Title: Relativity Lecture - 102523

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Collection: Relativity 2023/24

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b) MATTER IN CURVED SPACETIME

"MINIMAL COUPLING PRINCIPLE"

$$\eta \to g \, , \quad \partial \to \nabla$$

$$\boxed{S_m[\phi, g] = \int d^dx \sqrt{-g}\, \mathcal{L}_m(\phi, \nabla\phi, g)}$$

SCALARS

- $\delta_\phi S_m = 0 \iff$ EULER-LAGRANGE EQUATIONS OF MOTION

$$\boxed{\frac{\partial \mathcal{L}_m}{\partial \phi} - \nabla_\mu \left(\frac{\partial \mathcal{L}_m}{\partial (\nabla_\mu \phi)} \right) = 0}$$

- WE CAN ALSO VARY W.R.T. g:

$$\boxed{\delta_g S_m \equiv -\frac{1}{2} \int d^dx \sqrt{-g}\, T_{\mu\nu}\, \delta g^{\mu\nu}}$$

DEFINITION OF ENERGY-MOM TENSOR $T_{\mu\nu}$.

- $T_{\mu\nu} = T_{\nu\mu}$. SYMMETRIC
- ON-SHELL: $\delta_\phi S_m = 0 \Rightarrow \boxed{\nabla_\mu T^{\mu\nu} = 0}$

NOTE: DIFFEOMORPHISM INVARIANCE

WE HAVE SCALAR ... ACTION INVARIANT UNDER CHANGE OF COORDS

INFINIT: $x^\mu \to x^\mu - \xi^\mu(x)$ — SMALL VECTOR.

$$g^{\mu\nu} \to g^{\mu\nu} + 2\nabla^{(\mu}\xi^{\nu)} = \delta g^{\mu\nu}$$

OR $T_{\mu\nu}$.

$$\bullet\ \delta S_m = \underbrace{\delta_\phi S_m}_{\substack{\| \\ 0 \ \text{IF ON-SHELL}}} + \delta_g S_m = 0 \qquad = 0 + \delta_g S_m = -\frac{1}{2}\int$$

IF THE VARIATION IS JUST A DIFFEOM.

$$= -\int d^d x \sqrt{g}\, T_{\mu\nu} D^\mu \xi^\nu = \int d^d x \sqrt{g}\, D^\mu T_{\mu\nu}\, \xi^\nu + \text{BOUN}$$

ξ... ARBITRARY.

ENERGY MOM TENSOR $T_{\mu\nu}$.

$$= 0 + \delta_g S_m = -\frac{1}{2}\int d^dx \sqrt{-g}\, T_{\mu\nu}\, \underbrace{2\cdot \nabla^{(\mu}\xi^{\nu)}}_{\delta g^{\mu\nu}}$$

N

OM.

$$\boxed{\nabla_\mu T^{\mu\nu} = 0}$$

$$\sqrt{g}\, \underbrace{\nabla^\mu T_{\mu\nu}}_{= \emptyset}\, \xi^\nu + \underbrace{\text{BOUNDARY TERM}}_{\to 0}$$

RY.

$$\xi/_{\partial M} = 0$$

$$\delta_{\xi} S_m = -\frac{1}{2}\int d^4x\sqrt{-g}\, T_{\mu\nu}\, \underbrace{2\cdot D^{(\mu}\xi^{\nu)}}_{\delta g^{\mu\nu}}$$

$$\xi^{\nu} + \underbrace{\text{BOUNDARY TERM}}_{\to\, 0}.$$

$$\xi/_{\partial M} = 0$$

$$\boxed{D_\mu T^{\mu\nu} = 0} \overset{?}{\Longleftrightarrow} \boxed{\delta_\phi S_m = 0}$$

EQ. OF MOTION ?

