma_0} \rho \, d\Sigma_0$
∂M
$\int J^{\mu t} d\Sigma$
Σ_0
Σ
t

NECESSARY CONDITION FOR FINDING SOLS.

PROPOSAL:

$$\boxed{J^\mu = T^{\mu\nu}\xi_\nu}$$

$$D_\mu J^\mu = \underbrace{(D_\mu T^{\mu\nu})}_{0}\xi_\nu + T^{\mu\nu}\underbrace{D_{(\mu}\xi_{\nu)}}_{0} = \underline{0}$$

$\rightarrow \underline{\underline{Q}}$

. NOTE IN FLAT SPACE:

WE HAVE LOTS OF KVS.

$$\xi = \{\partial_t, \partial_x, \partial_y, \partial_z, \ldots\}$$

$J^\mu_{\partial_t} = T^{\mu t}$ .... E

$J^\mu_{\partial_x} = T^{\mu x}$ .... P_x

$J^\mu_{(\nu)} = T^\mu{}_\nu$ ... 4 QUANTITIES .. $\underline{P_\nu}$

2) ENERGY CONDITIONS

- POOH'S WAY OF SOLVING EE

TAKE ANY $g_{\mu\nu}$ WE LIKE

(WARP DRIVE, WORMHOLE, ...)

PLUG IN & CALCULATE $G_{\mu\nu}$

VICTORY: EXACT SOL OF EE WITH $T^{\mu\nu} = \frac{1}{8\pi G} G^{\mu\nu}$

Y YIELD $T^{\mu\nu}$ THAT LOOKS CRAZY:

$$= \begin{pmatrix} \rho & \text{ENERGY FLUX } j^i \\ j^i & \text{PRESSURES } P_{ij} \end{pmatrix}$$

NATURAL CONDITIONS $\rho \geq 0$

FLUX OF ENERGY PROPAGATES WITH $v \leq c$

TYPICALLY VIOLATED

ENERGY CONDITIONS: WEAK, STRONG, DOMINANT,

MATTER MUST BEHAVE REASONABLY?

WHERE DO THESE COME FROM?

BACK TO HOMEWORK. LINEARIZED GRAVITY:

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu{}_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} = 0$$

WHAT HAPPENS IN "NEWTONIAN APPROX"

• LINEARIZED

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} + O(h^2)$$

$$\boxed{\Box \bar{h}^{\mu\nu} = -16\pi G T^{\mu\nu}}$$

IN SPECIAL GAUGE

$$\boxed{\partial_\mu \bar{h}^{\mu\nu} = 0}$$

· LINEARIZED

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} + O(h^2)$$

$$\boxed{\Box \bar{h}^{\mu\nu} = -16\pi G T^{\mu\nu}}$$

IN SPECIAL GAUGE

$$\boxed{\partial_\mu \bar{h}^{\mu\nu} = 0}$$

WHERE

$$\bar{h}^{\mu\nu} = h^{\mu\nu} - \frac{1}{2} h \eta^{\mu\nu}$$

· WEAK FIELD

· NOW:

$$T^{\mu\nu} = \begin{pmatrix} \rho & \vec{0} \\ \vec{0} & \overleftrightarrow{0} \end{pmatrix} \Rightarrow \text{NEWTON!}$$

BEYND NEWTON:

$$T^{\mu\nu} = \begin{pmatrix} \rho & j^i \\ j^i & \sigma \end{pmatrix} \qquad \boxed{J^\mu \equiv T^{\mu 0}}$$

$$\boxed{\bar{h}^{\mu 0} \equiv A^\mu}$$

$$\boxed{\begin{aligned} \Box A^\mu &= -16\pi G J^\mu \\ \partial_\mu A^\mu &= 0 \end{aligned}}$$

GEOD: $\dfrac{d^2x^\mu}{d\tau^2} + 2\Gamma^\mu{}_{\nu 0}\underline{\dfrac{dx^\nu}{d\tau}\dfrac{dt}{d\tau}} = 0 \qquad =$

LINEAR IN VEL. TERM,

$$E_i = -\partial_i A_0 + 0$$

$$F_{ij} = \partial_i A_j - \partial_j A_i$$

GEOD: $$\frac{d^2x^\mu}{d\tau^2} + 2\Gamma^\mu{}_{\nu 0}\underbrace{\frac{dx^\nu}{d\tau}\frac{dt}{d\tau}}_{\text{LINEAR IN VEL. TERM.}} = 0 \Rightarrow \boxed{\frac{d^2x^i}{dt^2} = E^i - 4F^i{}_j v^j}$$

GRAVITOMAGNETISM

$$E_i = -\partial_i A_0 + 0$$

$$F_{ij} = \partial_i A_j - \partial_j A_i \quad \ldots \text{ B-FIELD}$$

GEOD: $\frac{d^2x^\mu}{d\tau^2} + 2\Gamma^\mu{}_{\nu 0} \underbrace{\frac{dx^\nu}{d\tau}\frac{dt}{d\tau}} = 0 \Rightarrow \boxed{\frac{d^2x^i}{dt^2} = E^i - 4F^i{}_j v^j}$

LINEAR IN VEL. TERM,

GRAVITOMAGNETISM

$$E_i = -\partial_i A_0 + 0$$

$$F_{ij} = \partial_i A_j - \partial_j A_i \quad \ldots \quad \text{B-FIELD}$$