Title: Relativity Lecture - 101623

Speakers: David Kubiznak

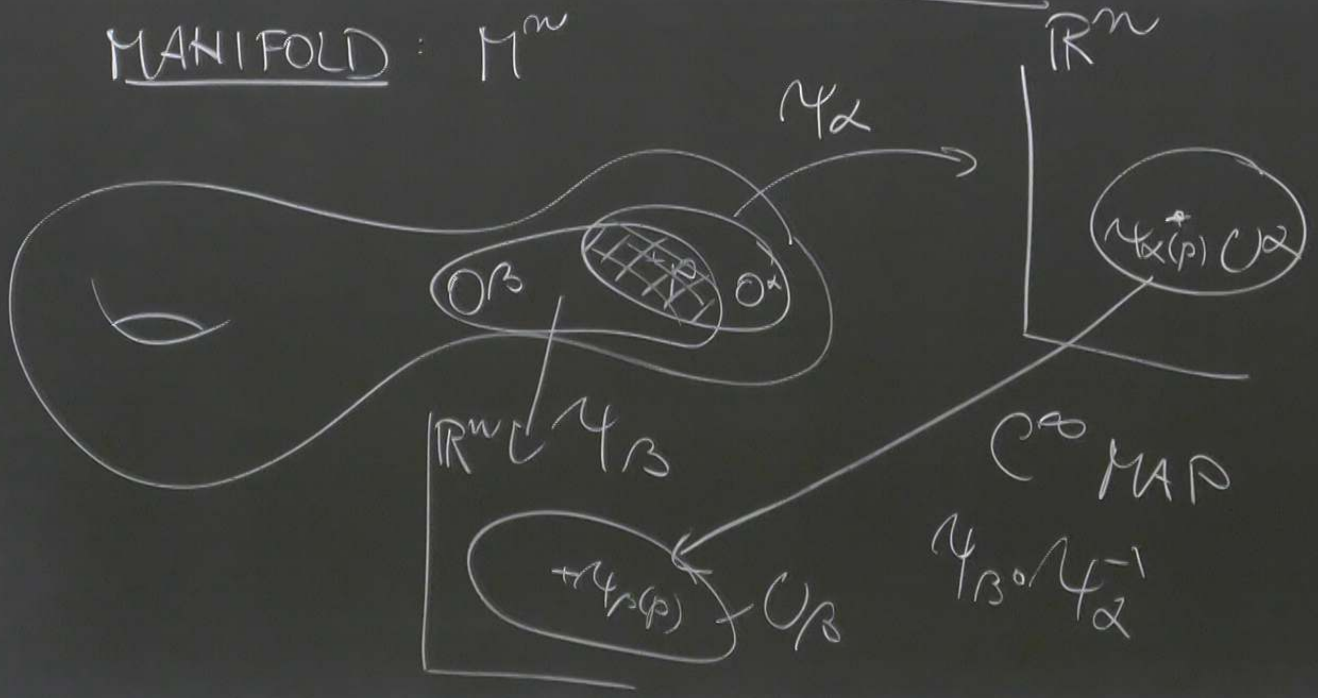
Collection: Relativity 2023/24

Date: October 16, 2023 - 9:00 AM

URL: https://pirsa.org/23100038

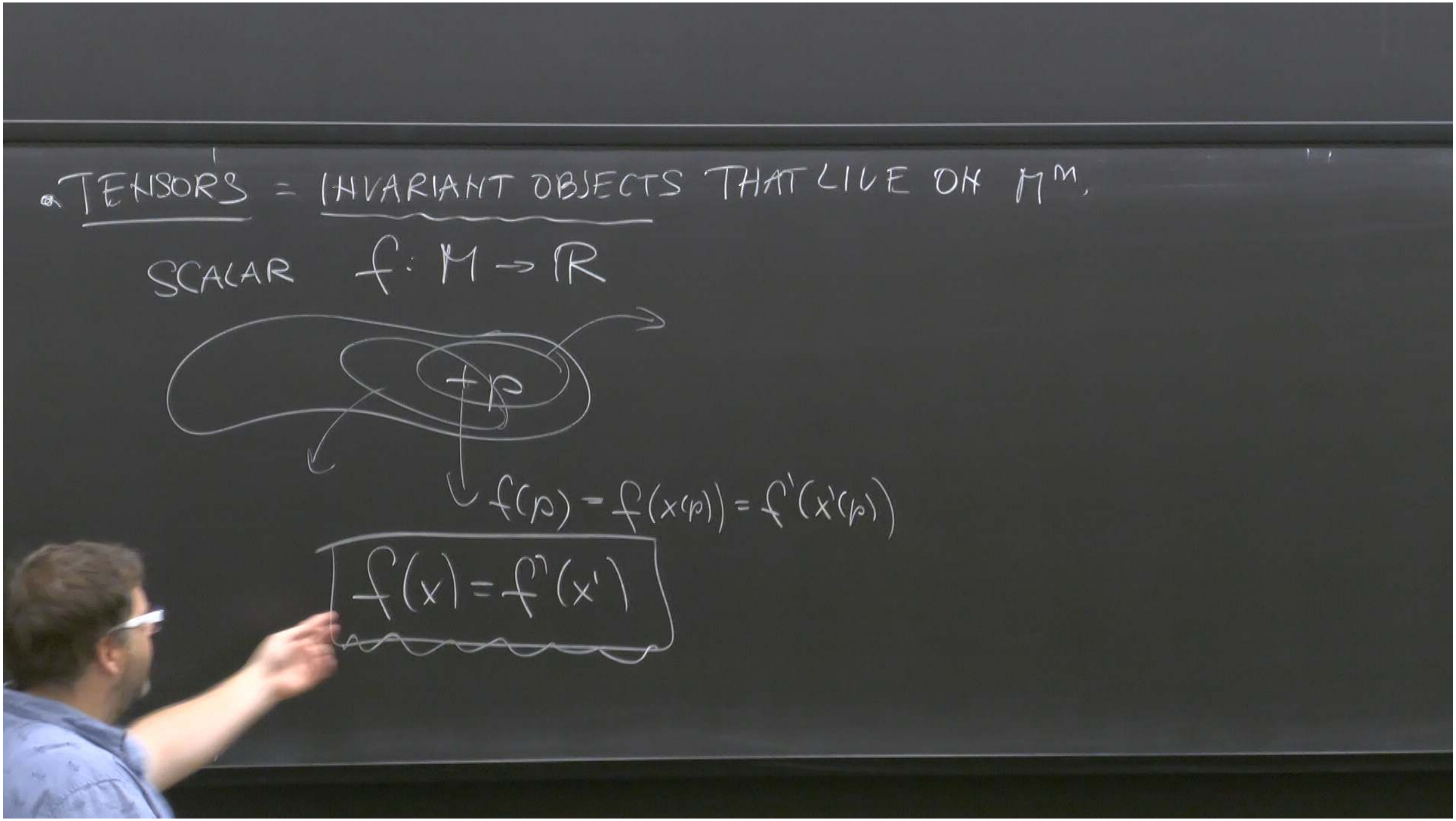
2) DIFFERENTIAL GEOMETRY

MANIFOLD: M^n



C^∞ MAP

$\psi_\beta \circ \psi_\alpha^{-1}$



- **TENSORS** = INVARIANT OBJECTS THAT LIVE ON M^m.

SCALAR $f : M \to \mathbb{R}$

$$f(p) = f(x(p)) = f'(x'(p))$$

$$\boxed{f(x) = f'(x')}$$

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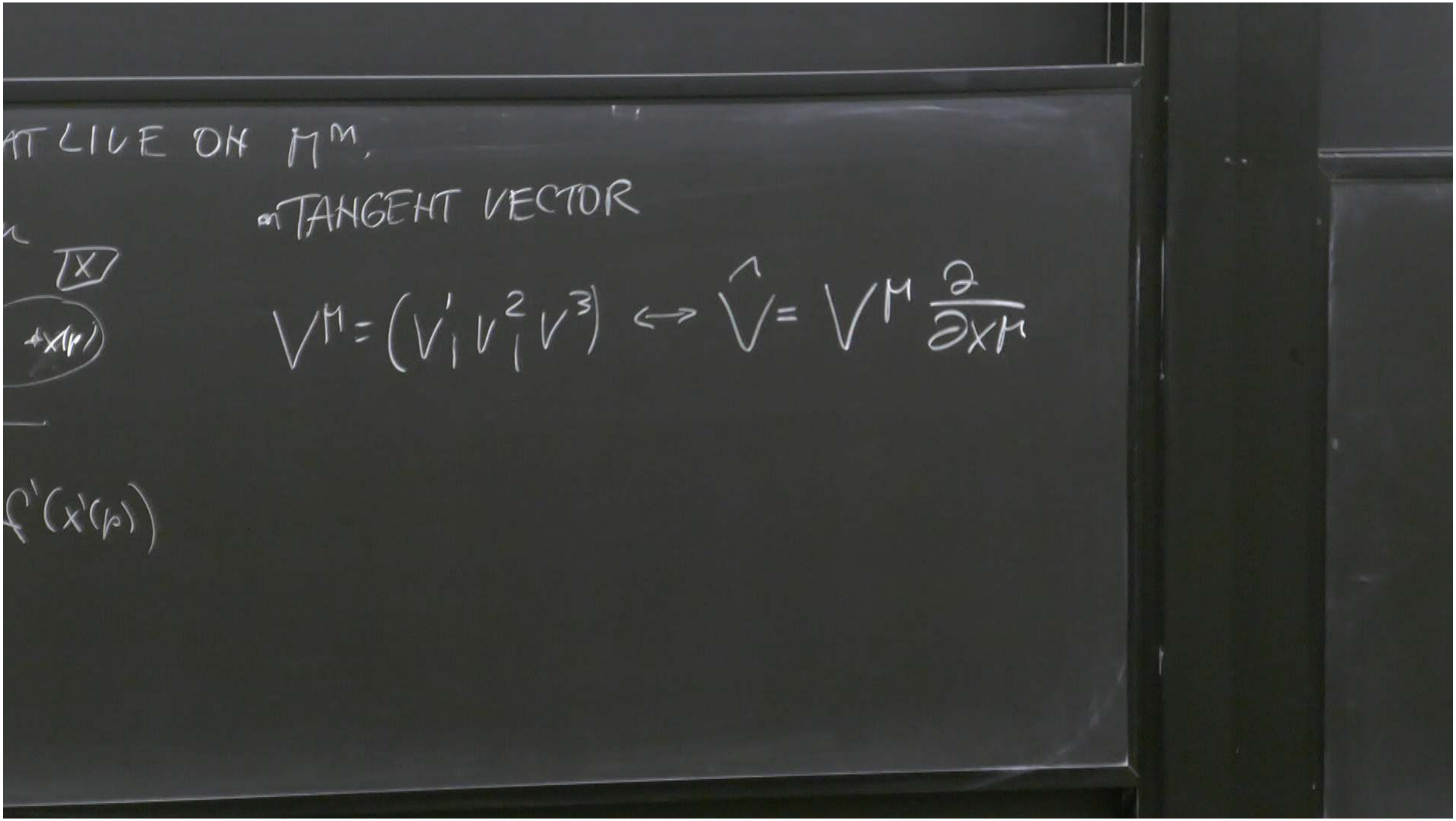


AT LIVE ON M^m.

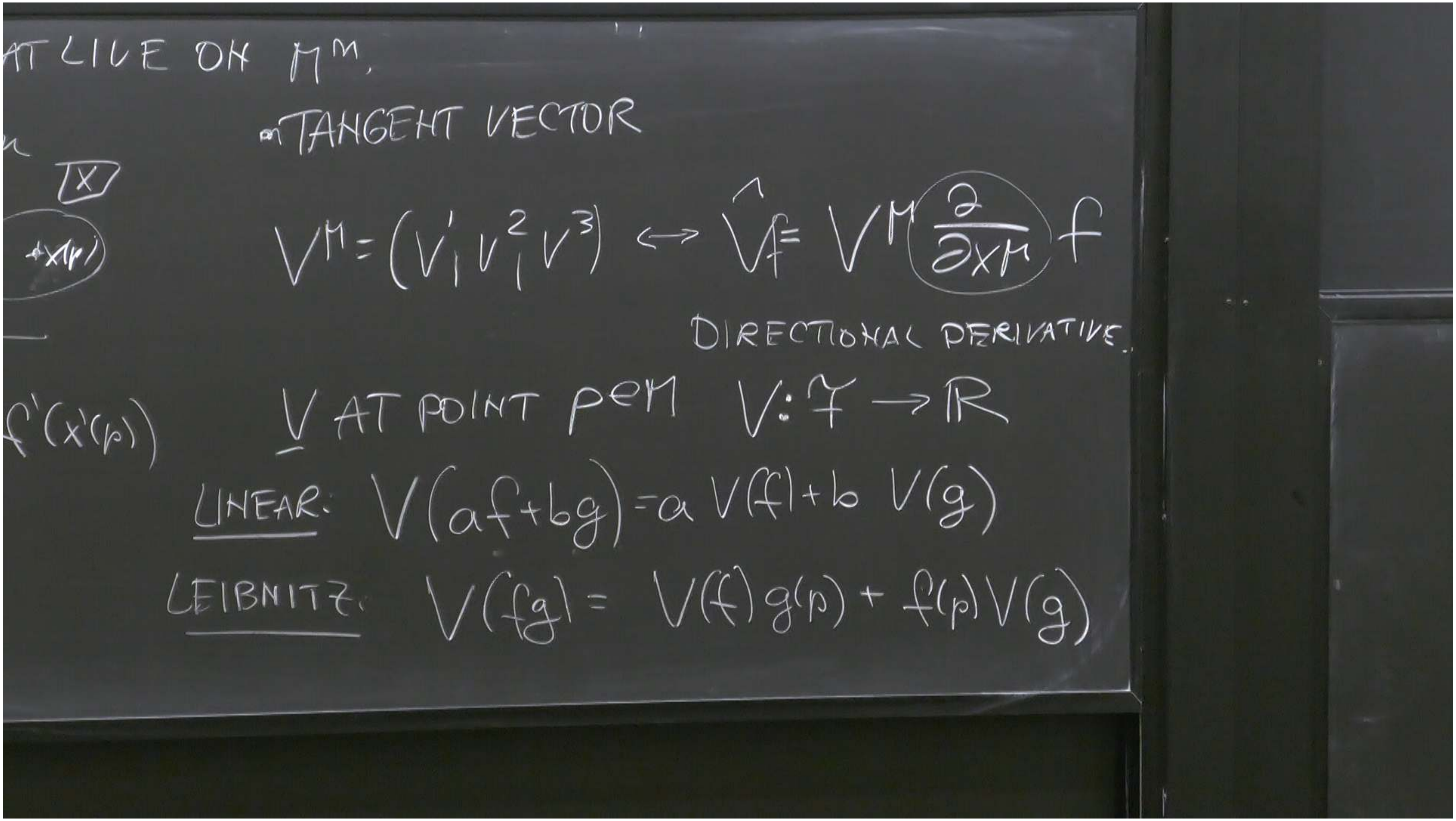
TANGENT VECTOR

$\boxed{X}$

$+x(p)$

$$V^\mu = (V^1, V^2, V^3) \leftrightarrow \hat{V} = V^\mu \frac{\partial}{\partial x^\mu}$$

$f'(x'(p))$



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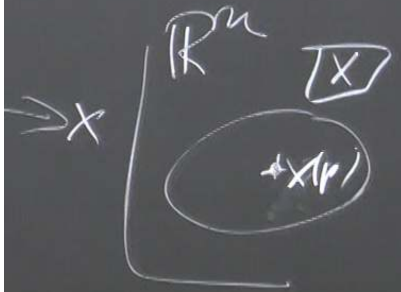
DIRECTIONAL DERIVATIVE.