HELPFUL:

$$\boxed{\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}\, g_{\mu\nu}\,\delta g^{\mu\nu}}, \quad \boxed{\delta g_{\mu\nu} = -g_{\mu\rho}\, g_{\nu\sigma}\,\delta g^{\rho\sigma}}$$

EX: SCALAR FIELD:

$$S = \int d^4x \sqrt{-g} \left(\underbrace{-\frac{1}{2} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - V(\phi)}_{\mathcal{L}_m} \right)$$

EOM:

$$-\frac{dV}{d\phi} - \nabla_\alpha \left(\underbrace{\frac{\partial \mathcal{L}_m}{\partial (\nabla_\alpha \phi)}}_{-g^{\alpha\nu} \nabla_\nu \phi} \right) = \boxed{-\frac{dV}{d\phi} + \nabla^2 \phi = 0}$$

• $\underline{T^{\mu\nu}}$:

$$\delta_g S = \int d^4x \underbrace{\left(\delta\sqrt{-g}\right)}_{-\frac{1}{2}\sqrt{-g}\, g_{\mu\nu}\,\delta g^{\mu\nu}} \mathcal{L}_m + \int d^4x \sqrt{-g}\left(-\frac{1}{2}\nabla_\mu\phi\,\nabla_\nu\phi\,\delta g^{\mu\nu}\right)$$

$$= -\frac{1}{2}\int d^4x\sqrt{-g}\left(\underbrace{g_{\mu\nu}\mathcal{L}_m + \nabla_\mu\phi\,\nabla_\nu\phi}_{T_{\mu\nu}}\right)\delta g^{\mu\nu}$$

• IF $\nabla_\mu T^{\mu\nu}=0$ WHAT DO WE GET ?

$$g^{\mu\nu}\nabla_\mu \mathcal{L}_m + \nabla^2\phi\, \nabla^\nu\phi + \nabla^\mu\phi\, \nabla_\mu\nabla^\nu\phi = \nabla_\mu T^{\mu\nu} = 0$$

$$\nabla_\mu \mathcal{L}_m = -g^{\alpha\beta}\nabla_\mu\nabla_\alpha\phi\, \nabla_\beta\phi - \frac{dV}{d\phi}\nabla_\mu\phi$$

$$-\underbrace{\nabla^\nu\nabla^\beta\phi\, \nabla_\beta\phi}_{-\cancel{\nabla^\nu\nabla_\mu\phi\, \nabla^\mu\phi}} - \frac{dV}{d\phi}\nabla^\nu\phi + \nabla^2\phi\, \nabla^\nu\phi + \cancel{\nabla^\mu\phi\, \nabla_\mu\nabla^\nu\phi} = 0$$

$$\nabla^\nu\phi\left(\nabla^2\phi - \frac{dV}{d\phi}\right) = 0$$

IF ϕ NOT VERY SPECIAL

$$\underbrace{\nabla_\mu T^{\mu\nu} = 0 \Rightarrow \nabla^2\phi - \frac{dV}{d\phi} = 0}$$

$\Leftarrow$ ALWAYS TRUE.

