Title: Relativity Lecture - 102023

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Collection: Relativity 2023/24

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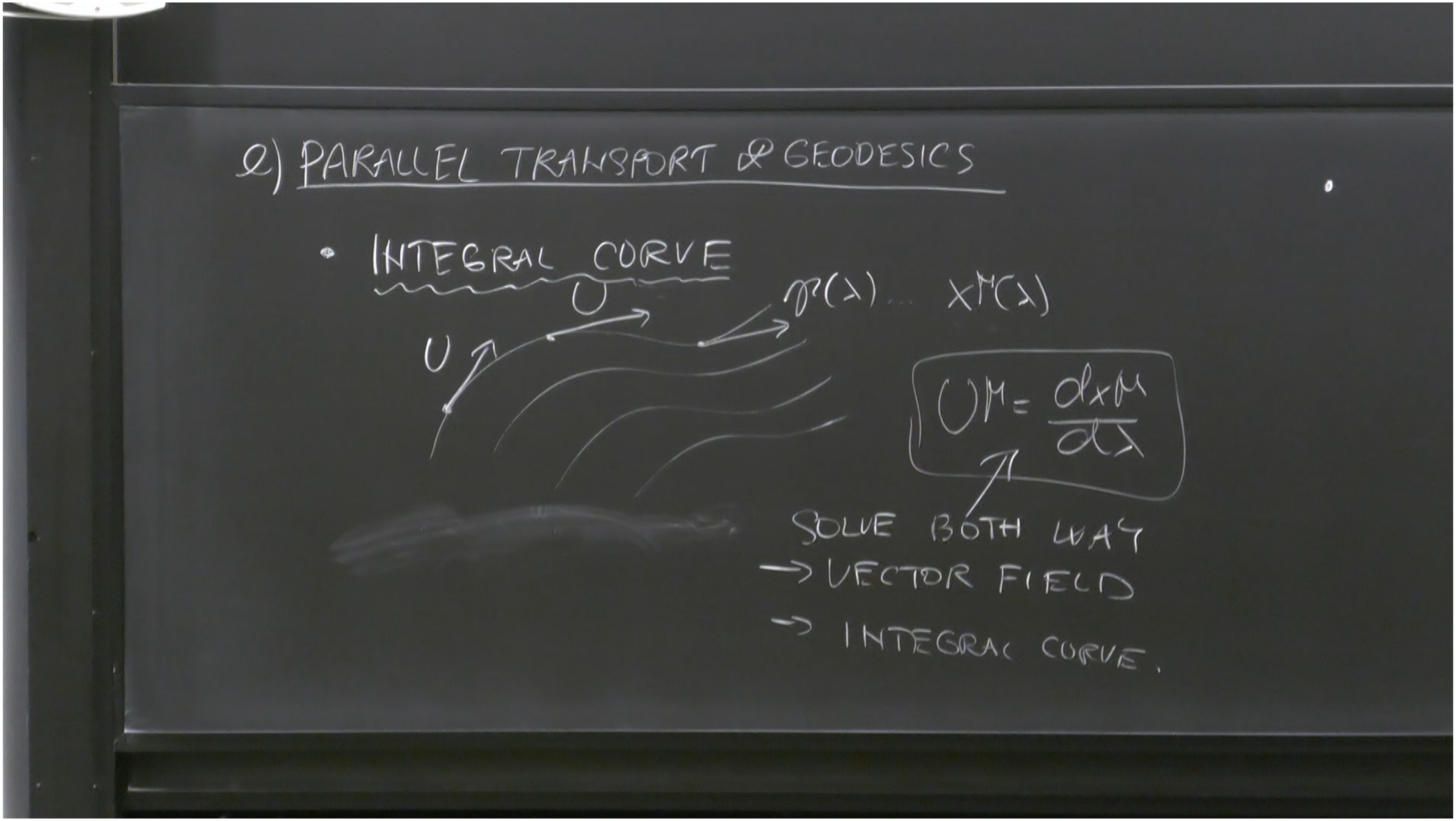

2) PARALLEL TRANSPORT & GEODESICS
• INTEGRAL CURVE
U
U
$\gamma(\lambda)$ ... $x^\mu(\lambda)$
$U^\mu = \frac{dx^\mu}{d\lambda}$
SOLVE BOTH WAY
→ VECTOR FIELD
→ INTEGRAL CURVE.

