Title: Relativity Lecture - 101823

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WE WANT TO DIFFERENTIATE TENSORS

- LIE DERIVATIVE (NEED v)
- EXTERIOR DERIVATIVE (RESTRICTED)
- COVARIANT DERIVATIVE (NEED Γ)

} RUTH'S COURSE

OBSERVATION: $\partial_\mu \phi(x)$ ... IS A (0,1) TENSOR

$$\underbrace{\frac{\partial \phi'}{\partial x'^\mu}} = \frac{\partial x^\nu}{\partial x'^\mu} \frac{\partial}{\partial x^\nu} \phi = \frac{\partial x^\nu}{\partial x^{\mu'}} \underbrace{\partial_\nu \phi}$$

• TAKE VECTOR $\partial_\mu V^\nu$ ... IS THIS A $(1,1)$ TENSOR ?

$$\frac{\partial V^{\nu'}}{\partial x^{\mu'}} = \frac{\partial x^\alpha}{\partial x^{\mu'}} \frac{\partial}{\partial x^\alpha}\left(\frac{\partial x^{\nu'}}{\partial x^\beta} V^\beta \right)$$

$$= \underbrace{\frac{\partial x^\alpha}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\beta} \partial_\alpha V^\beta}_{\text{THIS IS WHAT WE WOULD LIKE TO HAPPEN}} + \underbrace{\frac{\partial x^\alpha}{\partial x^{\mu'}} \frac{\partial^2 x^{\nu'}}{\partial x^\alpha \partial x^\beta} V^\beta}_{\text{PROBLEM.}}$$

(UNLESS SR)

(RIEMANN - PHD 1830)

CONNECTION

$$\boxed{\nabla_\mu V^\nu \equiv \partial_\mu V^\nu + \Gamma^\nu_{\alpha\mu} V^\alpha}$$

$$\boxed{{}_{\beta\gamma} = \underbrace{\frac{\partial x'^\alpha}{\partial x^\varkappa}\frac{\partial x^\delta}{\partial x'^\beta}\frac{\partial x^z}{\partial x'^\gamma}\Gamma^\varkappa_{\delta z}}_{\text{WOULD BE OK FOR (1,2) TENSOR}} - \underbrace{\frac{\partial^2 x'^\alpha}{\partial x^\varkappa \partial x^\delta}\frac{\partial x^\varkappa}{\partial x'^\beta}\frac{\partial x^\delta}{\partial x'^\gamma}}_{\text{SPOILER.}}}$$

TO DEAL WITH THIS (RIEMANN - PHD 1830)

CONNECTION

COVARIANT DERIVATIVE

$$\boxed{\nabla_\mu V^\nu \equiv \partial_\mu V^\nu + \Gamma^\nu{}_{\alpha\mu} V^\alpha}$$

WHERE

$$\Gamma'^{\alpha}{}_{\beta\gamma} = \underbrace{\frac{\partial x'^\alpha}{\partial x^\kappa} \frac{\partial x^\delta}{\partial x'^\beta} \frac{\partial x^2}{\partial x'^\gamma} \Gamma^\kappa{}_{\delta 2}}_{\text{WOULD BE OK FOR (1,2) TENSOR}} - \frac{\partial^2}{\partial x}$$

GENERAL CONNECTION

ION

$$\cdot\ \nabla_\mu \phi = \partial_\mu \phi$$

$$\cdot\ \boxed{\nabla_\mu \omega_\nu = \partial_\mu \omega_\nu - \Gamma^{\rho}_{\mu\nu}\,\omega_\rho}$$

$$- \underbrace{\frac{\partial^2 x'^\alpha}{\partial x^\kappa \partial x^\delta} \frac{\partial x^\kappa}{\partial x'^\beta} \frac{\partial x^\delta}{\partial x'^\gamma}}_{\text{SPOILER.}}$$

PROOF: $\nabla_\mu (V^\alpha \omega_\alpha) = \partial_\mu (V^\alpha \omega_\alpha)$

$$= (\partial V + \Gamma V)\omega + V(\partial\omega - \Gamma\omega)$$

$$= (\partial V)\omega + V\partial\omega = \partial(V\omega)$$

GENERAL CONNECTION

WOULD BE OK FOR (1,2) TENSOR

$$\nabla_\alpha T_{\beta\ldots}{}^{\gamma\ldots} = \partial_\alpha T_{\beta\ldots}{}^{\gamma\ldots} + \Gamma^\gamma_{\alpha\delta}\, T_{\beta\ldots}{}^{\delta\ldots} + \ldots -$$

- "ANTISYMMETRIZED VERSION OF Γ" IS A TENSOR

$$T^\alpha{}_{\beta\gamma} = -2\,\Gamma^\alpha{}_{[\beta\gamma]} = \Gamma^\alpha{}_{\gamma\beta} - \Gamma^\alpha{}_{\beta\gamma}$$

TORSION

$$\tfrac{1}{2}(\beta\gamma - \gamma\beta)$$

$$\ldots - \Gamma^{\varkappa}_{\alpha\beta}\, T^{\eta\ldots}_{\varkappa\ldots}$$

T IS ZERO IN GR (ENERGY MOM IS SOURCE OF METRIC)

CAN BE NON-TRIVIAL IN ST, SUGRA, ..