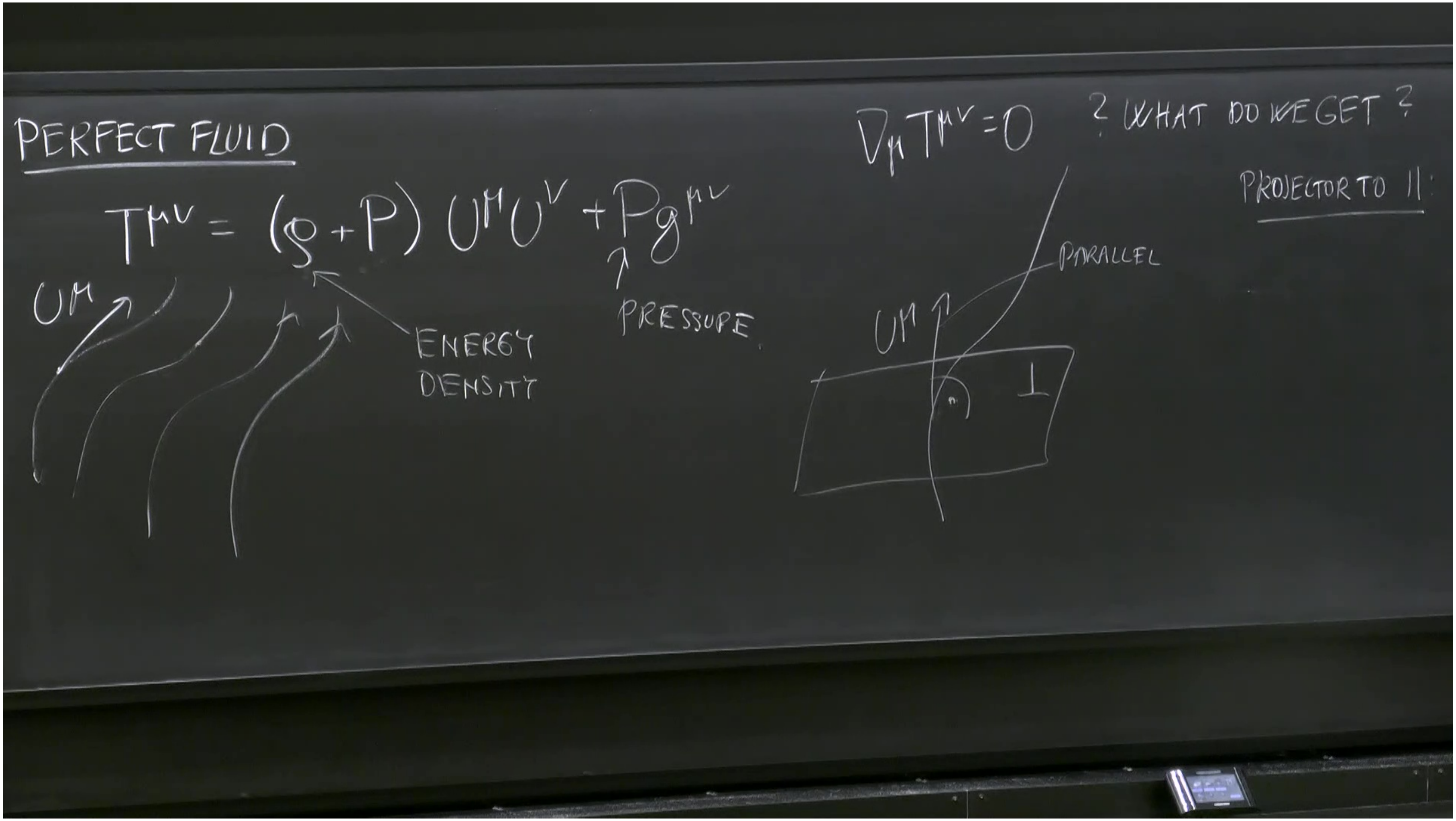
$\nabla^\nu \phi = \emptyset$

$\nabla_\mu T^{\mu\nu} = 0$ IS EQUIVALENT TO EOM UP TO SPECIAL CASES.

MAXWELL: $\nabla_\mu F^{\mu\nu} = J^\nu$

$\nabla_\mu T^{\mu\nu} = 0 \Rightarrow$

$\nabla_\mu F^{\mu\nu} F_{\nu\alpha} = J^\nu F_{\nu\alpha}$



PERFECT FLUID
$T^{\mu\nu} = (\rho + P) U^\mu U^\nu + P g^{\mu\nu}$
U^μ
ENERGY
DENSITY
PRESSURE
$D_\mu T^{\mu\nu} = 0$
? WHAT DO WE GET ?
PROJECTOR TO ∥:
PARALLEL
U^μ
$\perp$

WHAT DO WE GET ?

PROJECTOR TO $\parallel$: $\boxed{P_{\parallel}{}^{\alpha}{}_{\beta} \overset{?}{=} -U^{\alpha} U_{\beta}}$ $\quad P_{\parallel}{}^{\alpha}{}_{\beta} P_{\parallel}{}^{\beta}{}_{\gamma} = P_{\parallel}{}^{\alpha}{}_{\gamma}$

ALLEL

$$P_{\parallel}{}^{\alpha}{}_{\beta} U^{\beta} = -U^{\alpha} \underbrace{U_{\beta} U^{\beta}}_{U^2} = +U^{\alpha}$$

PROJECTOR TO $\perp$: $P_{\perp}{}^{\alpha}{}_{\beta} = \mathbb{1}^{\alpha}{}_{\beta} - P_{\parallel}{}^{\alpha}{}_{\beta} = \delta^{\alpha}_{\beta} + U^{\alpha} U_{\beta}$

$$U^{\nu} + \underbrace{(\rho + P)}\, U^{\mu} D_{\mu} U^{\nu} + P_{,\mu} g^{\mu\nu} + 0 \qquad 2 U_{\nu} D_{\mu} U^{\nu} = 0$$

WHAT DO WE GET ?