- CONSIDER CURVE $\gamma = \gamma(\lambda)$ .. INTEGRAL CURVE OF U

$\Rightarrow$ COVARIANT DERIVATIVE ALONG γ OF VECTOR V AS

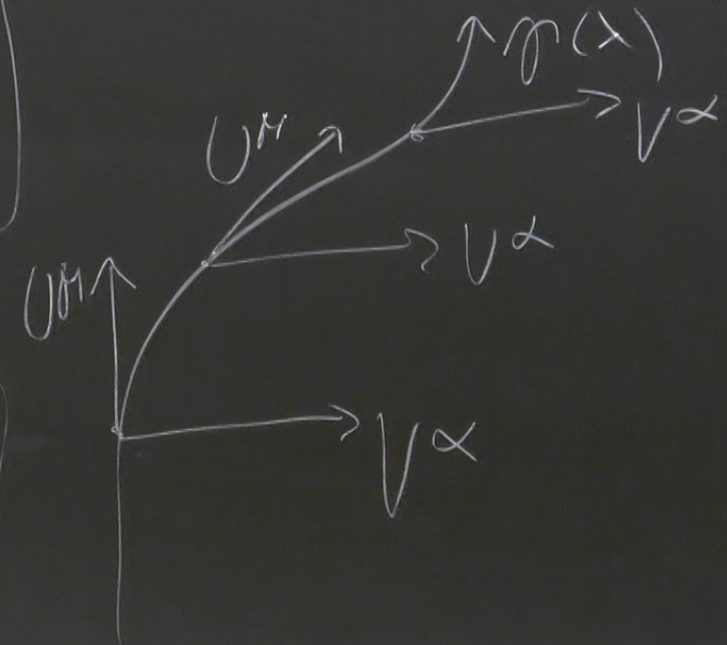
$$\frac{DV^\alpha}{d\lambda} \equiv U^\mu \nabla_\mu V^\alpha \equiv \nabla_U V^\alpha \equiv \dot{V}^\alpha$$

$$= \underbrace{U^\mu}_{\frac{dx^\mu}{d\lambda}} \Big(\underbrace{\partial_\mu V^\alpha}_{\frac{\partial V^\alpha}{\partial x^\mu}} + \Gamma^\alpha{}_{\mu\beta} V^\beta \Big) = \frac{dV^\alpha}{d\lambda} + \Gamma^\alpha{}_{\mu\beta} U^\mu V^\beta$$

V^α IS PARALLEL TRANSPORTED $\iff$

$$\boxed{\frac{DV^\alpha}{d\lambda}=0}$$

WE WANT TO PRESERVE V^α ALONG γ AS MUCH AS POSSIBLE



DEF: GEODESIC IS A CURVE ALONG WHICH THE TANGENT VECTOR IS PARALLEL TRANSPORTED

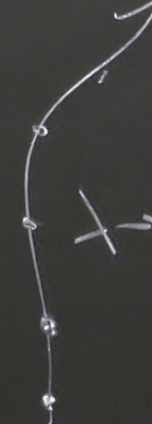
$$\boxed{\frac{DV^\alpha}{d\lambda} = 0 = V^\beta D_\beta V^\alpha = 0}$$

"GENERALIZATION OF STRAIGHT LINE"

F: GEODESIC IS A CURVE ALONG WHICH THE TANGENT VECTOR IS PARALLEL TRANSPORTED

$$\boxed{\frac{DV^\alpha}{d\lambda} = 0 = V^\beta \nabla_\beta V^\alpha = 0}$$

$x^\alpha(\lambda)$



$\lambda \to \tilde{\lambda} = f(\lambda)$

"GENERALIZATION OF STRAIGHT LINE"

$$\frac{DV^\alpha}{d\tilde{\lambda}} = \# V^\alpha$$

... CAN ALWAYS BE REPARAMETRIZED.