$\underline{V}$ AT POINT $p \in M$ $\quad V: \mathcal{F} \to \mathbb{R}$

LINEAR: $V(af+bg) = aV(f) + bV(g)$

LEIBNITZ: $V(fg) = V(f)g(p) + f(p)V(g)$

CTS THAT LIVE ON M^m.



TANGENT VECTOR

$$V^\mu = (V^1, V^2, V^3) \leftrightarrow \hat{V}f = V^\mu \boxed{\frac{\partial}{\partial x^\mu}} f$$

DIRECTIONAL DERIVATIVE.

$f(x(p)) = f'(x'(p))$

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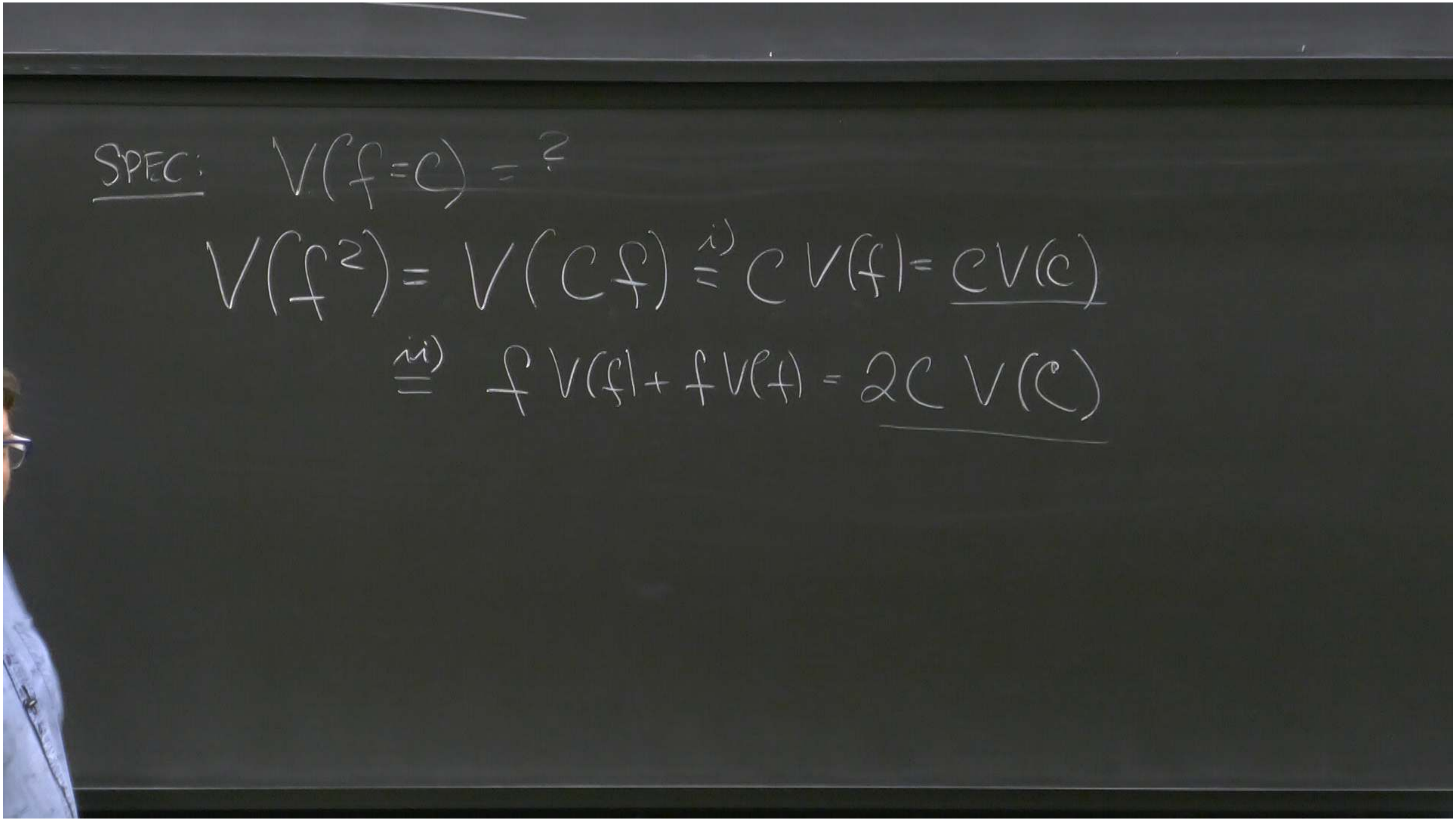

SPEC: $V(f=c) = ?$

$$V(f^2) = V(cf) \overset{i)}{=} cV(f) = \underline{cV(c)}$$

$$\overset{ii)}{=} fV(f) + fV(f) = \underline{2cV(c)}$$

SPEC: $V(f=c) = ?$

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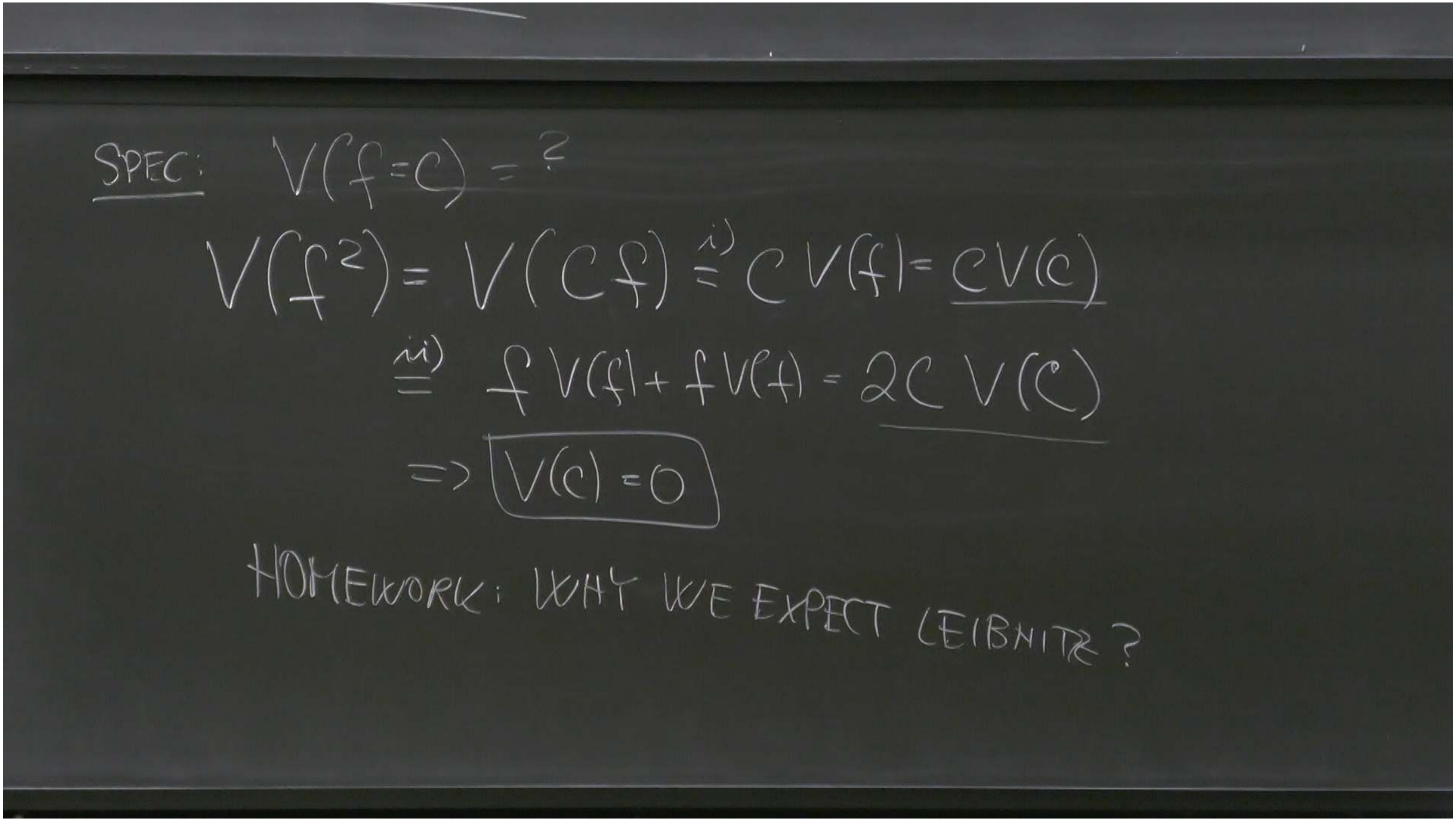
$$\Rightarrow \boxed{V(c) = 0}$$



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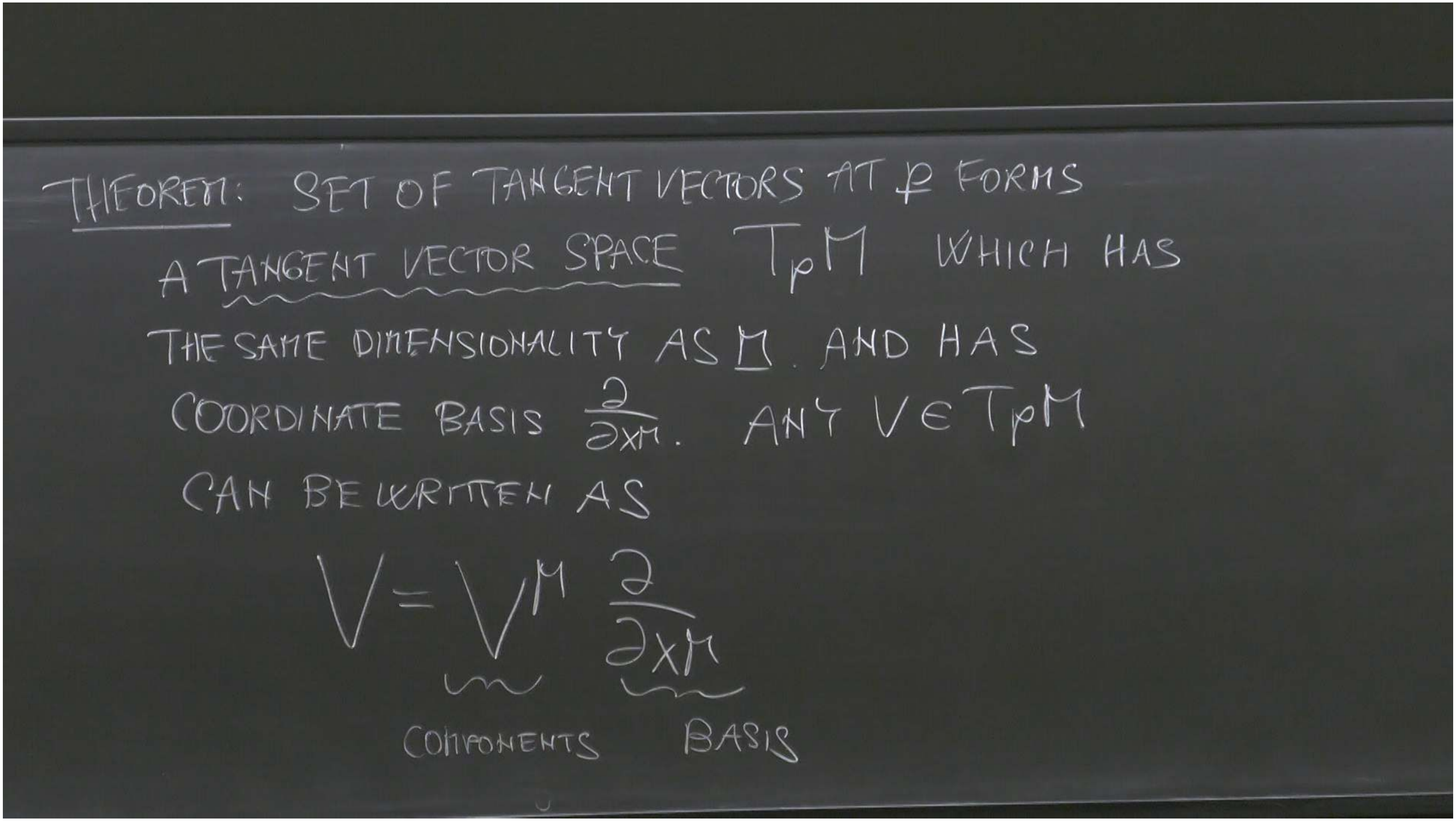
HOMEWORK: WHY WE EXPECT LEIBNITZ?



THEOREM: SET OF TANGENT VECTORS AT p FORMS A TANGENT VECTOR SPACE T_pM WHICH HAS THE SAME DIMENSIONALITY AS M. AND HAS COORDINATE BASIS $\frac{\partial}{\partial x^\mu}$. ANY $V \in T_pM$ CAN BE WRITTEN AS

$$V = \underbrace{V^\mu}_{\text{COMPONENTS}} \underbrace{\frac{\partial}{\partial x^\mu}}_{\text{BASIS}}$$

GENT VECTORS AT p FORMS

SPACE T_pM WHICH HAS

ITY AS $\underline{M}$. AND HAS

$\frac{\partial}{\partial x^\mu}$. ANY $V \in T_pM$

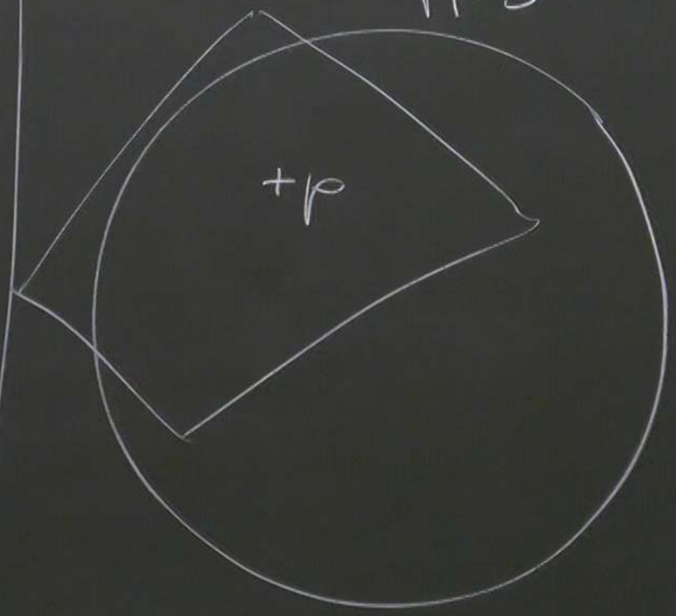
AS

$$1 \underbrace{\frac{\partial}{\partial x^\mu}}$$

TS BASIS

EX:

$M = S^2$



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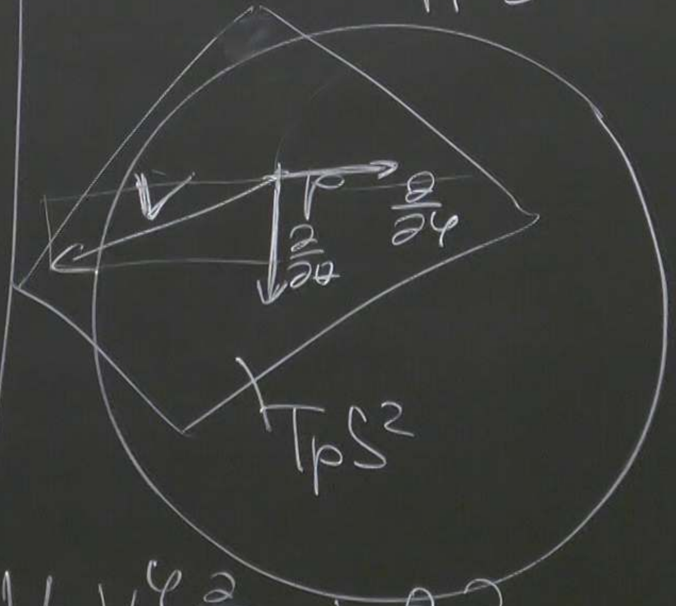


$$V = V^\varphi \frac{\partial}{\partial \varphi} + V^\theta \frac{\partial}{\partial \theta}$$

COMPON

UNDER CHANGE OF COORDINATES, V REMAINS INVARIANT!