PROJECTOR TO $\parallel$: $\boxed{P_{\parallel}{}^{\alpha}{}_{\beta} \overset{?}{=} -U^{\alpha}U_{\beta}}$ $\quad P_{\parallel}{}^{\alpha}{}_{\beta} P_{\parallel}{}^{\beta}{}_{\gamma} = P_{\parallel}{}^{\alpha}{}_{\gamma}$

ALLEL

$$P_{\parallel}{}^{\alpha}{}_{\beta} U^{\beta} = -U^{\alpha} \underbrace{U_{\beta}U^{\beta}}_{U^2} = +U^{\alpha}$$

PROJECTOR TO $\perp$: $P_{\perp}{}^{\alpha}{}_{\beta} = \mathbb{1}^{\alpha}{}_{\beta} - P_{\parallel}{}^{\alpha}{}_{\beta} = \delta^{\alpha}{}_{\beta} + U^{\alpha}U_{\beta}$

$$U^{\nu} + \underbrace{(\rho+P)U^{\mu}D_{\mu}U^{\nu}} + P_{,\mu}g^{\mu\nu} + 0 \qquad 2U_{\nu}D_{\mu}U^{\nu} = 0 = D_{\mu}(U^2)$$

1) PROJECT $\parallel$ TO U^μ:

$$P_{\parallel}{}^{\alpha}{}_{\nu}\left(\nabla_\mu T^{\mu\nu}\right) = 0 = -U^\alpha U_\nu \nabla_\mu T^{\mu\nu}$$

$$0 = U_\nu \nabla_\mu T^{\mu\nu} = -(\rho+P)_{,\mu} U^\mu - (\rho+P)\nabla_\mu U^\mu + \emptyset + U_\nu P_{,}{}^{\nu}$$

$$\boxed{\frac{d\rho}{d\tau} + (\rho+P)\nabla\cdot U = 0}$$

CONTINUITY EQ.

2) PROJECT $\perp$ TO U^μ:

$$= -U^\alpha U_\nu D_\mu T^{\mu\nu}$$

$$P_{\perp}{}^{\alpha}{}_{\nu}\left(D_\mu T^{\mu\nu}\right) = \cancel{0}$$

$$-(\rho+P)_{,\mu} U^\mu - (\rho+P) D_\mu U^\mu + \cancel{0} + U_\nu P_{,}{}^{\nu}$$

$$0 = 0 + 0 + (\rho+P) U^\mu D_\mu U^\alpha (\ldots + U$$

$$\boxed{D \cdot U = 0}$$

$$\boxed{(\rho+P)\, U^\mu D_\mu U^\alpha = -\left(\nabla^\alpha P\right)_\perp}$$

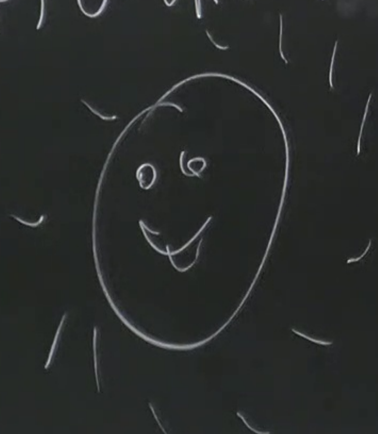
NAVIER-STOKES EQ.

2) PROJECT $\perp$ TO U^μ:

$$P_\perp{}^\alpha{}_\nu \left(\nabla_\mu T^{\mu\nu} \right) = 0$$

$$0 = 0 + 0 + (\rho + P)\, U^\mu \nabla_\mu U^\alpha \left(\;\; + \cancel{U^\alpha U} \right) + \underbrace{P_\perp{}^\alpha{}_\nu \left(P_{,}{}^\nu \right)}_{(\nabla^\alpha P)_\perp}$$

$$\boxed{= -\left(\nabla^\alpha P\right)_\perp}$$



$\mathcal{L}(g,\partial g)$ THIS DOES NOT EXIST

SCALAR.

2ND E-L - g

GIVE UP DIFFEO INV.