· FIRST LOOK AT CONSERVED QUANTITIES (FOR GEODESICS). LET'S HAVE A GEOD. WITH U^μ

$$U^\mu \nabla_\mu U^\nu = 0$$

VECTOR FIELD

$$C = U^\mu \xi_\mu$$

IF C TO BE A CONSTANT FOR EVERY GEODESIC

THEN

$$0 = \dot{C} = U^\mu \nabla_\mu (C) = U^\mu \nabla_\mu \left(U^\nu \xi_\nu \right)$$

$$= U^\mu U^\nu \nabla_{(\mu} \xi_{\nu)} + \underbrace{\left(U^\mu \nabla_\mu U^\nu \right)}_{0} \xi_\nu$$

$$\Rightarrow \boxed{\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0}$$

KILLING VECTOR EQ

R EVERY GEODESIC

$$\nabla_\mu (C) = U^\mu \nabla_\mu (U^\nu \xi_\nu)$$

$$\xi_\nu) + \underbrace{(U^\mu D_\mu U^\nu)}_{\cancel{0}} \xi_\nu$$

$$\boxed{\xi_\mu = 0}$$

KILLING VECTOR EQ .. DESCRIBE SYMMETRIES OF SPACETIME.

IF C TO BE A CONSTANT FOR EVERY GEODESIC

THEN

OD-

$$0 = \dot{C} = U^\mu \nabla_\mu (C) = U^\mu \nabla_\mu (U$$

$$= U^\mu U^\nu \nabla_{(\mu} \xi_{\nu)} + \underbrace{\left(U^\mu \nabla_\mu U^\nu\right)}_{0} \xi_\nu$$

$$\Rightarrow \boxed{\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0}$$

KILLING VECTOR

CHARGE

SPACETIME SYMMETRY (TUTORIAL)

...H FOR EVERY GEODESIC

$$U^\mu \nabla_\mu (C) = U^\mu \nabla_\mu \left(U^\nu \xi_\nu \right)$$

$$\ldots{}^\nu \nabla_{(\mu} \xi_{\nu)} + \underbrace{\left(U^\mu \nabla_\mu U^\nu \right)}_{0} \xi_\nu$$

$$S^{\alpha\beta} A_{\alpha\beta} = 0$$

$$= -S^{\beta\alpha} A_{\beta\alpha} = -S^{\alpha\beta} A_{\alpha\beta}$$

$$\boxed{\ldots \nabla_\nu \xi_\mu = 0}$$

KILLING VECTOR EQ ... DESCRIBE SYMMETRIES OF SPACETIME.

SYMMETRY (TUTORIAL)

NSERVED QUANTITIES

ESICS). LET'S HAVE A GEOD.

WITH U^μ

$$\boxed{U^\mu \nabla_\mu U^\nu = 0}$$

VECTOR FIELD

$U^\mu \xi_\mu$ ← NOETHER CHARGE

ξ^μ

IF C TO BE A CONSTAN

THEN

$0 = \dot{C} = ($

$= U^\mu U^\nu \nabla$

$\Rightarrow \nabla_\mu \xi_\nu + \nabla_\nu$

SPACETIME SYM

- BACK TO METRICITY: LET V & W BE TWO PARALLEL TRANSPORTED VECTORS ALONG $\gamma(U)$

$$\nabla_U V = 0 = \nabla_U W$$

$$\Rightarrow \frac{D}{d\lambda}(V\cdot W) = \frac{D}{d\lambda}\left(V^\alpha W^\beta g_{\alpha\beta}\right) =$$

$$= V^\alpha W^\beta U^\gamma \underbrace{D_\gamma g_{\alpha\beta}}_{M_{\gamma\alpha\beta}} \overset{\text{IF } M=0}{=} 0$$

$\Rightarrow$ ANGLE DOES NOT CHANGE!