$x \to x'(x)$

$$V = V^{\mu} \frac{\partial}{\partial x^{\mu}} \underset{\uparrow \text{ CHAIN RULE}}{=} \underbrace{V^{\mu} \frac{\partial x^{\nu'}}{\partial x^{\mu}}}_{V^{\nu'}} \frac{\partial}{\partial x^{\nu'}}$$

$$\boxed{V^{\nu'} = \frac{\partial x'^{\nu}}{\partial x^{\mu}} V^{\mu}}$$

TRANSF. OF COMPONENTS!

$\varphi_{\beta}(p)$ U_β φ_α

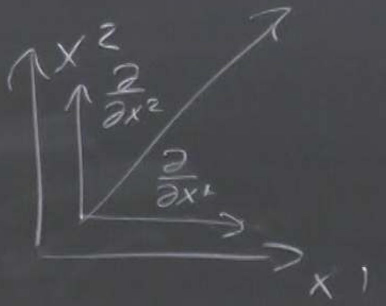
SPEC: $V(f = C) =$

$$V(f^2) = V(Cf) \overset{i)}{=} C\,V(f) = \underline{C\,V(C)}$$

$$\overset{ii)}{=} f\,V(f) + f\,V(f) = \underline{2C\,V(C)}$$

$$\Rightarrow \boxed{V(C) = 0}$$

HOMEWORK: WHY WE EXPE



THEOREM:

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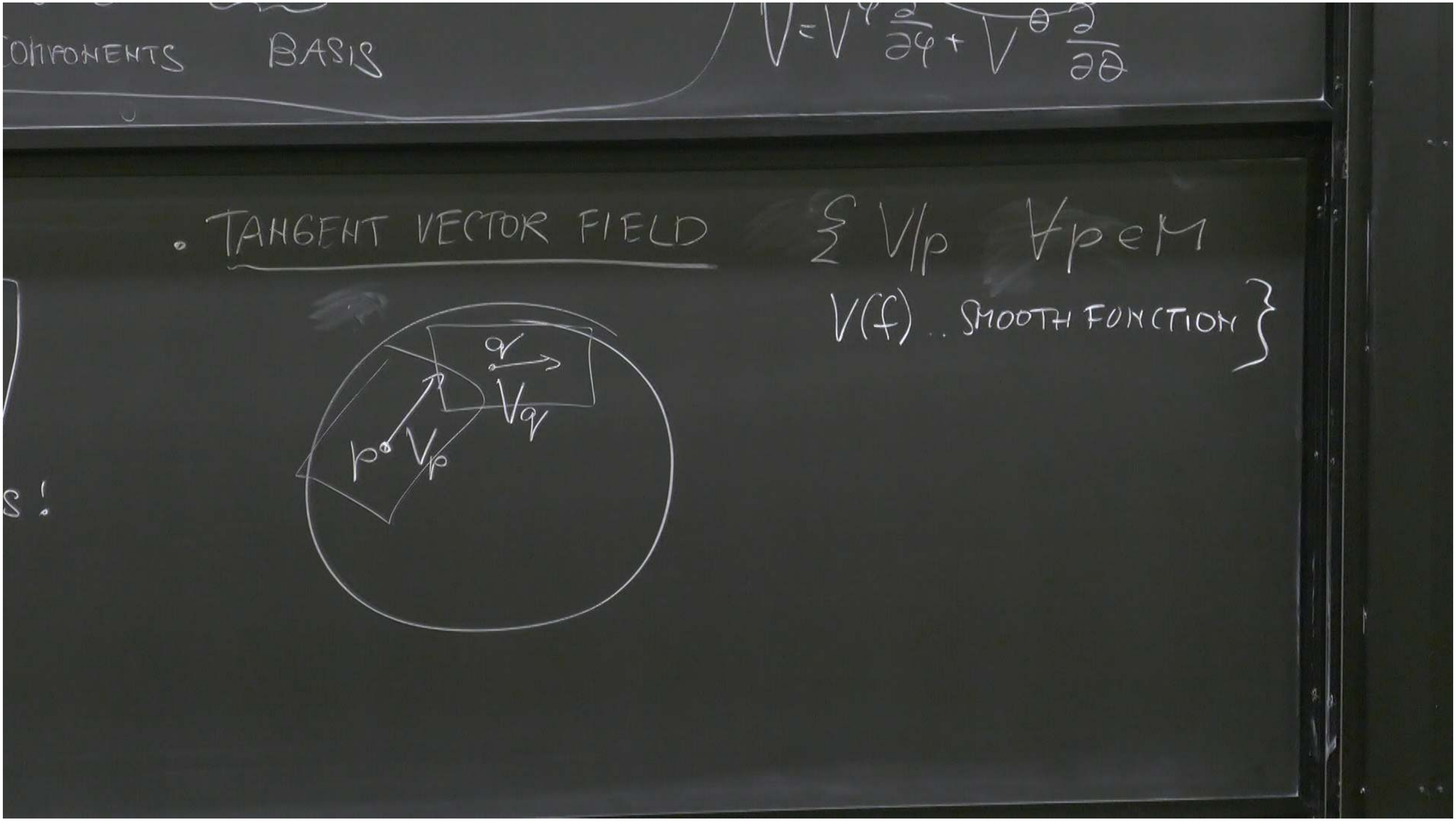
UNDER CHANGE OF COORDINATES, V REMAINS INVARIANT!



OMPONENTS BASIS

$$V = V^{\varphi}\frac{\partial}{\partial\varphi} + V^{\theta}\frac{\partial}{\partial\theta}$$

. TANGENT VECTOR FIELD

$\{ V|_p \quad \forall p \in M$

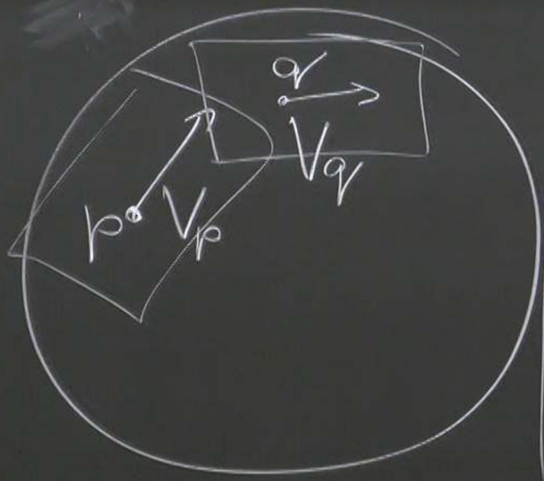
$V(f)$ .. SMOOTH FUNCTION $\}$

s !

s!

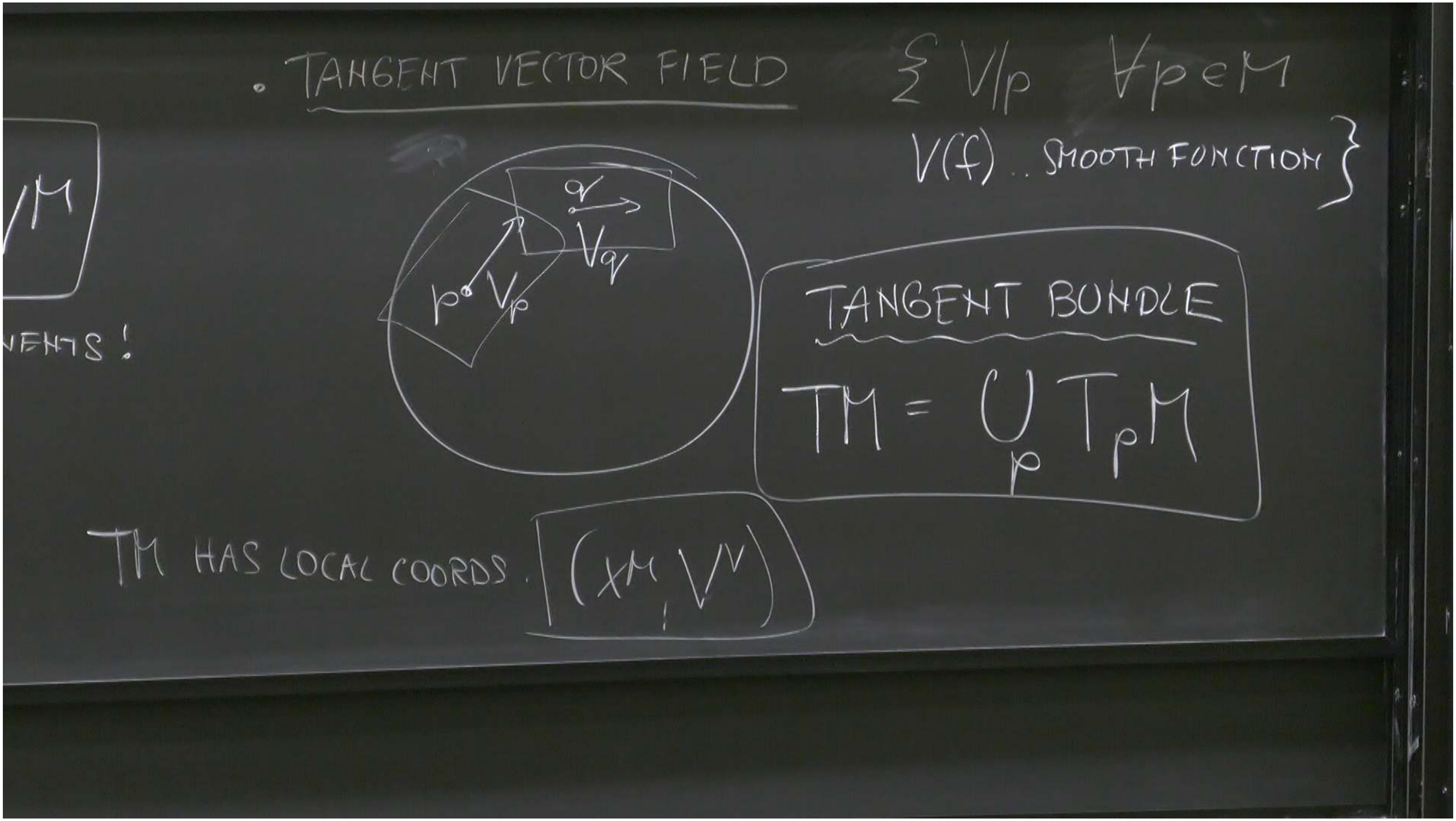
• TANGENT VECTOR FIELD $\{ V|_p \quad \forall p \in M$

$V(f)$ .. SMOOTH FUNCTION $\}$



TANGENT BUNDLE

$$TM = \bigcup_p T_p M$$



- TANGENT VECTOR FIELD $\{ V|_p \ \forall p \in M$

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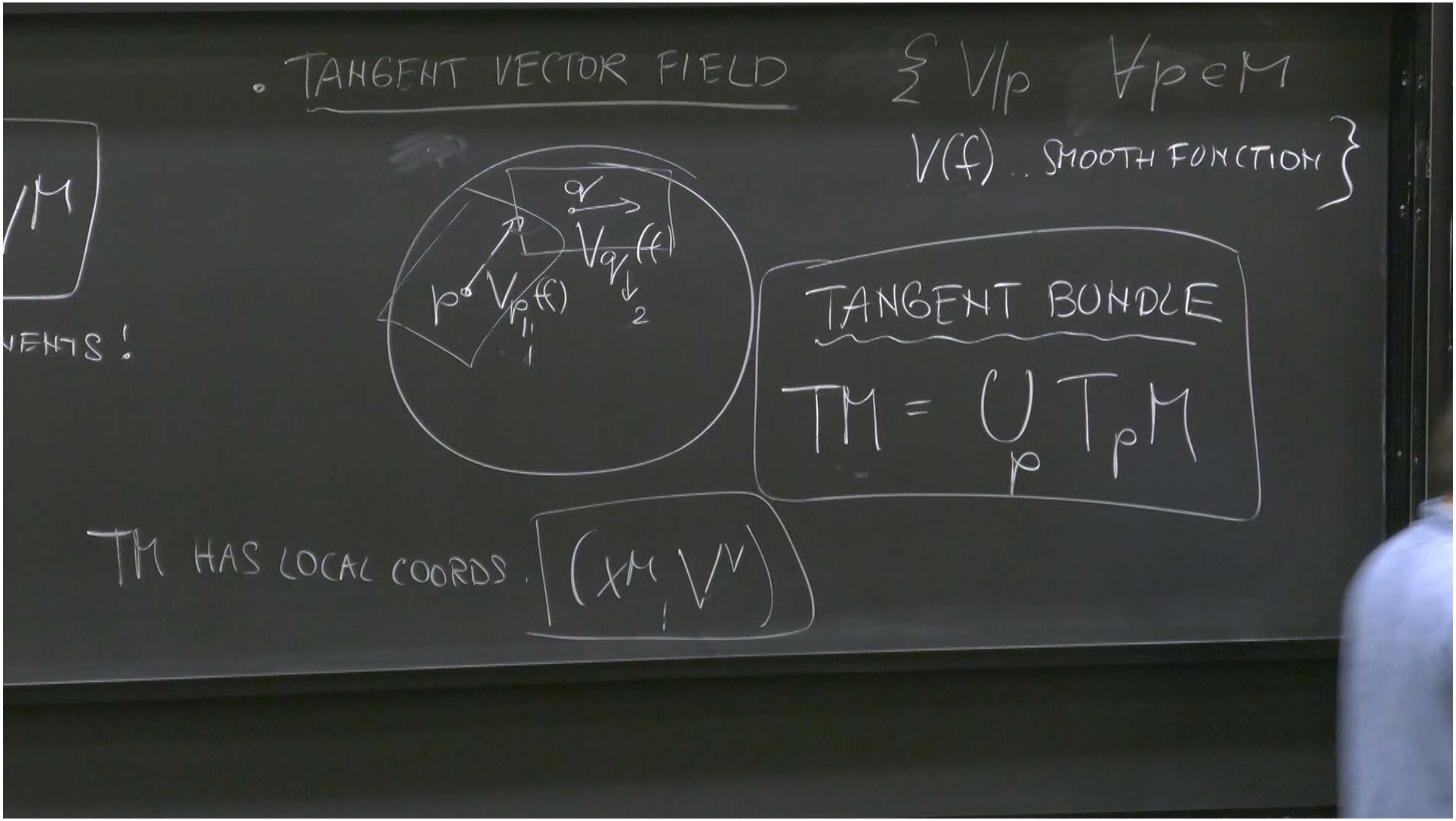
/M

NENTS !