SPIN IS THE SOURCE OF TORSION

DEF: COVARIANT DERIVATIVE ∇ IS A MAP: (k,l) TENSORS $\rightarrow$ $(k,l+1)$ TENSORS.

i) IT IS A DERIVATIVE: LINEAR & LEIBNITZ

ii) REDUCES TO ∂ ON SCALARS: $\nabla_\mu \phi = \partial_\mu \phi$

iii) COMMUTES WITH CONTRACTION $\nabla\left(T^{\alpha\dots}{}_{\alpha\dots}\right) = \nabla T_{\text{CONTR.}}$

iv) WE HAVE

$$[\nabla_\mu, \nabla_\nu]\phi = \left(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu\right)\phi = -T^\alpha{}_{\mu\nu} \nabla_\alpha \phi$$

SORS.

d) $\boxed{\text{METRIC}}$ g ... IS A SYMMETRIC, NON-DEGENERATE (0,2) TENSOR

$$g = g_{\mu\nu}\, dx^\mu \otimes dx^\nu \cong g_{\mu\nu}\, dx^\mu dx^\nu \cong ds^2$$

$\nabla_\alpha \phi$

SORS.

$\alpha \phi$

d) $\boxed{\text{METRIC}}$ g ... IS A SYMMETRIC, NON-DEGENERATE (0,2) TENSOR

$$g = g_{\mu\nu}\, dx^\mu \otimes dx^\nu \cong g_{\mu\nu}\, dx^\mu dx^\nu \cong ds^2$$

IN GR: 3 POSITIVE & 1 NEGATIVE EIGENVALUES

$\Rightarrow$ PSEUDO RIEMANNIAN GEOMETRY !

(FOR EXAMPLE: CAN HAVE NON-TRIVIAL k: $k^2 = 0$)

DEF: METRICITY TENSOR

$$\boxed{\nabla_\mu g_{\alpha\beta} = M_{\mu\alpha\beta}}$$

? WHAT WOULD BE THE PHYSICAL SOURCE OF METRICITY ?

IN GR: $M_{\mu\alpha\beta} = 0$

THEOREM: IF $M_{\alpha\beta\gamma} = 0 = T^{\alpha}{}_{\beta\gamma}$

$\Rightarrow$ THERE EXISTS A UNIQUE CONNECTION ... CHRISTOFFEL

$$\boxed{\Gamma^{\alpha}{}_{\beta\gamma} = \frac{1}{2} g^{\alpha\delta}\left(g_{\delta\beta,\gamma} + g_{\delta\gamma,\beta} - g_{\beta\gamma,\delta}\right)}$$

PRINCIPLE OF EQUIVALENCE (MATH VERSION)

$(\beta\gamma,\delta)$

AT EVERY SPACETIME POINT IT IS POSSIBLE TO CHOOSE COORD. SYSTEM (FREELY FALLING LOCAL INERT. SYSTEM) SUCH THAT THE LAWS REDUCE TO SR.

$$g_{\mu\nu}\big|_p = \eta_{\mu\nu}, \quad g_{\mu\nu,\alpha}\big|_p = 0$$

$$\Gamma\big|_p = 0$$

PROOF: (HINT THAT IT MIGHT WORK)

LET'S START FROM $\{x^\alpha\}$ $g_{\alpha\beta}$, $\Gamma^\alpha{}_{\beta\gamma}$ ARE NON-TRIVIAL. DO A COORD. TRANSF. TO NEW $\{x'^\alpha\}$ WHER $g=\eta$, $\Gamma=0$

- $x'^\alpha = A^\alpha{}_\beta x^\beta + O(x^2) \iff x^\alpha = \tilde{A}^\alpha{}_\beta x'^\beta + O(x'^2)$

$$g'_{\mu\nu} = \frac{\partial x^\gamma}{\partial x'^\mu}\frac{\partial x^\delta}{\partial x'^\nu} g_{\gamma\delta} = \tilde{A}^\gamma{}_\mu \tilde{A}^\delta{}_\nu g_{\gamma\delta}$$

WE HAVE $\underline{\text{10 EQS}}$, BUT WE HAVE 16 COMPTS OF $\tilde{A}^{\alpha}{}_{\beta}$ TO PLAY WITH.

→ CAN BE DONE.

IN FACT WE HAVE 6 MORE DOF. → CAN STILL DO LORENTZ TRANSF.

- WHAT ABOUT IF I GO TO NEXT ORDER?

$$x'^{\alpha} = \underbrace{A^{\alpha}{}_{\beta} x^{\beta}}_{\text{FIXED}} + \underbrace{B^{\alpha}{}_{(\beta\gamma)}}_{\text{40 COMPTS}} x^{\beta} x^{\gamma} + O(x^3)$$

$g_{\mu\nu,\alpha}$ ... 40 COMPTS. $\boxed{g_{\mu\nu,\alpha} = 0}$ CAN BE DONE.