V & W BE TWO

-CTORS ALONG $\gamma(O)$

$\nabla_U W$

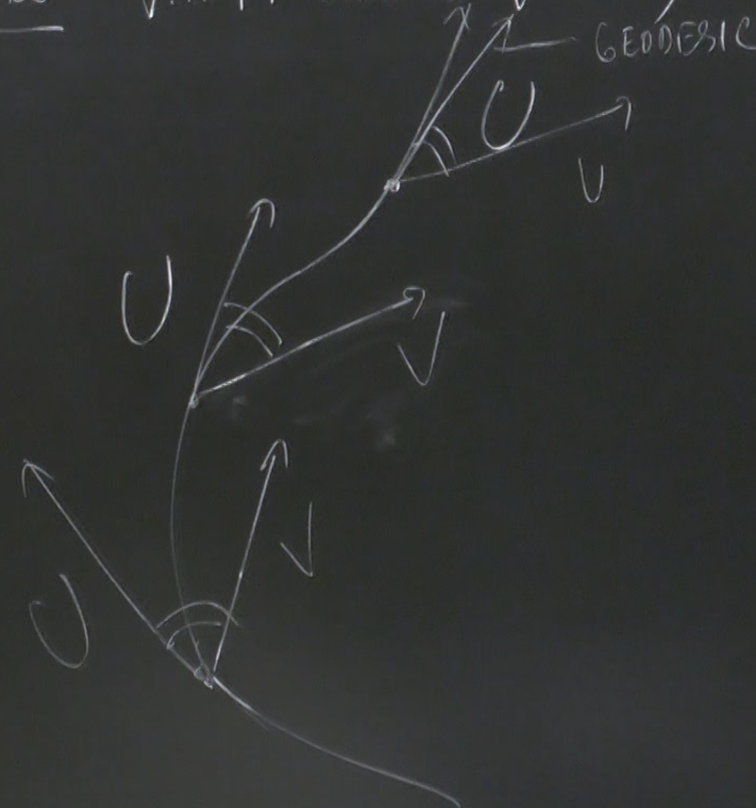
$(V^\alpha W^\beta g_{\alpha\beta}) =$ BETWEEN V & W