TANGENT BUNDLE

$$\boxed{TM = \bigcup_p T_pM}$$

TM HAS LOCAL COORDS. $\boxed{(x^\mu, V^\nu)}$



V/M

NENTS !

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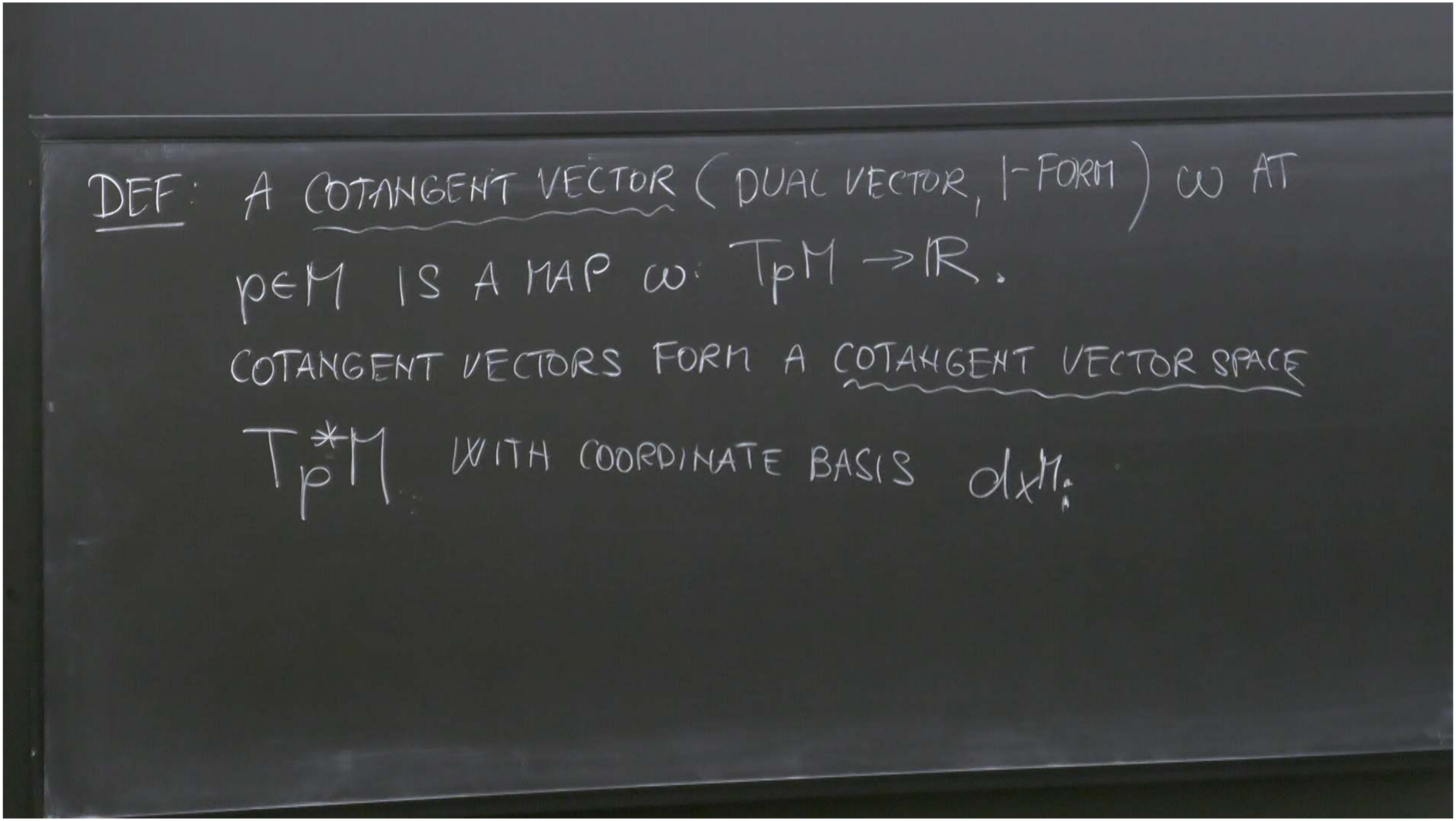
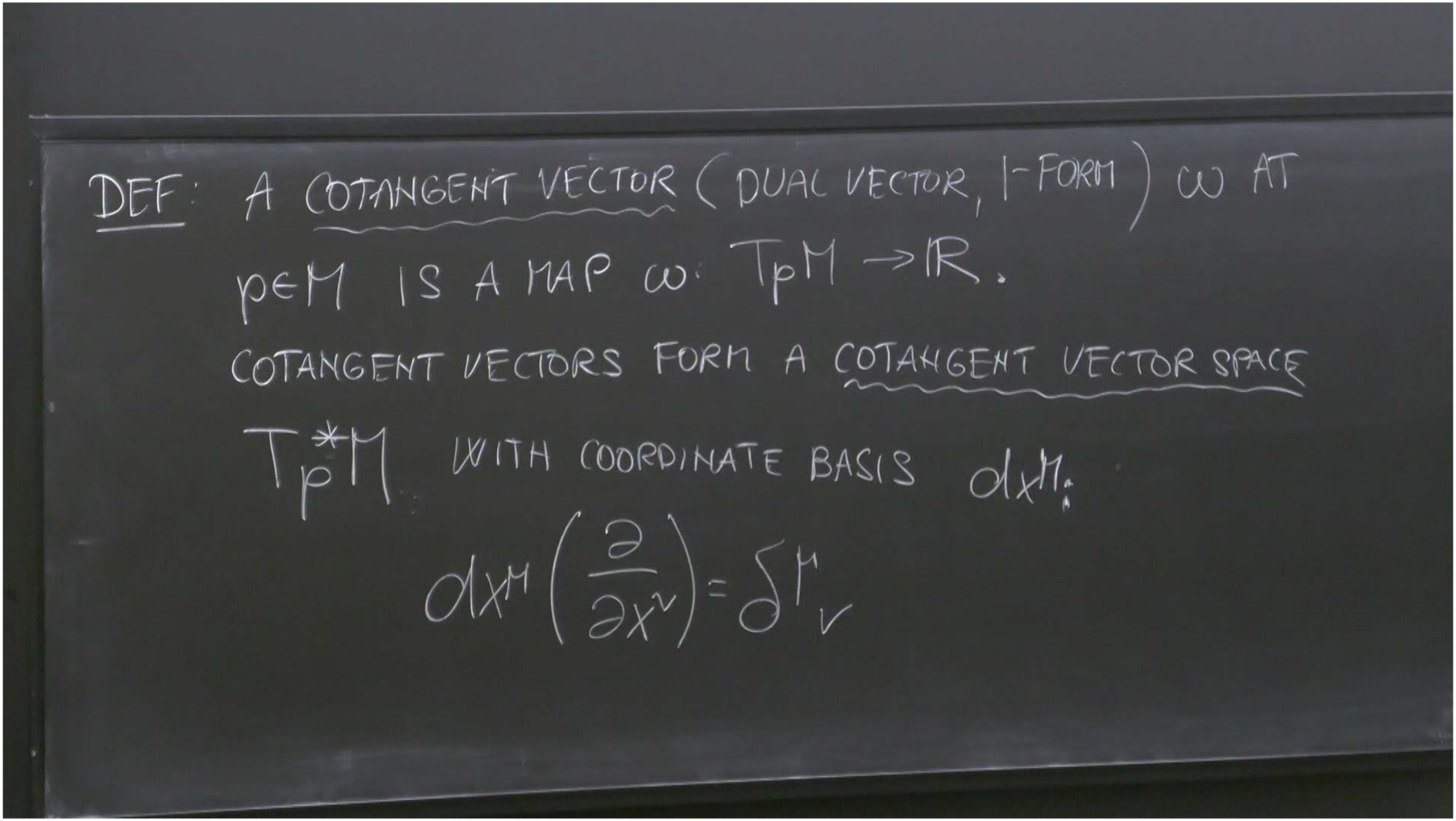


DEF: A COTANGENT VECTOR (DUAL VECTOR, 1-FORM) ω AT $p \in M$ IS A MAP $\omega : T_pM \to \mathbb{R}$.

COTANGENT VECTORS FORM A COTANGENT VECTOR SPACE T_p^*M WITH COORDINATE BASIS dx^μ.



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$$dx^\mu\left(\frac{\partial}{\partial x^\nu}\right) = \delta^\mu{}_\nu$$

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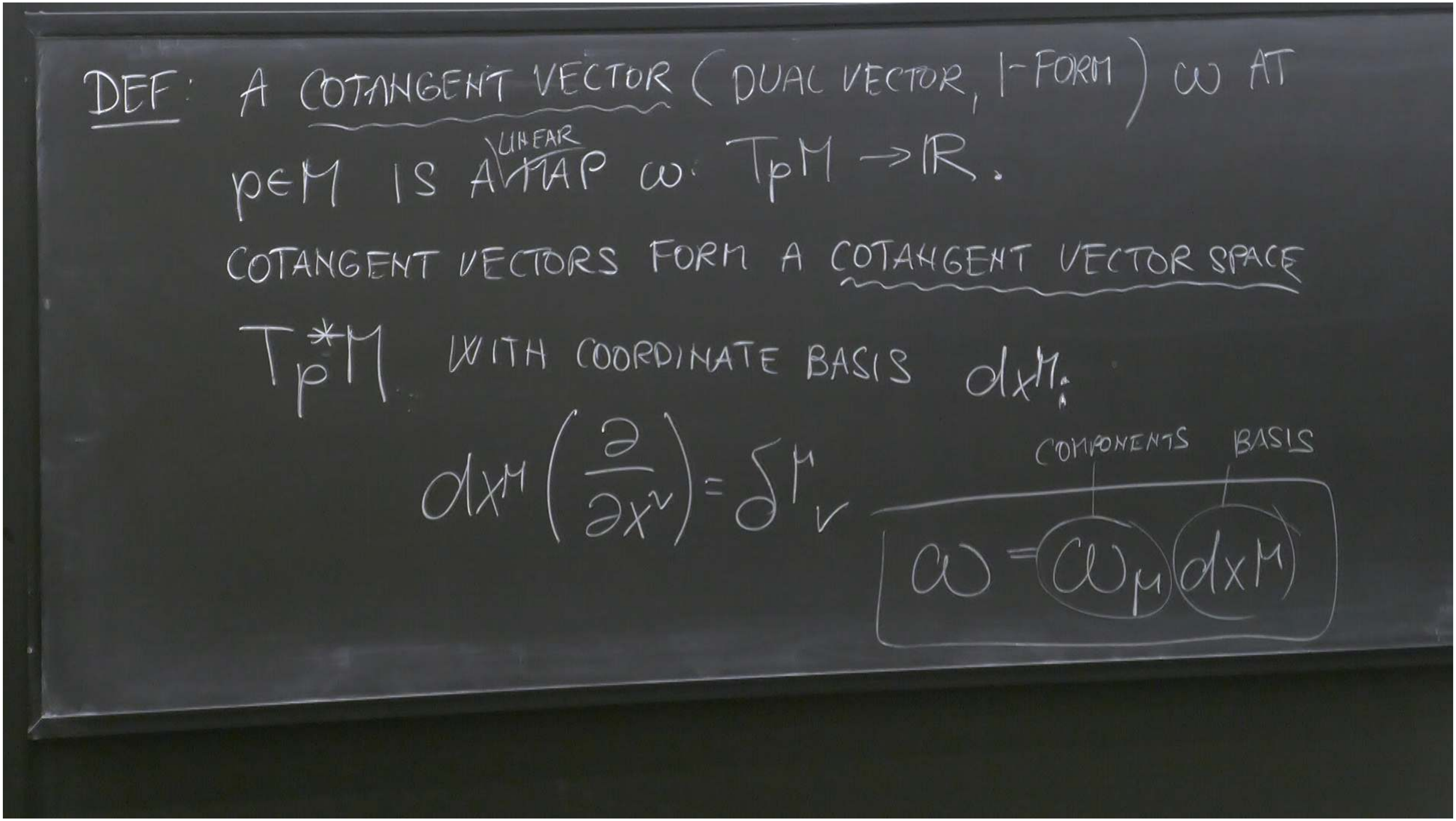