∘ NEXT ORDER.

$$X'^{\alpha} = \underbrace{A^{\alpha}{}_{\beta}\, x^{\beta}}_{\text{FIXED}} + \underbrace{B^{\alpha}{}_{\beta\gamma}\, x^{\beta} x^{\gamma}}_{\text{FIXED}} + C^{\alpha}{}_{(\beta\gamma\delta)}\, x^{\beta} x^{\gamma} x^{\delta} + O(x^{4})$$

$g = \eta$ $\qquad$ $\partial g = 0$

$$\boxed{4 \mid 5 \mid 6} = \frac{4\cdot 5\cdot 6}{1\cdot 2\cdot 3} = 20$$

$C^{\alpha}{}_{(\beta\gamma\delta)}$. $4 \times 20 = 80$ COMPTS.

WE HAVE $\underline{10 \text{ EQS}}$, BUT WE HAVE 16 COMPTS OF $\tilde{A}^{\alpha}{}_{\beta}$ TO PLAY WITH.

→ CAN BE DONE !

IN FACT WE HAVE 6 MORE DOF. → CAN STILL DO LORENTZ TRANSF.

• WHAT ABOUT IF I GO TO NEXT ORDER?

$$X'^{\alpha} = \underbrace{A^{\alpha}{}_{\beta} X^{\beta}}_{\text{FIXED}} + \underbrace{B^{\alpha}{}_{(\beta\gamma)}}_{4\times 10} X^{\beta} X^{\gamma} + O(x^3)$$

40 COMPTS.

WE WANT $\overset{\checkmark}{=} \eta_{\mu\nu}$

$$\boxed{0 = g_{\mu\nu,\alpha}} \quad (10\times 4) \ \ldots \ 40 \text{ COMPTS.} \qquad \boxed{g_{\mu\nu,\alpha} = 0}$$

CAN BE DONE.

$\partial^2 g$

$O(x^3))$

$= 20$

80 COMPTS

NTRIVIAL $\partial^2 g$

$$R^\alpha{}_{\beta\gamma\delta}(\partial^2 g, \partial g, g) \quad \ldots \text{ 20 COMPTS.}$$

RIEMANN TENSOR.

- INVERSE METRIC

$$\boxed{g^{\alpha\beta} g_{\beta\gamma} = \delta^\alpha{}_\gamma}$$

$(g^{-1})^{\alpha\beta}$ $(0,2)$

$$\boxed{V^\alpha = g^{\alpha\beta} V_\beta}$$

$$g'_{\mu\nu} = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta} \qquad / \det$$

$$\det g'_{\mu\nu} = g' = \underbrace{\left(\det \frac{\partial x}{\partial x'}\right)}_{\text{JACOBIAN}} \det\left(\frac{\partial x}{\partial x'}\right) g$$

$$\boxed{g' = \left|\frac{\partial x'}{\partial x}\right|^{-2} g} \qquad \left|\frac{\partial x}{\partial x'}\right| = \left|\frac{\partial x'}{\partial x}\right|^{-1}$$

- g ... SCALAR DENSITY OF WEIGHT +2

$$\phi' = \left|\frac{\partial x'}{\partial x}\right|^{-w}\phi$$

... ϕ ... SCALAR DENSITY OF WEIGHT $\underline{w}$

- $$dV' = d^n x' = \left|\frac{\partial x'}{\partial x}\right| d^n x = \left|\frac{\partial x'}{\partial x}\right| dV$$

LET $[\alpha\beta\gamma\delta]$ BE A PERMUTATION SYMBOL

$$[\alpha\beta\gamma\delta] = \begin{cases} +1 & [0,1,2,3] \text{ \& EVEN PERMS.} \\ -1 & \text{FOR ODD PERMS.} \\ 0 & \text{IF 2 OR MORE ENTRIES ARE THE SAME,} \end{cases}$$

<u>LEVI-CIVITA TENSOR</u>

$$\varepsilon_{\alpha\beta\gamma\delta} = \sqrt{-g}\,[\alpha\beta\gamma\delta], \quad \varepsilon^{\alpha\beta\gamma\delta} = -\frac{1}{\sqrt{-g}}$$

$\ldots]$ BE A PERMUTATION SYMBOL

$$[\ldots] = \begin{cases} +1 & [0,1,2,3] \text{ \& EVEN PERMS.} \\ -1 & \text{FOR ODD PERMS.} \\ 0 & \text{IF 2 OR MORE ENTRIES ARE THE SAME.} \end{cases}$$

TENSOR

$$\ldots\gamma\delta = \sqrt{-g}\,[\alpha\beta\gamma\delta], \quad \varepsilon^{\alpha\beta\gamma\delta} = -\frac{1}{\sqrt{-g}}[\alpha\beta\gamma\delta]$$