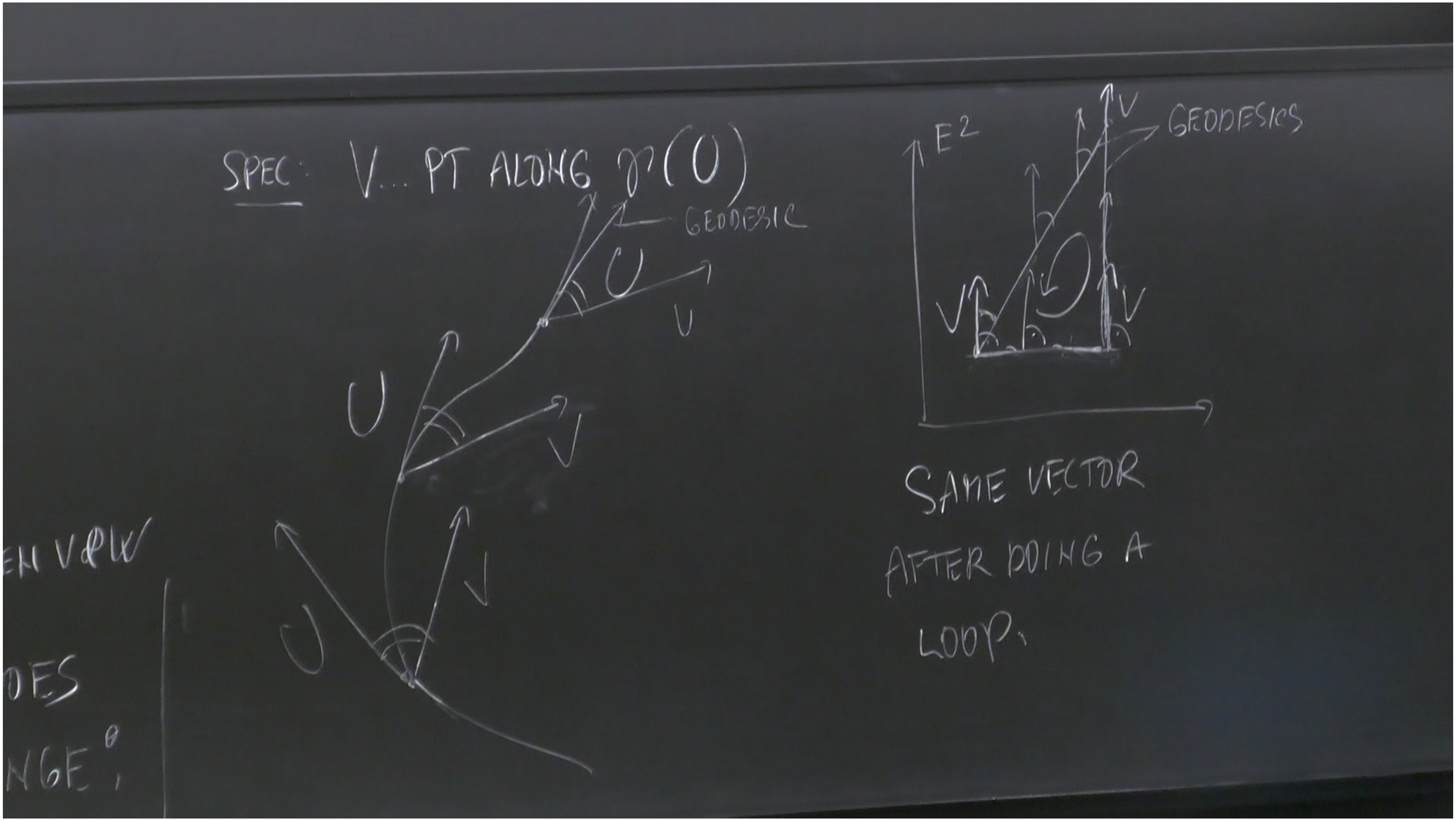
IF $M=0$

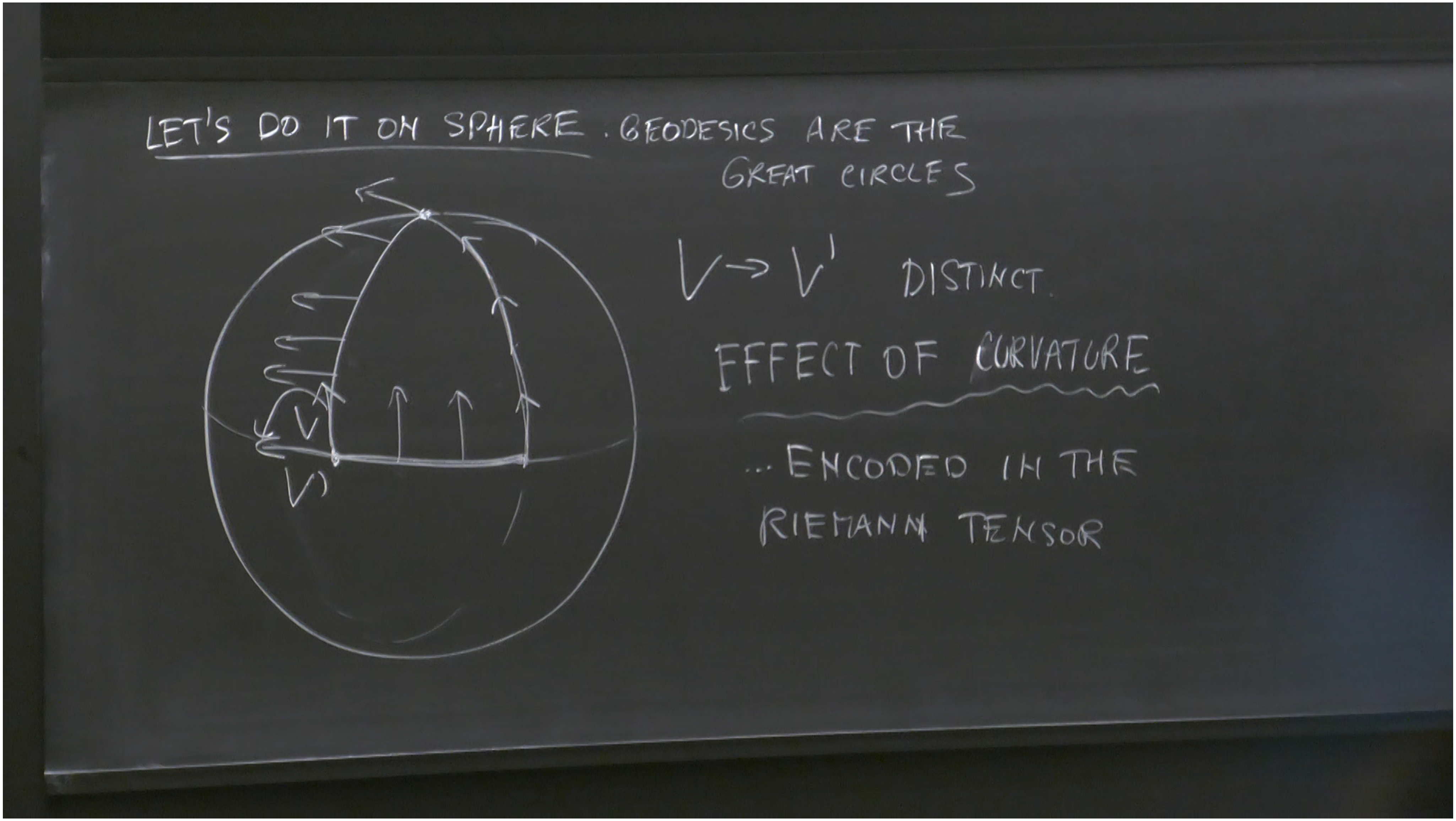
$g_{\alpha\beta} = 0$

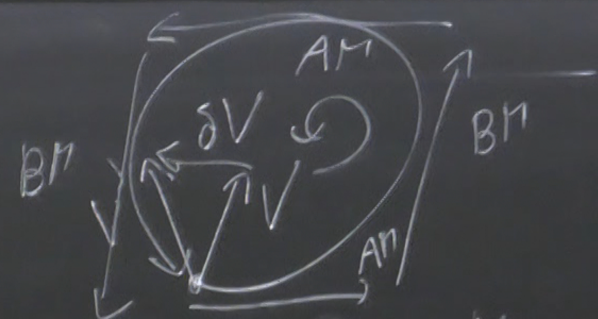
$\Rightarrow$ ANGLE DOES NOT CHANGE!

SPEC: V... PT ALONG $\gamma(O)$




SPEC: V... PT ALONG γ(U)
GEODESIC
U
V
E 2
V
GEODESICS
SAME VECTOR
AFTER DOING A
LOOP.


LET'S DO IT ON SPHERE . GEODESICS ARE THE
GREAT CIRCLES
V
V'
V → V' DISTINCT.
EFFECT OF CURVATURE
... ENCODED IN THE
RIEMANN TENSOR



URE

$$\delta V^\mu = R(A,B)^\mu{}_\nu V^\nu$$

HE

RIEMANN TENSOR

R

"COMMUTATOR"

5) CURVATURE $\simeq$ MEASURES HOW MUCH THE SPACETIME IS CURVED

DEF: IN ABSENCE OF TORSION, RIEMANN TENSOR IS DEFINED AS

$$\boxed{[D_\mu, D_\nu] V^\alpha = R^\alpha{}_{\beta\mu\nu} V^\beta}$$

OR

$$(\partial_\mu + \Gamma_\mu)(\partial_\nu + \Gamma_\nu) - (\partial_\nu + \Gamma_\nu)(\partial_\mu + \Gamma_\mu)(\cdot)$$

$$= \underbrace{\partial_\mu \partial_\nu - \partial_\nu \partial_\mu}_{\cancel{\partial}} + \underbrace{\partial_\mu}_{\text{TORSION}} + \underbrace{\text{NO DERIVATIVES}}_{\text{RIEMANN!}}$$