DEMAND SCALAR

$\mathcal{L}=\mathcal{L}(g,\partial g,\partial^2 g)$

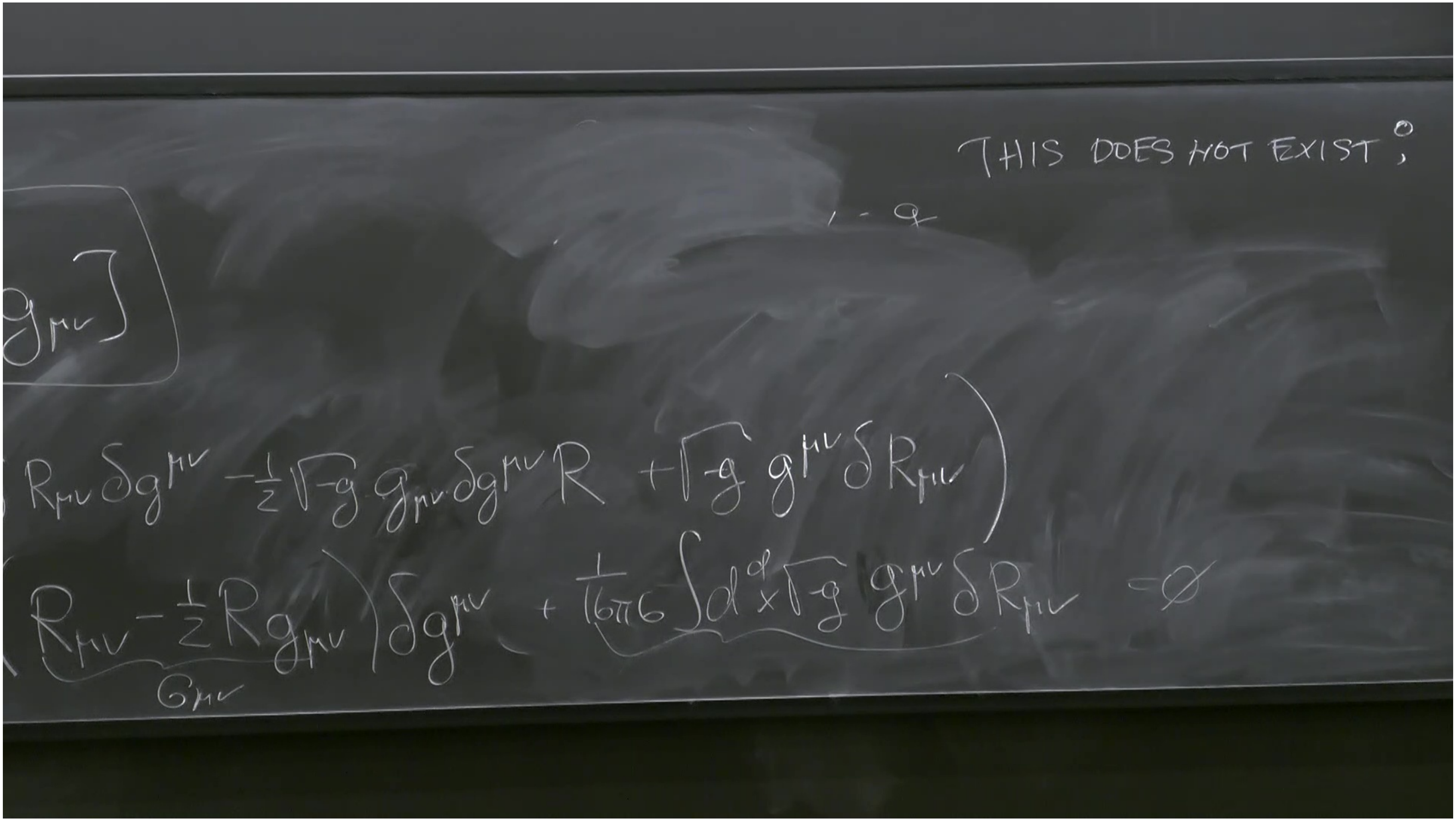
R

THIS

$$\boxed{\ldots, \partial^2 g) = S_{EH}[g_{\mu\nu}]}$$

$$\ldots) = \frac{1}{16\pi G}\int d^4x \left(\sqrt{-g}\, R_{\mu\nu}\, \delta g^{\mu\nu} - \frac{1}{2}\sqrt{-g}\, g_{\mu\nu}\, \delta g^{\mu\nu} R + \sqrt{-g}\, g^{\mu\nu}\, \delta R_{\mu\nu} \right)$$

$$\frac{1}{16\pi G}\int d^4x \sqrt{-g} \left(R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} \right) \delta g^{\mu\nu} + \frac{1}{16\pi G}\int d^4x \sqrt{-g}\, g^{\mu\nu}\, \delta R_{\mu\nu}$$