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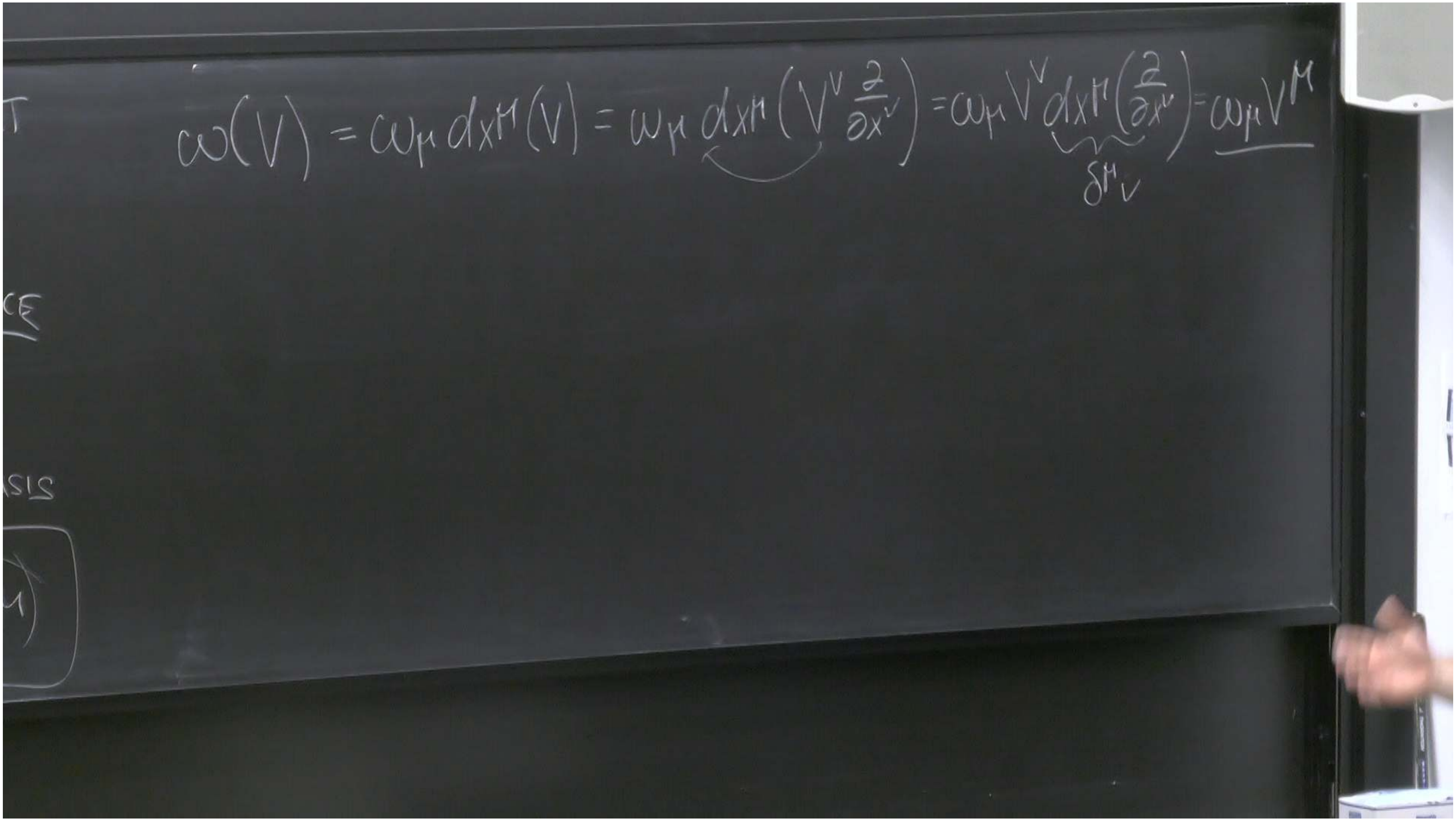
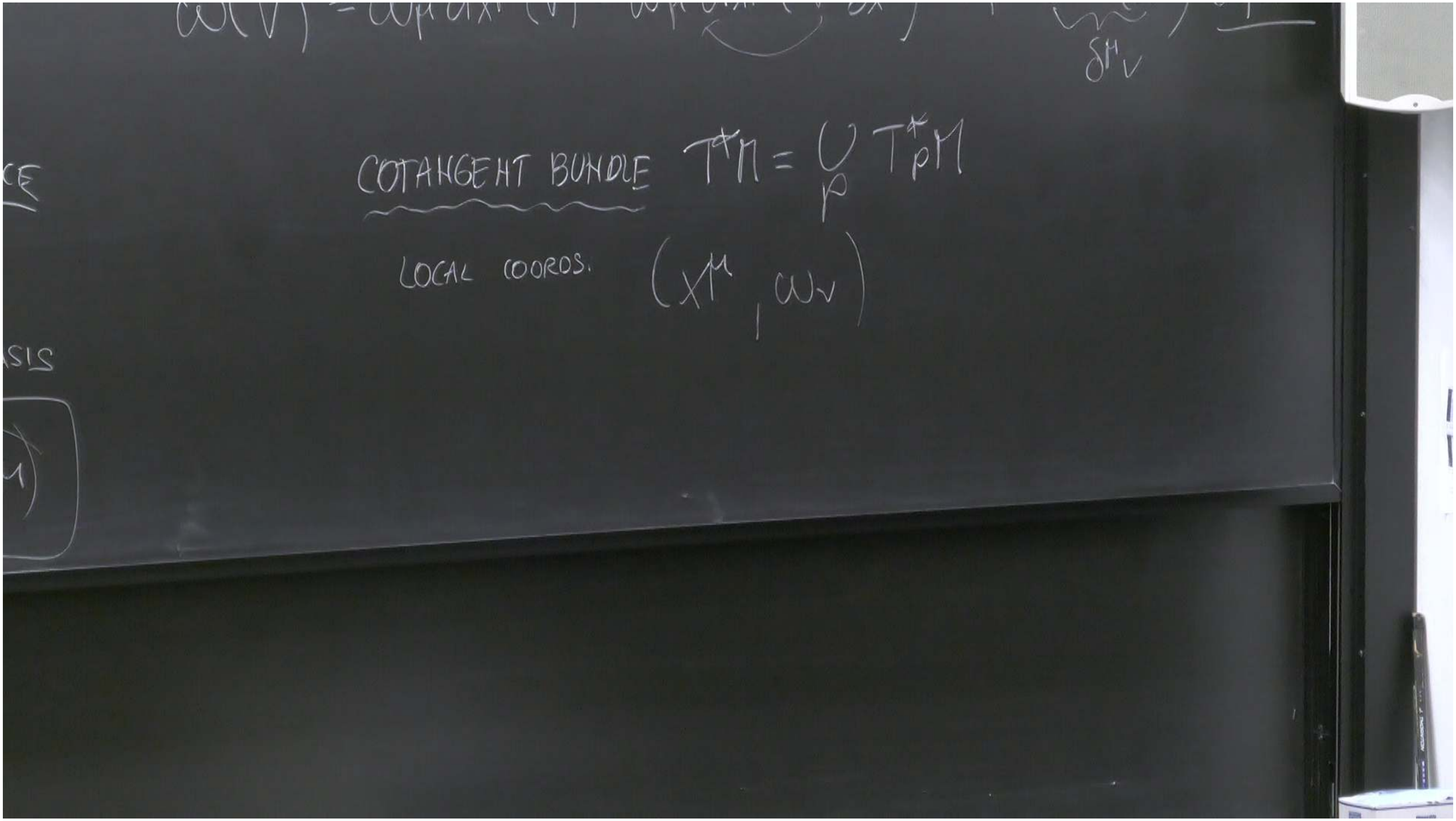
COTANGENT VECTORS FORM A COTANGENT VECTOR SPACE T_p^*M WITH COORDINATE BASIS dx^μ.

$$dx^\mu\left(\frac{\partial}{\partial x^\nu}\right) = \delta^\mu{}_\nu$$

$$\boxed{\omega = \overset{\text{COMPONENTS}}{\omega_\mu}\;\overset{\text{BASIS}}{dx^\mu}}$$



$$\omega(V) = \omega_\mu dx^\mu(V) = \omega_\mu dx^\mu\left(V^\nu \frac{\partial}{\partial x^\nu}\right) = \omega_\mu V^\nu \underbrace{dx^\mu\left(\frac{\partial}{\partial x^\nu}\right)}_{\delta^\mu{}_\nu} = \underline{\omega_\mu V^\mu}$$



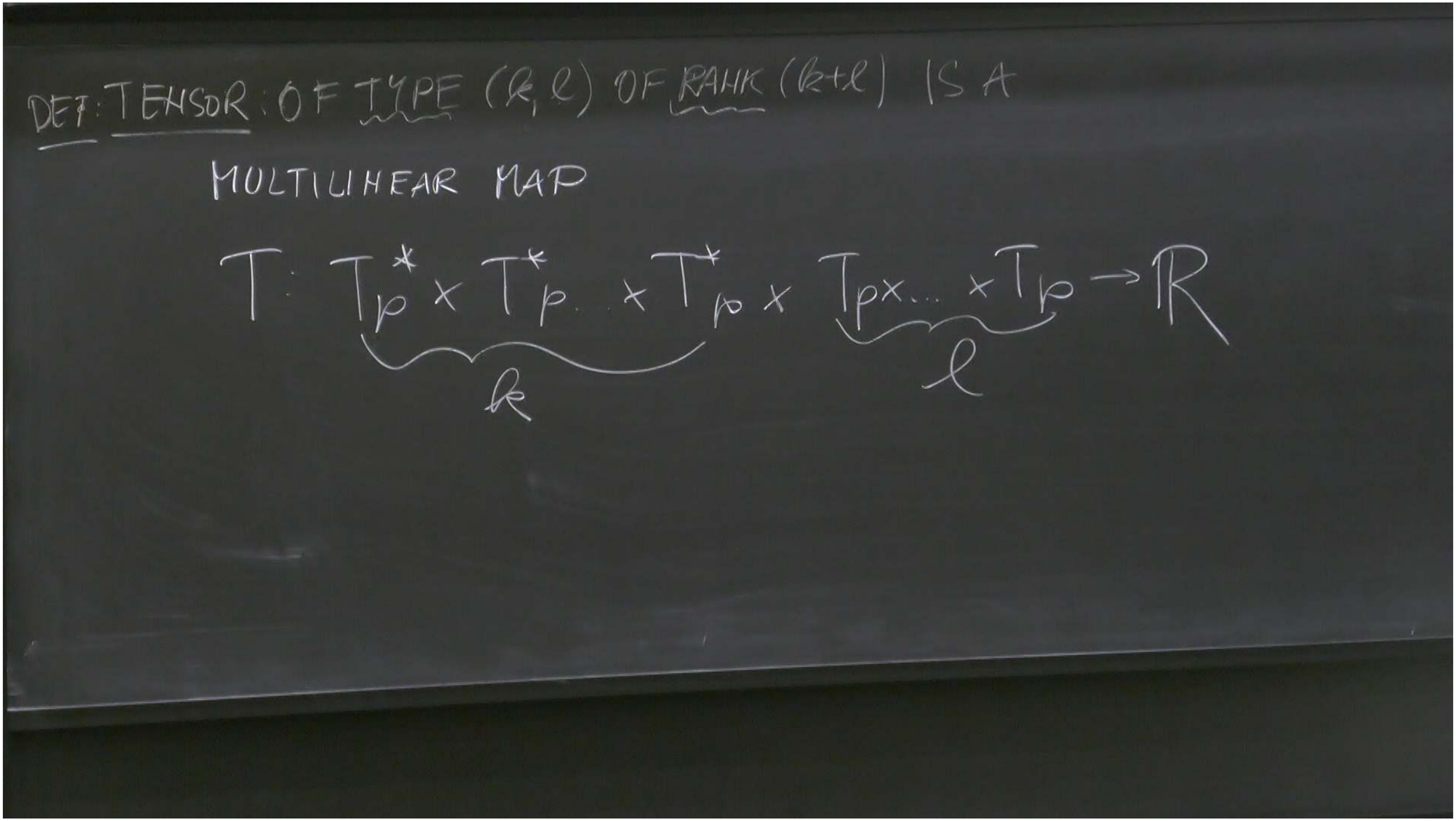
$\omega(V) = \omega_\mu dx^\mu(V) = \omega_\mu dx^\mu(\ldots)$ … $\delta^\mu{}_\nu$

CE

COTANGENT BUNDLE $T^*M = \bigcup_p T^*_p M$

LOCAL COORDS. (x^μ, ω_ν)

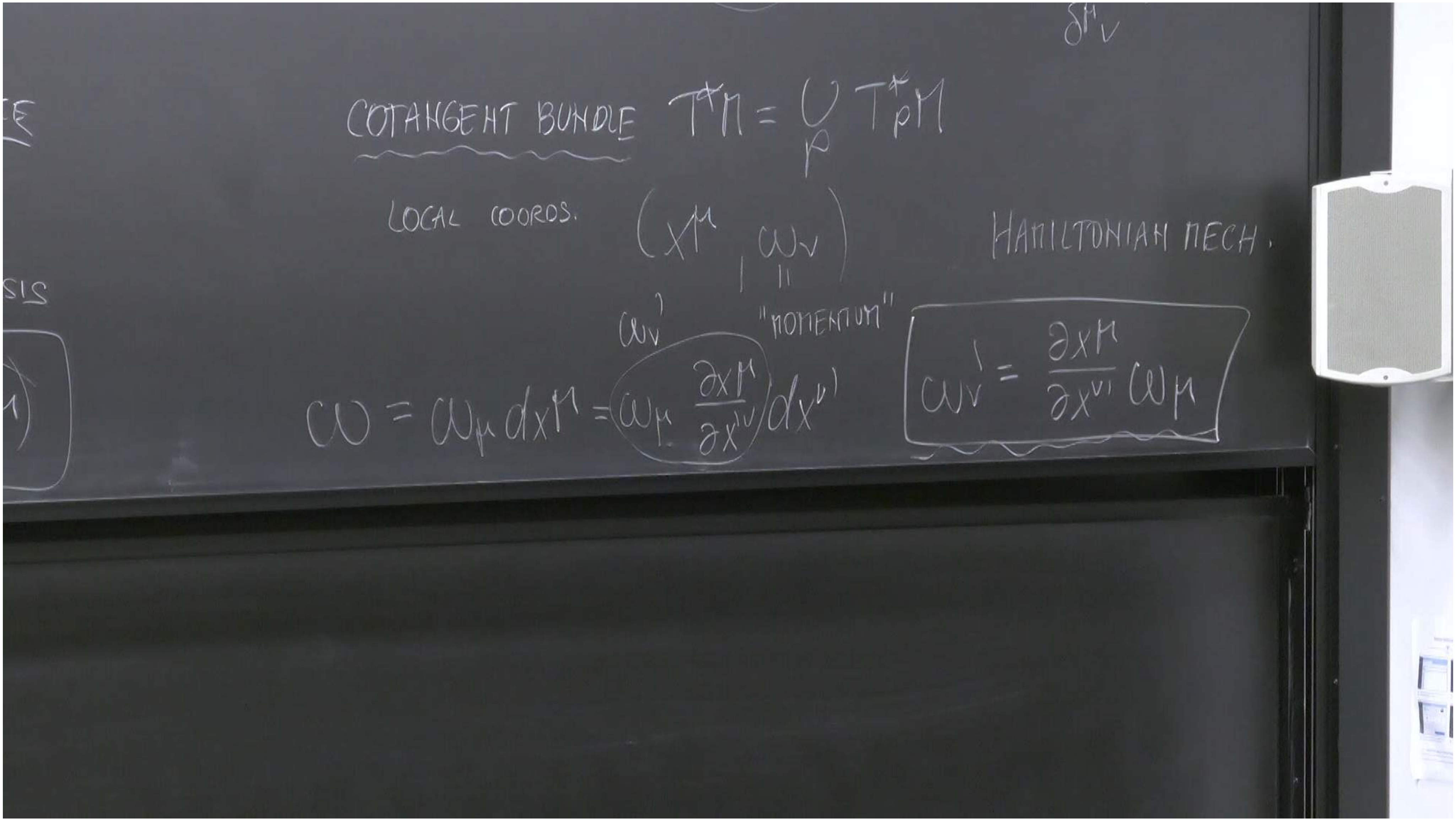
SIS



DEF: TENSOR: OF TYPE (k, l) OF RANK $(k+l)$ IS A

MULTILINEAR MAP

$$T: \underbrace{T_p^* \times T_p^* \ldots \times T_p^*}_{k} \times \underbrace{T_p \times \ldots \times T_p}_{l} \to \mathbb{R}$$



$\delta^\mu{}_\nu$

E

COTANGENT BUNDLE $T^*M = \bigcup_p T^*_p M$

LOCAL COORDS. (x^μ, ω_ν)

ω'_ν "MOMENTUM"

HAMILTONIAN MECH.