MORE EXPLICITELY :

$$R^{\rho}{}_{\sigma\mu\nu} = \partial_\mu \Gamma^{\rho}_{\nu\sigma} - \partial_\nu \Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma}$$

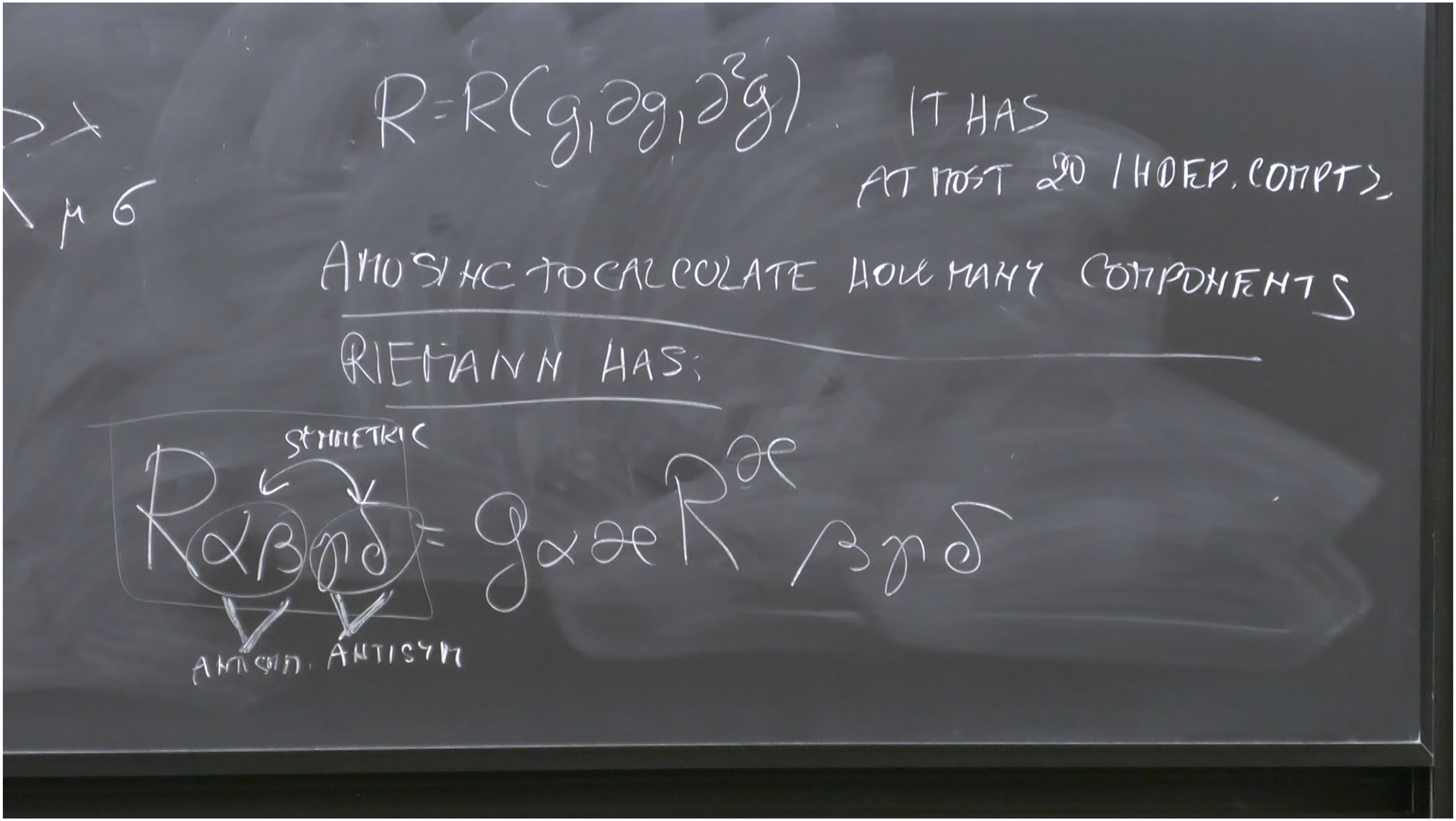
↑ ANTISYMMETRIC !