THIS DOES NOT EXIST ;
$g_{\mu\nu}]$
$R_{\mu\nu}\delta g^{\mu\nu} - \frac{1}{2}\sqrt{-g}\, g_{\mu\nu}\delta g^{\mu\nu} R + \sqrt{-g}\, g^{\mu\nu}\delta R_{\mu\nu})$
$(\underbrace{R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}}_{G_{\mu\nu}})\delta g^{\mu\nu} + \frac{1}{16\pi G}\int d^d x\sqrt{-g}\, g^{\mu\nu}\delta R_{\mu\nu} = \emptyset$

c) EINSTEIN HILBERT ACTION

$$\boxed{S_{EH} = \frac{1}{16\pi G}\int d^dx \sqrt{-g}\, R(g, \partial g, \partial^2 g)} = S_{EH}[g_{\mu\nu}]$$

$$\delta_g S_{EH} = \frac{1}{16\pi G}\int d^dx\, \delta\Big(\underbrace{\sqrt{-g}}_{2}\ \underbrace{g^{\mu\nu}}_{1} R_{\mu\nu}\Big) = \frac{1}{16\pi G}\int d^dx \Big(\sqrt{-g}\, R_{\mu\nu}\, \delta \ldots$$

$$= \frac{1}{16\pi G}\int d^dx \sqrt{-g}\, \Big(R_{\mu\nu} \ldots$$

IT CAN B...

$$16\pi G \int d^4x \sqrt{g}\,\big(\underbrace{R_{\mu\nu} - \tfrac{1}{2} R g_{\mu\nu}}_{G_{\mu\nu}}\big)\,\delta g$$

IT CAN BE SHOWN THAT

$$g^{\mu\nu}\delta R_{\mu\nu} = \nabla_\mu V^\mu, \qquad V_\mu = \nabla^\beta(\delta g_{\mu\beta}) - g^{\alpha\beta}\nabla_\mu(\delta g_{\alpha\beta})$$

BOUNDARY TERM THAT DOES NOT CONTRIBUTE TO EE.

VACUUM EE: $\boxed{G_{\mu\nu} = 0}$

$-g^{\alpha\beta} D_\mu (\delta g_{\alpha\beta})$ ... THIS IS NOT DIRICHLET BOUNDARY VALUE PROBLEM.

SEE.

IF WE WANT DIRICHLET BOUNDARY PROBLEM.

$$\boxed{S_{EH} + \int d^3x \sqrt{h}\, K}$$

EXTRINSIC CURVATURE.

YORK-GIBBONS-HAWKING TERM.

$G_{\mu\nu}$

IT CAN BE SHOWN THAT

$$g^{\mu\nu}\delta R_{\mu\nu} = D_\mu V^\mu, \qquad V_\mu = D^\beta(\delta g_{\mu\beta}) - g^{\alpha\beta} D_\mu(\delta g_{\alpha\beta})$$

BOUNDARY TERM THAT DOES NOT CONTRIBUTE TO EE.