SIS

$$\omega = \omega_\mu dx^\mu = \omega_\mu \frac{\partial x^\mu}{\partial x'^\nu} dx'^\nu$$

$$\boxed{\omega'_\nu = \frac{\partial x^\mu}{\partial x'^\nu} \omega_\mu}$$

$\omega \quad \omega_\mu \, dx^\mu$

DEF: TENSOR OF TYPE (k, l) OF RANK $(k+l)$ IS A

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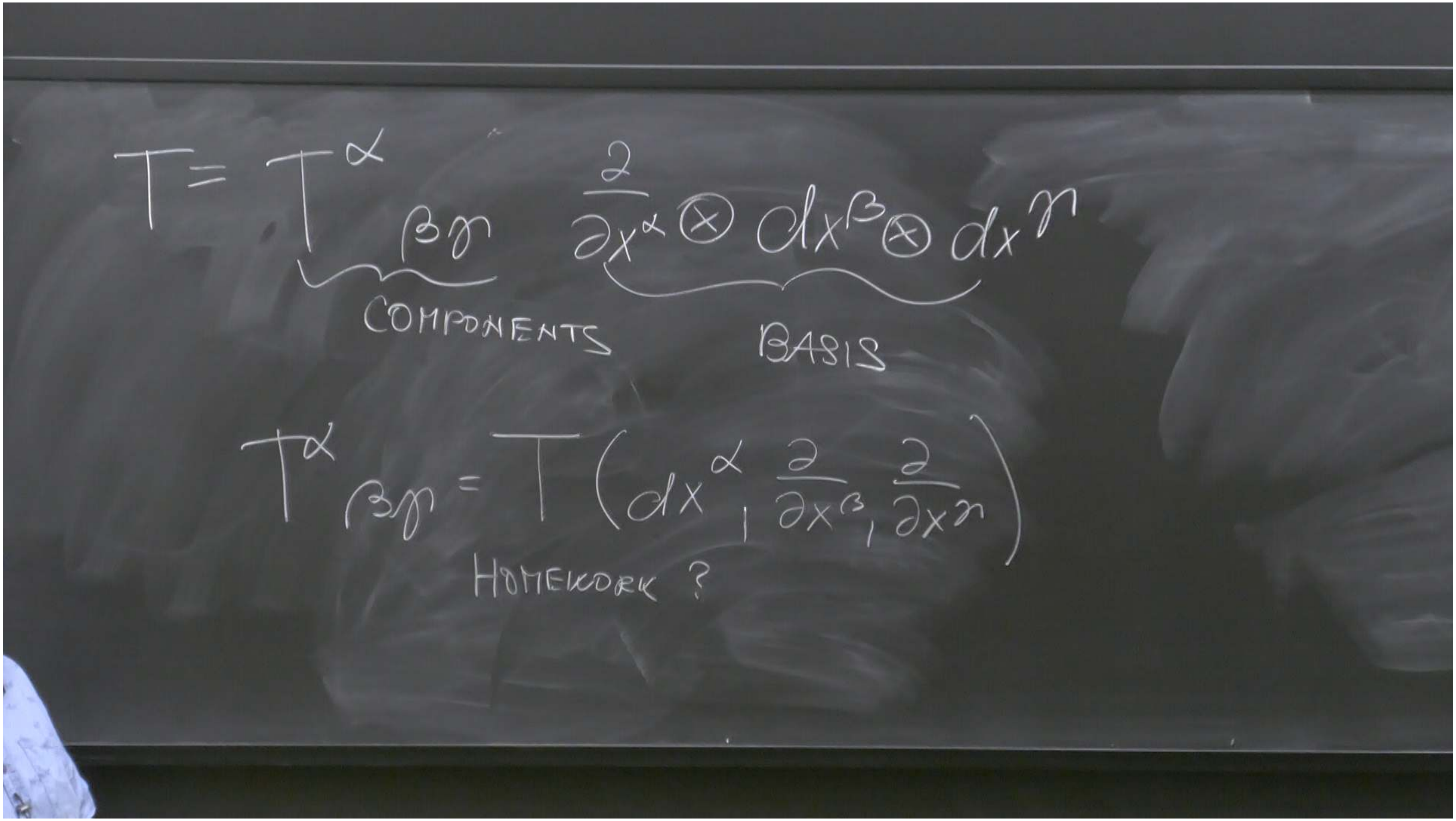
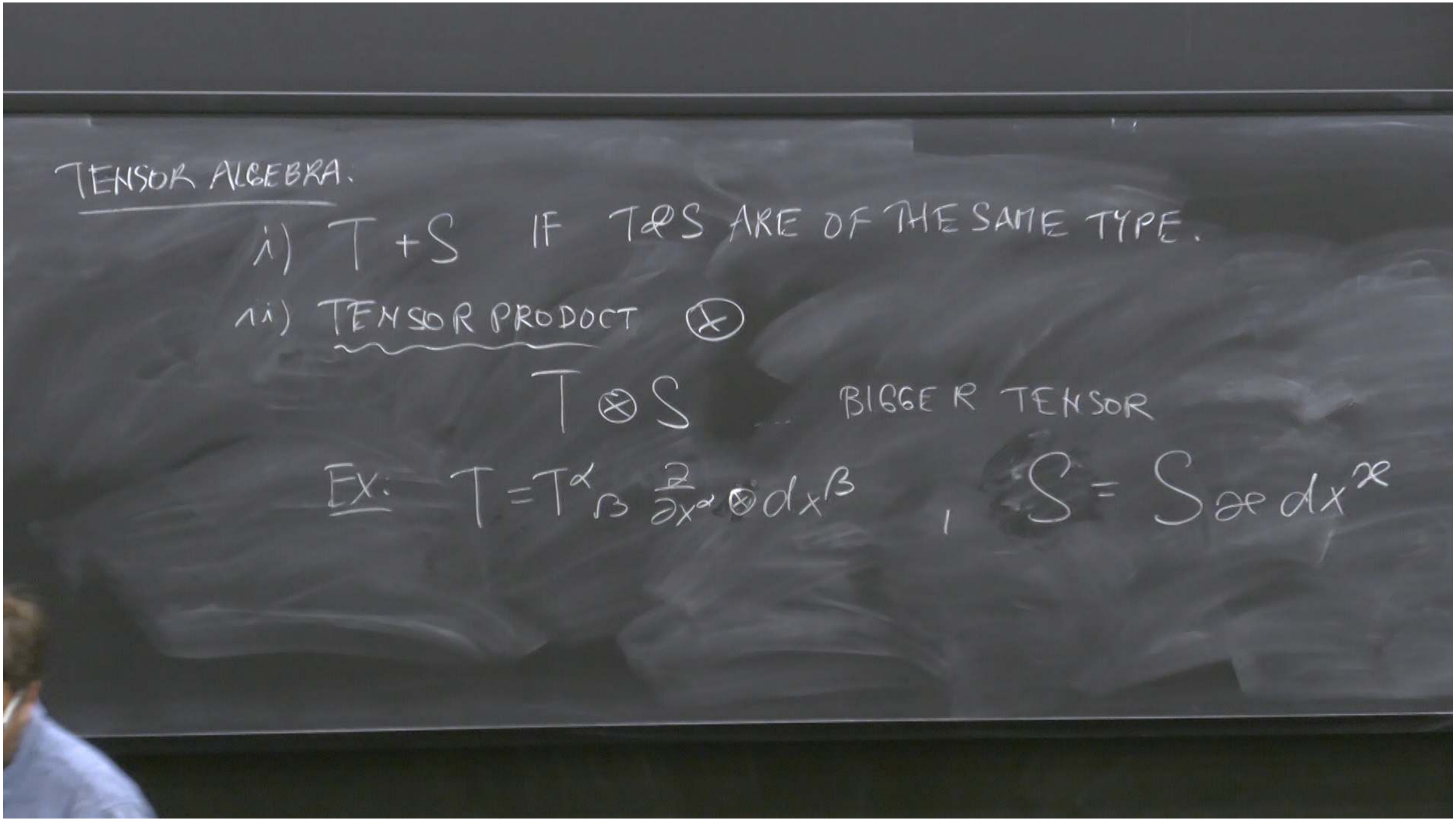
$$\boxed{T'^{\alpha \ldots}{}_{\mu \ldots} = \frac{\partial x'^{\alpha}}{\partial x^{\delta}} \ldots \frac{\partial x^{\varkappa}}{\partial x'^{\mu}} \ldots T^{\delta \ldots}{}_{\varkappa \ldots}}$$



$$T = \underbrace{T^{\alpha}{}_{\beta\gamma}}_{\text{COMPONENTS}} \underbrace{\frac{\partial}{\partial x^{\alpha}} \otimes dx^{\beta} \otimes dx^{\gamma}}_{\text{BASIS}}$$

$$T^{\alpha}{}_{\beta\gamma} = T\left(dx^{\alpha}, \frac{\partial}{\partial x^{\beta}}, \frac{\partial}{\partial x^{\gamma}}\right)$$

HOMEWORK ?



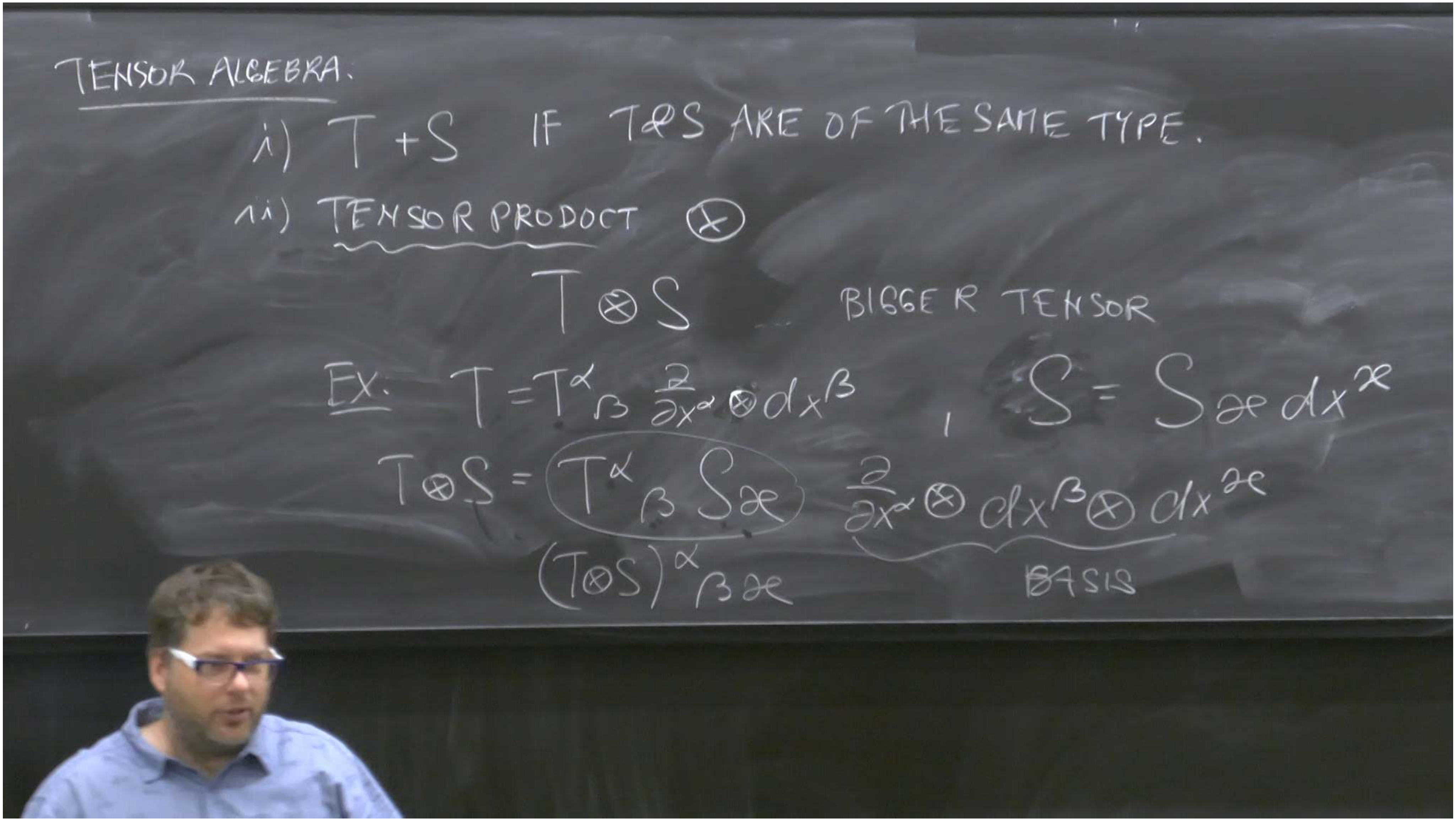
TENSOR ALGEBRA.

i) $T+S$ IF T & S ARE OF THE SAME TYPE.

ii) TENSOR PRODUCT $\otimes$

$$T \otimes S \quad \ldots \quad \text{BIGGER TENSOR}$$

Ex: $T = T^{\alpha}{}_{\beta} \frac{\partial}{\partial x^{\alpha}} \otimes dx^{\beta}$, $S = S_{\varkappa} dx^{\varkappa}$



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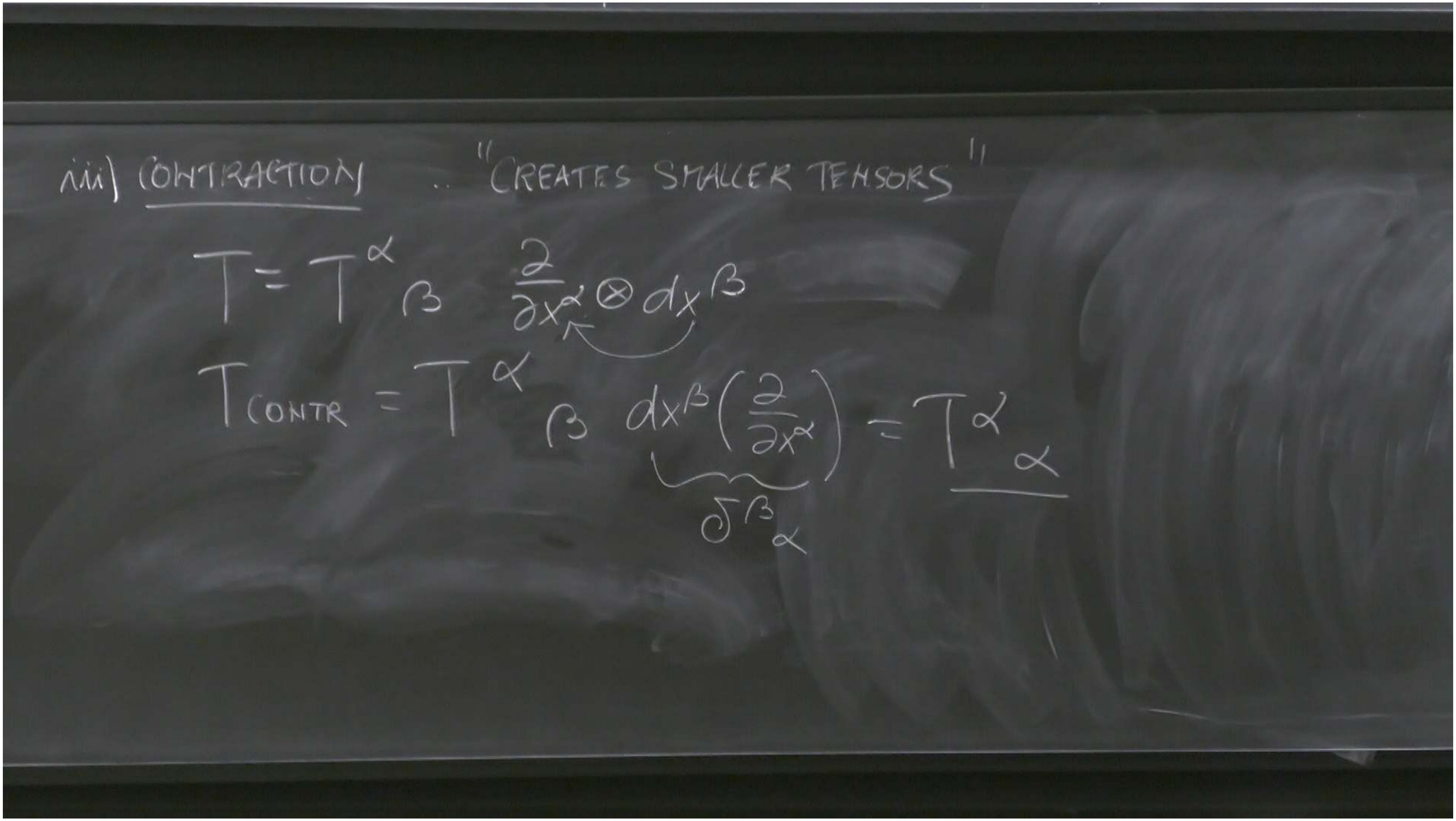
$T \otimes S$ BIGGER TENSOR