! ONLY DEPENDS ON CONNECTION !

REMEMBER: EQ. PRINCIPLE. $g = \eta$, $\partial g = 0 = \Gamma$

$\partial^2 g$? ... 20 NON-ZERO LEFT ?

THESE ARE ENCODED IN RIEMANN !


$R = R(g, \partial g, \partial^2 g)$ IT HAS
AT MOST 20 INDEP. COMPTS.
AMUSING TO CALCULATE HOW MANY COMPONENTS
RIEMANN HAS:
SYMMETRIC
$R_{\alpha\beta\gamma\delta} = g_{\alpha\kappa} R^{\kappa}{}_{\beta\gamma\delta}$
ANTISYM. ANTISYM

SYMMETRIES

$$R_{\alpha\beta\gamma\delta} = -R_{\beta\alpha\gamma\delta}$$

$$\underbrace{\alpha\beta}_{A}\ \underbrace{\gamma\delta}_{B} = -R_{\alpha\beta\delta\gamma}$$

$$= R_{\underbrace{\gamma\delta}_{B}\underbrace{\alpha\beta}_{A}}$$

$$R_{[\alpha\beta\gamma\delta]} = 0$$

ANTISYMMETRIZE OVER ALL INDICES $\Rightarrow 0$

$$R_{\underbrace{\alpha\beta}_{A}\underbrace{\gamma\delta}_{B}} = R_{BA}$$

d <u>DIMENSIONS</u>

$\underset{\sim}{A}$ TAKES $\binom{d}{2} = \frac{d(d-1)}{2} = n$ VALUES.

$\frac{n(n+1)}{2}$ COMPTS

THESE ARE ENCODED IN RIEMANN?

SYMMETRIES

$$R_{\alpha\beta\gamma\delta} = -R_{\beta\alpha\gamma\delta}$$
$$\quad A \quad B = -R_{\alpha\beta\delta\gamma}$$
$$= R_{\gamma\delta\alpha\beta}$$
$$\quad B \quad A$$

$R_{[\alpha\beta\gamma\delta]} = 0$

ANTISYMMETRIZE OVER ALL INDICES $\Rightarrow 0$

GIVES

$$\binom{d}{4} = \frac{d(d-1)(d-2)(d-3)}{4\cdot3\cdot2\cdot1}$$

CONSTRAINTS

COMPTS OF

$$\#\text{COMPTS OF RIEMANN} = \frac{n(n+1)}{2} - \binom{d}{4} = \frac{\frac{d(d-1)}{2}\left(\frac{d(d-1)}{2}+1\right)}{2} - \binom{d}{4}$$

$$= \frac{d^2}{12}\left(d^2-1\right)$$

$d=4$

$$= 20 \text{ COMPTS}$$

RELATIONS TO RIEMANN.

$$+1) - \binom{d}{4}$$

$$R^{\underline{\alpha}}{}_{\beta\gamma\delta} \qquad \boxed{R_{\beta\delta} = R^{\alpha}{}_{\beta\alpha\delta}}$$

RICCI TENSOR

(STILL CONNECTION ONLY)

RELATIONS TO RIEMANN.

$$\left(\frac{d(d-1)}{2}+1\right)-\binom{d}{4}$$

$$R^{\alpha}{}_{\beta\gamma\delta}$$

$$\boxed{R_{\beta\delta}=R^{\alpha}{}_{\beta\alpha\delta}}$$

RICCI TENSOR

(STILL CONNECTION ONLY)

$$\boxed{R=R_{\alpha\beta}\,g^{\alpha\beta}}$$

RICCI SCALAR

$$\boxed{G^{\mu\nu}=R^{\mu\nu}-\frac{1}{2}R\,g^{\mu\nu}}$$

EINSTEIN TENSOR