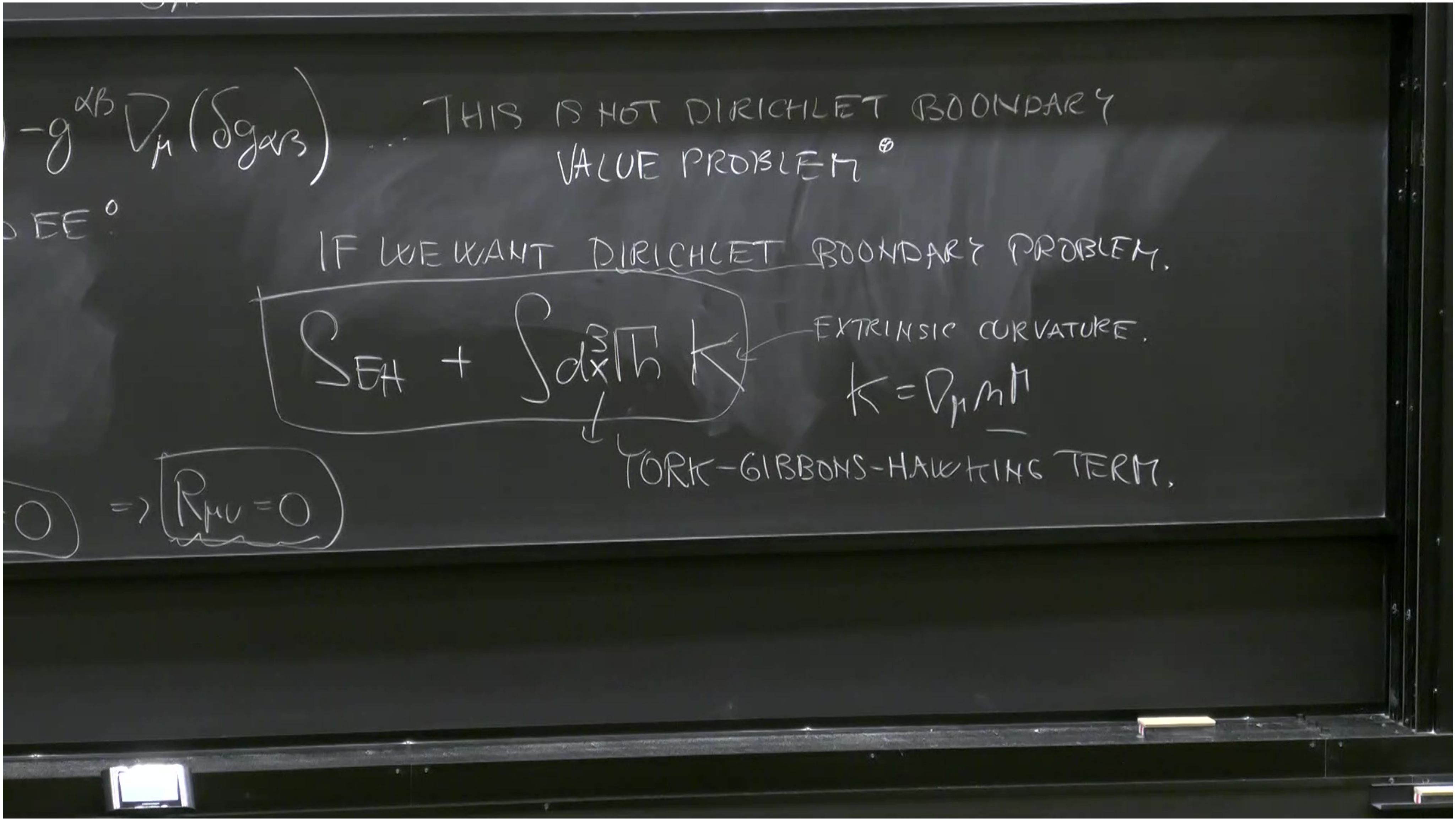
IF

VACUUM EE: $\boxed{G_{\mu\nu} = 0}$

$$\int d^2x\, \phi R$$

$$R_{\mu\nu} - \tfrac{1}{2} R g_{\mu\nu} = 0 \qquad / g^{\mu\nu}$$

$$R - \tfrac{1}{2} R d = 0 \overset{d\neq 2}{\Longrightarrow} \boxed{R=0} \Rightarrow \boxed{R_{\mu\nu}=0}$$


$-g^{\alpha\beta} D_\mu (\delta g_{\alpha\beta})$ ... THIS IS NOT DIRICHLET BOUNDARY VALUE PROBLEM
IF WE WANT DIRICHLET BOUNDARY PROBLEM.
$S_{EH} + \int d^3x \sqrt{h}\, K$
EXTRINSIC CURVATURE.
$K = D_\mu n^\mu$
YORK-GIBBONS-HAWKING TERM.
$\Rightarrow R_{\mu\nu} = 0$