Ex. $T = T^{\alpha}{}_{\beta} \frac{\partial}{\partial x^{\alpha}} \otimes dx^{\beta}$, $S = S_{\varkappa} dx^{\varkappa}$

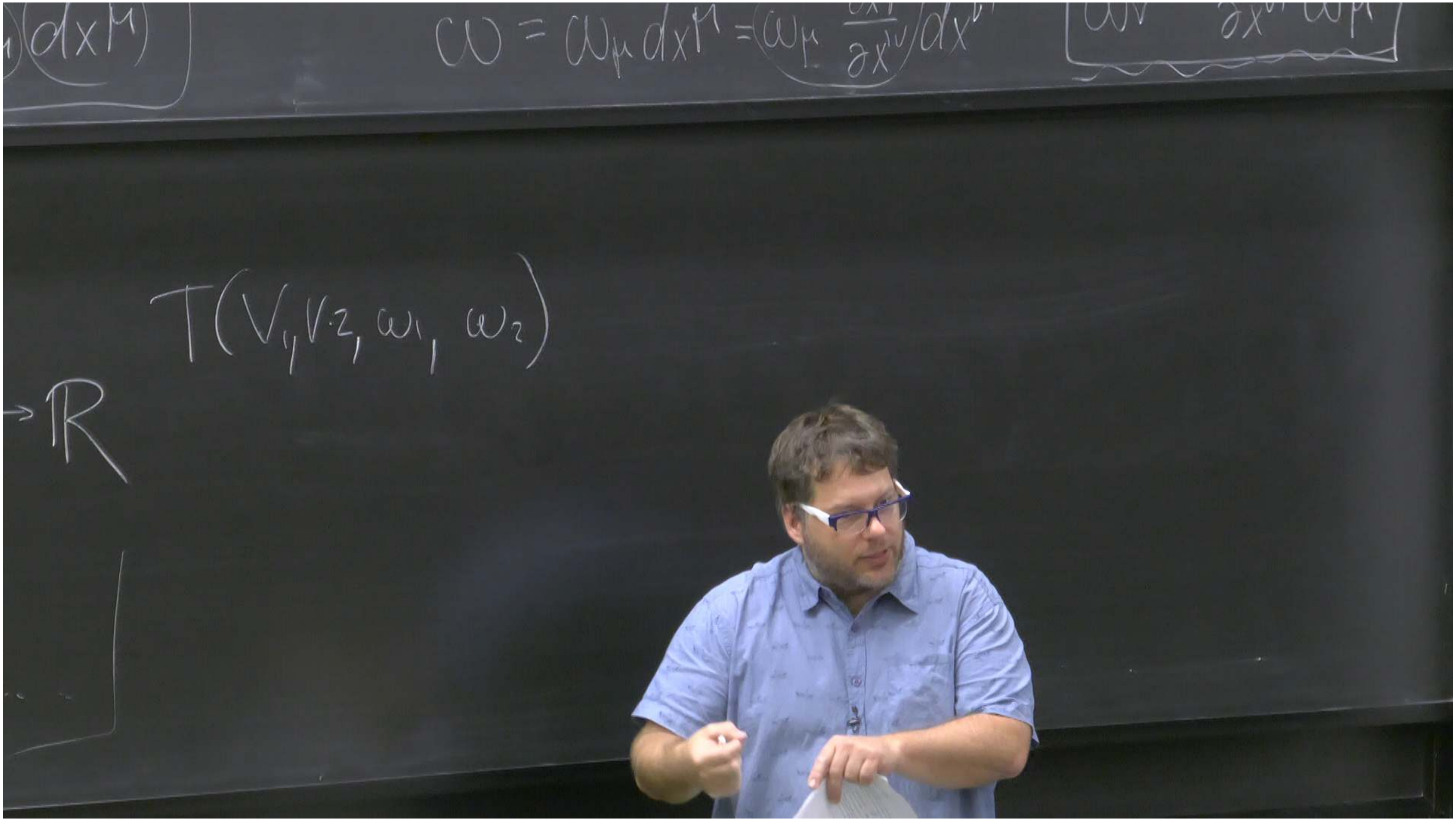
$$T \otimes S = \underbrace{T^{\alpha}{}_{\beta} S_{\varkappa}}_{(T \otimes S)^{\alpha}{}_{\beta\varkappa}} \underbrace{\frac{\partial}{\partial x^{\alpha}} \otimes dx^{\beta} \otimes dx^{\varkappa}}_{\text{BASIS}}$$

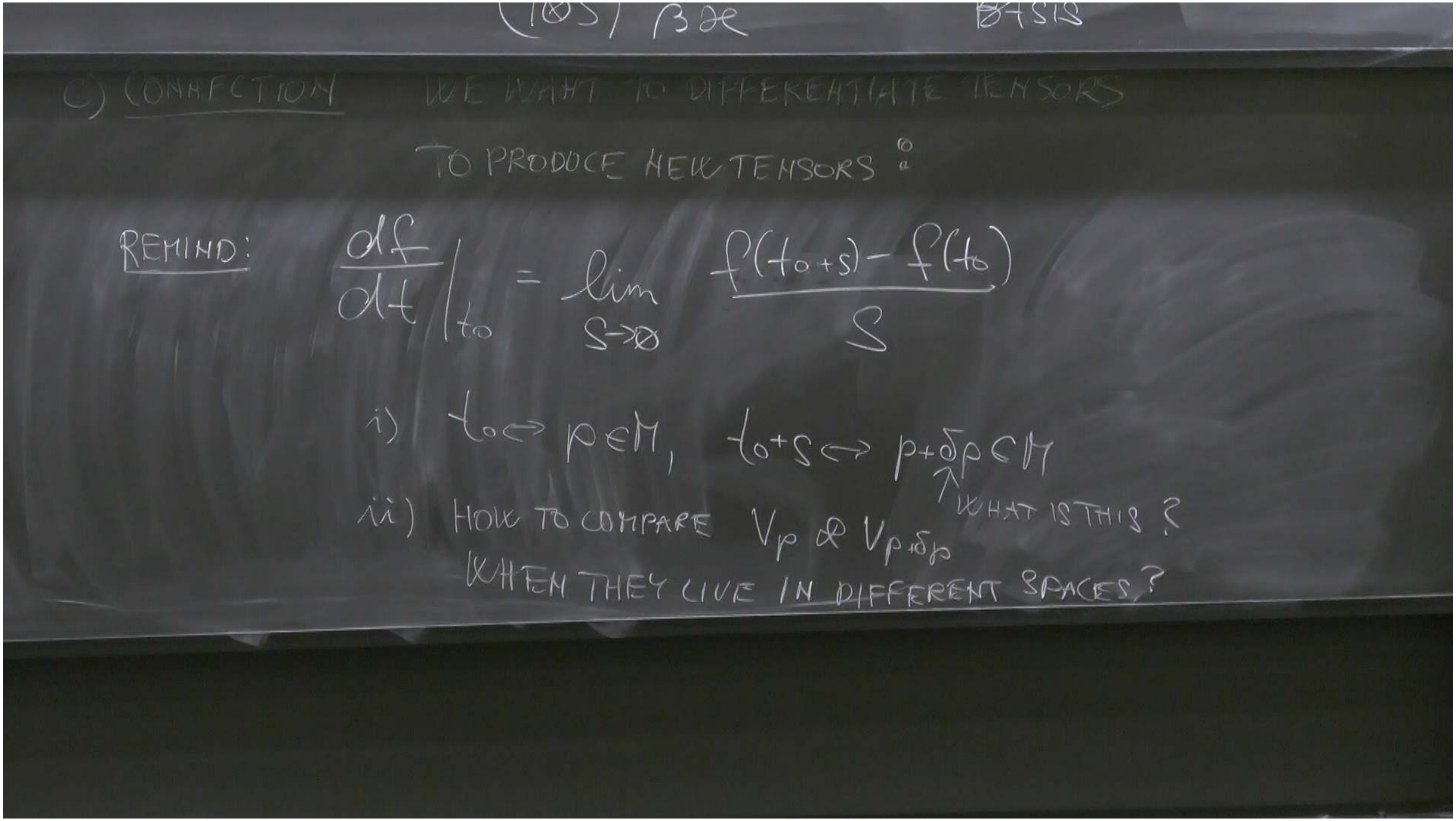


iii) CONTRACTION .. "CREATES SMALLER TENSORS"

$$T = T^{\alpha}{}_{\beta} \, \frac{\partial}{\partial x^{\alpha}} \otimes dx^{\beta}$$

$$T_{\mathrm{CONTR}} = T^{\alpha}{}_{\beta} \, \underbrace{dx^{\beta}\left(\frac{\partial}{\partial x^{\alpha}}\right)}_{\delta^{\beta}{}_{\alpha}} = \underline{T^{\alpha}{}_{\alpha}}$$


(dx^μ)
$\omega = \omega_\mu dx^\mu = \omega_\mu \frac{\partial x^\mu}{\partial x'^\nu} dx'^\nu$
$T(V_1, V_2, \omega_1, \omega_2)$
$\to \mathbb{R}$



$(1\otimes S)$ $\beta\,\mathfrak{X}$ BASIS

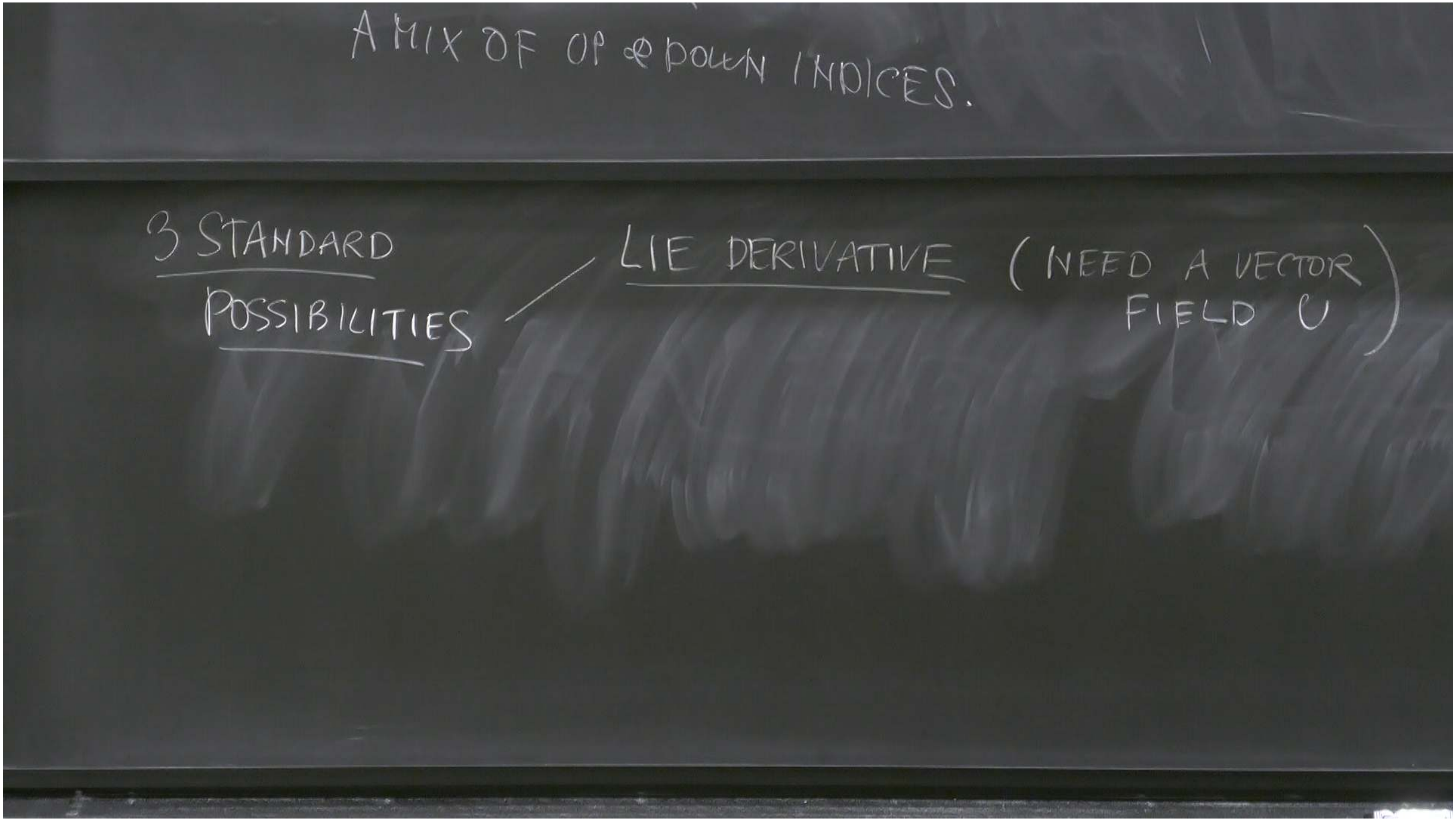
C) CONNECTION   WE WANT TO DIFFERENTIATE TENSORS

TO PRODUCE NEW TENSORS:

REMIND: $\frac{df}{dt}\Big|_{t_0} = \lim_{s\to 0} \frac{f(t_0+s)-f(t_0)}{s}$

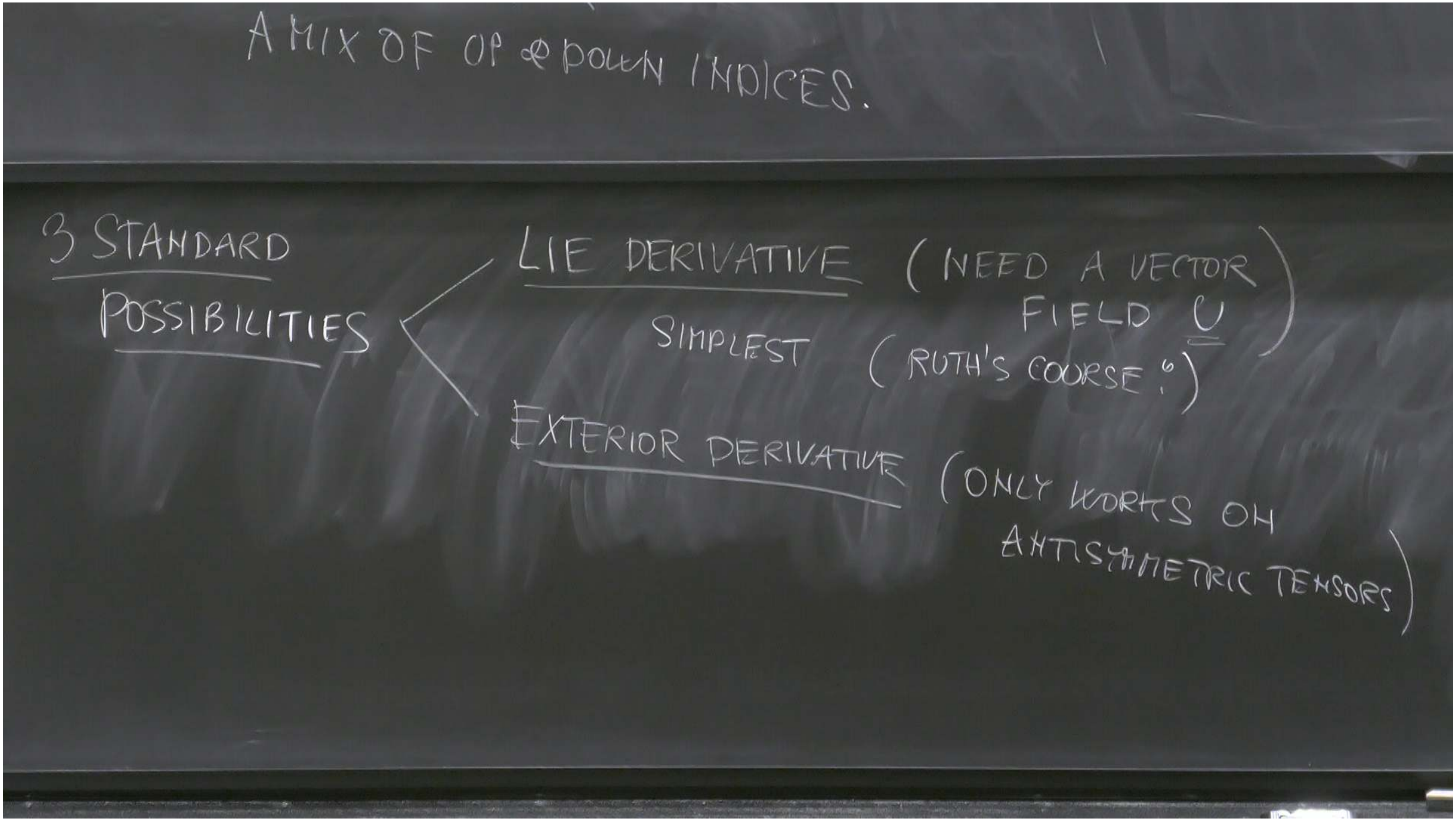
i) $t_0 \Leftrightarrow p\in M$, $t_0+s \Leftrightarrow p+\underset{\uparrow}{\delta p} \in M$ WHAT IS THIS?

ii) HOW TO COMPARE V_p & $V_{p+\delta p}$ WHEN THEY LIVE IN DIFFERENT SPACES?



A MIX OF UP & DOWN INDICES.

3 STANDARD POSSIBILITIES — LIE DERIVATIVE (NEED A VECTOR FIELD v)



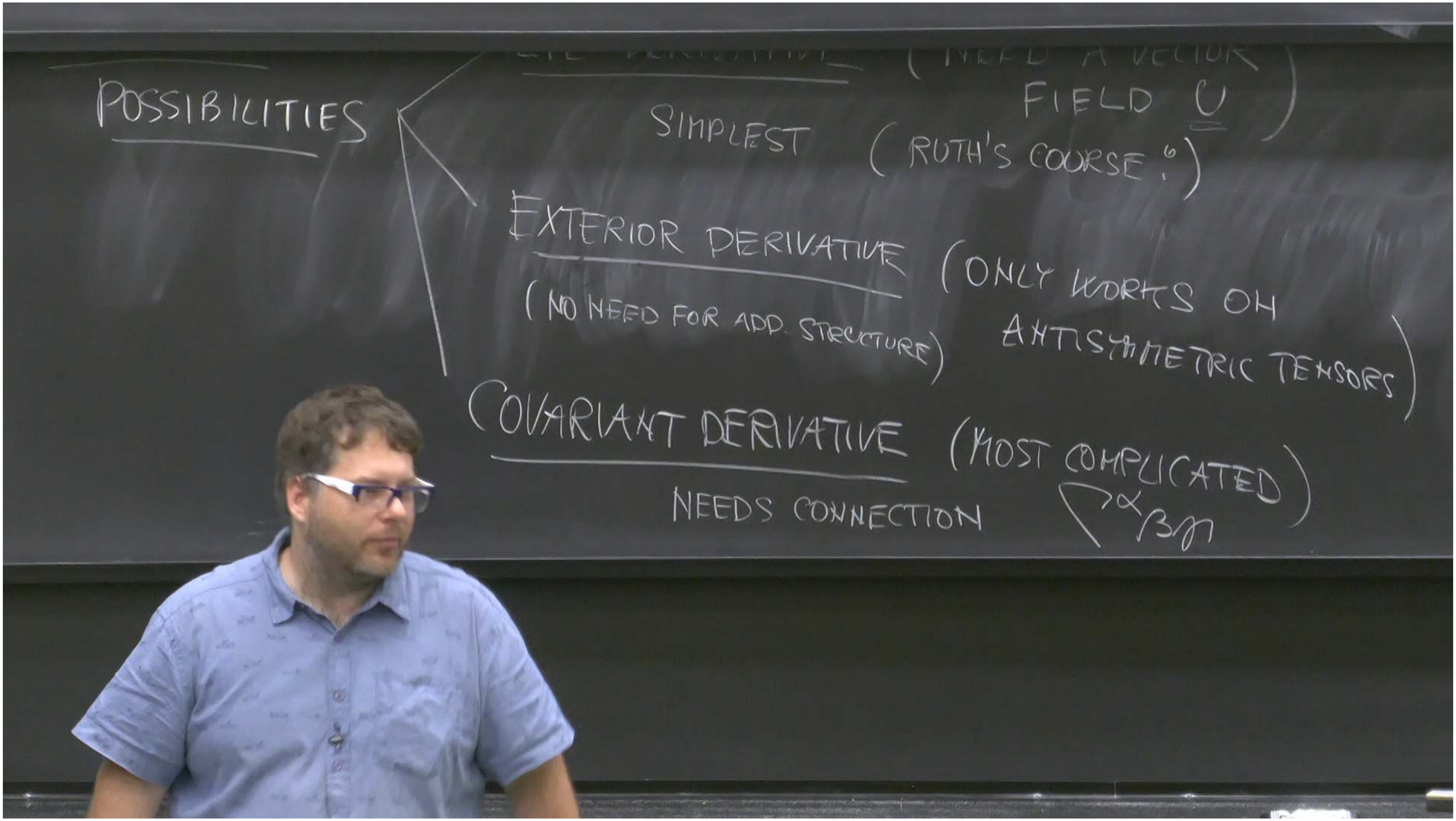
A MIX OF UP & DOWN INDICES.

3 STANDARD POSSIBILITIES

LIE DERIVATIVE (NEED A VECTOR FIELD $\underline{U}$)

SIMPLEST (RUTH'S COURSE :)

EXTERIOR DERIVATIVE (ONLY WORKS ON ANTISYMMETRIC TENSORS)



POSSIBILITIES

LIE DERIVATIVE (NEED A VECTOR FIELD U)

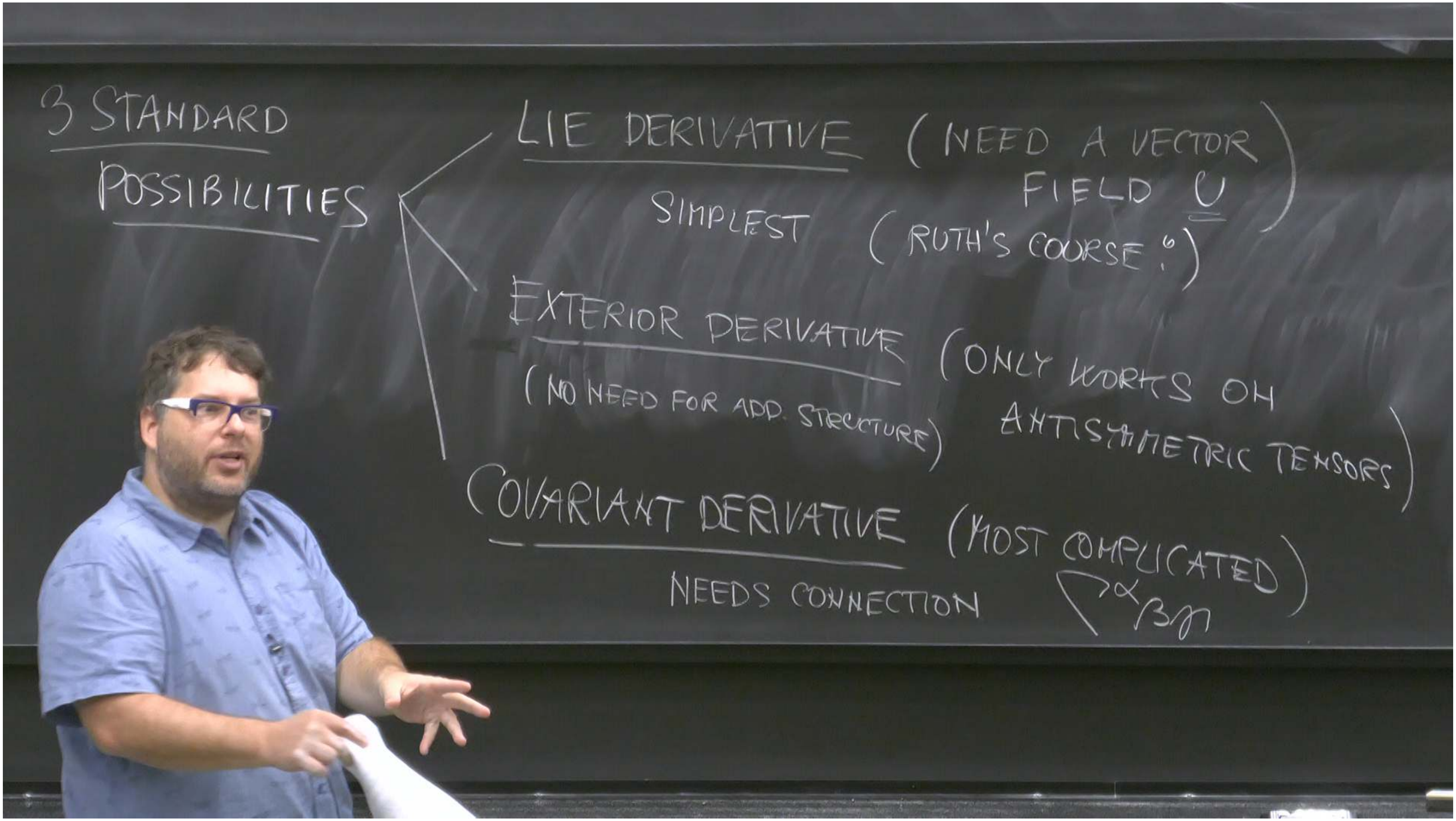
SIMPLEST (RUTH'S COURSE ☺)

EXTERIOR DERIVATIVE (ONLY WORKS ON ANTISYMMETRIC TENSORS)

(NO NEED FOR ADD. STRUCTURE)

COVARIANT DERIVATIVE (MOST COMPLICATED)

NEEDS CONNECTION $\Gamma^{\alpha}{}_{\beta\gamma}$



3 STANDARD POSSIBILITIES

LIE DERIVATIVE (NEED A VECTOR FIELD $\underline{v}$)

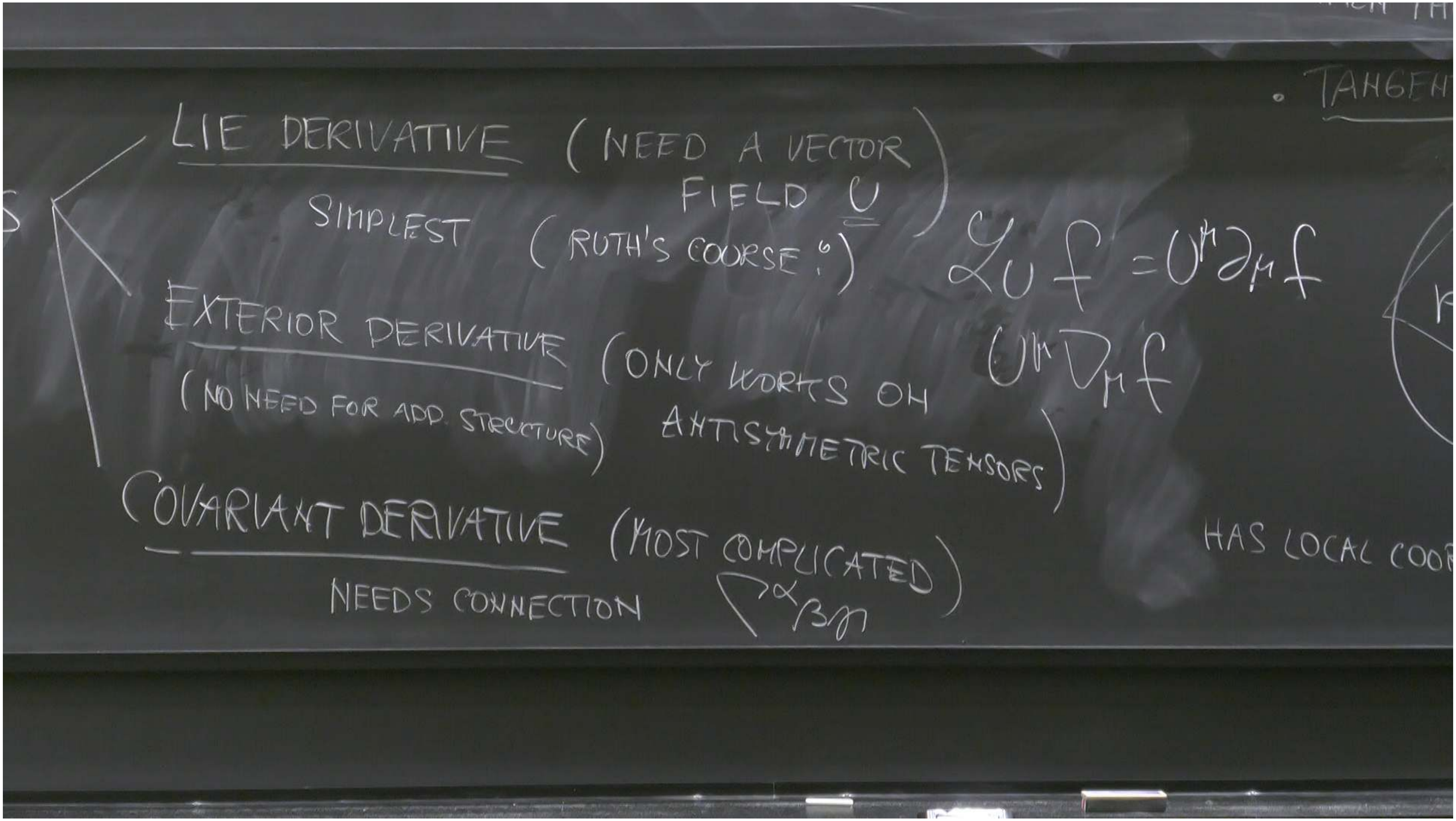
SIMPLEST (RUTH'S COURSE ☺)

EXTERIOR DERIVATIVE (ONLY WORKS ON ANTISYMMETRIC TENSORS)

(NO NEED FOR ADD. STRUCTURE)

COVARIANT DERIVATIVE (MOST COMPLICATED)

NEEDS CONNECTION $\Gamma^{\alpha}{}_{\beta\gamma}$



• TANGEN

LIE DERIVATIVE (NEED A VECTOR FIELD $\underline{U}$)

SIMPLEST (RUTH'S COURSE :))

$\mathcal{L}_U f = U^\mu \partial_\mu f$

$U^\mu \nabla_\mu f$

EXTERIOR DERIVATIVE (ONLY WORKS ON ANTISYMMETRIC TENSORS)

(NO NEED FOR ADD. STRUCTURE)

COVARIANT DERIVATIVE (MOST COMPLICATED)

NEEDS CONNECTION $\Gamma^{\alpha}{}_{\beta\gamma}$

HAS LOCAL COOR