Title: Relativity Lecture - 101623

Speakers: David Kubiznak

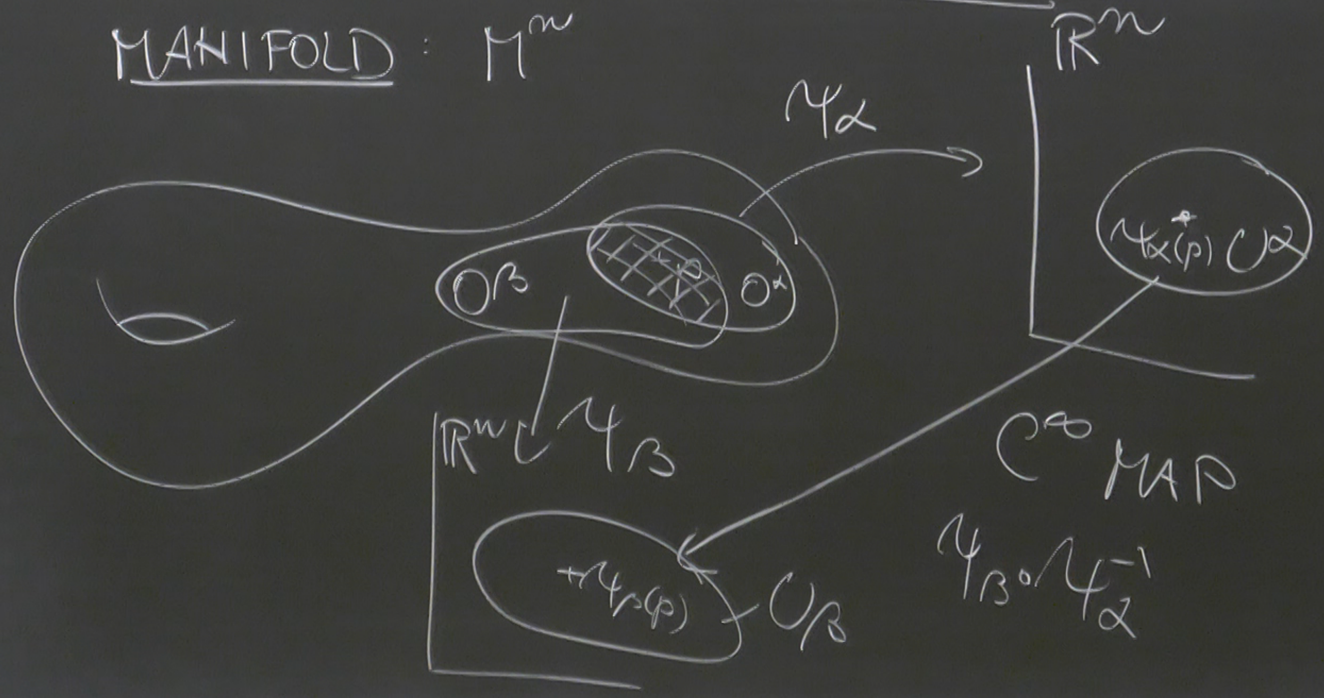
Collection: Relativity 2023/24

Date: October 16, 2023 - 9:00 AM

URL: https://pirsa.org/23100038

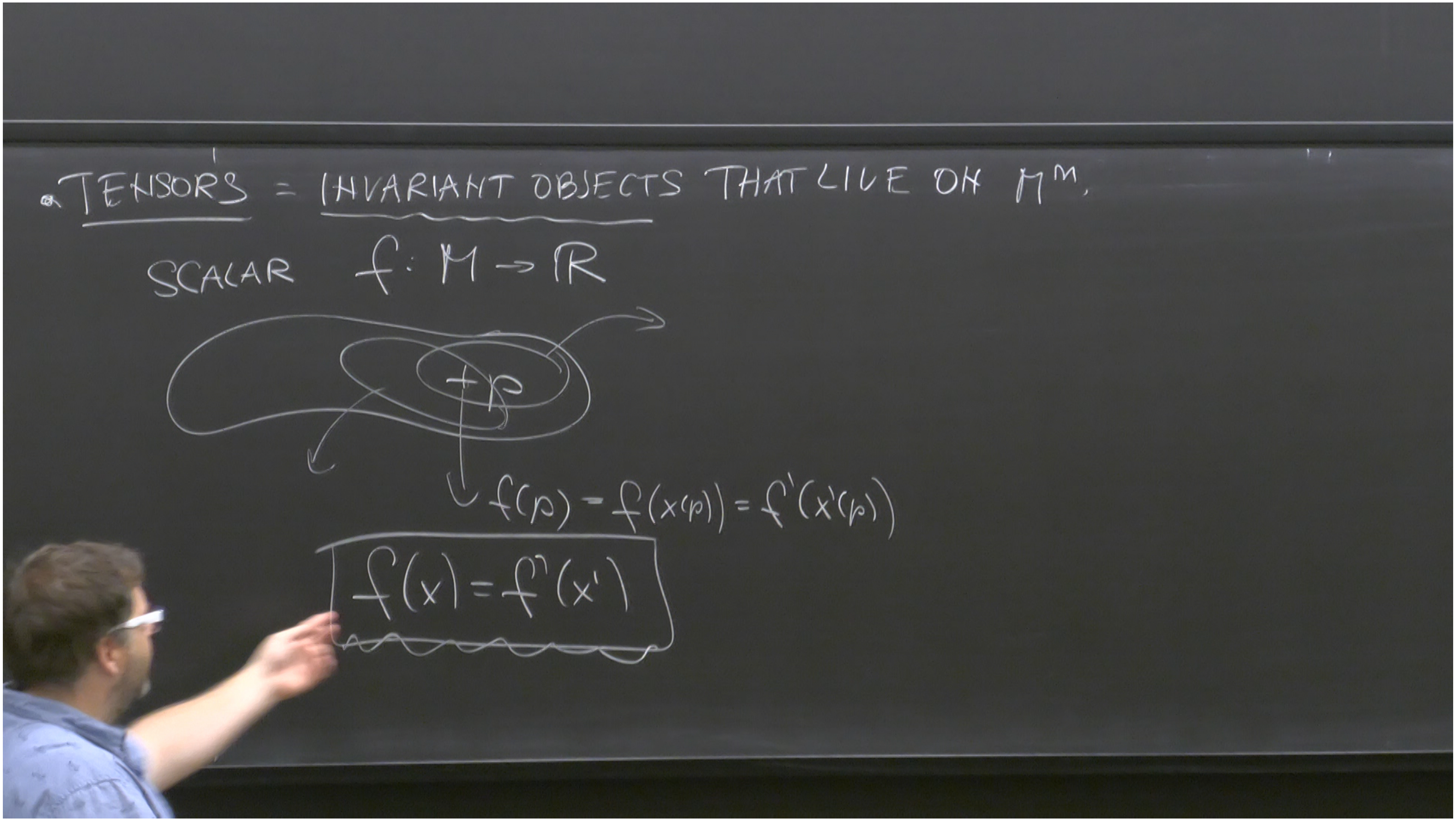
2) DIFFERENTIAL GEOMETRY

MANIFOLD: M^n



C^∞ MAP

$\psi_\beta \circ \psi_\alpha^{-1}$



- TENSORS = INVARIANT OBJECTS THAT LIVE ON M^m.

SCALAR $f: M \to \mathbb{R}$

$$f(p) = f(x(p)) = f'(x'(p))$$

$$\boxed{f(x) = f'(x')}$$

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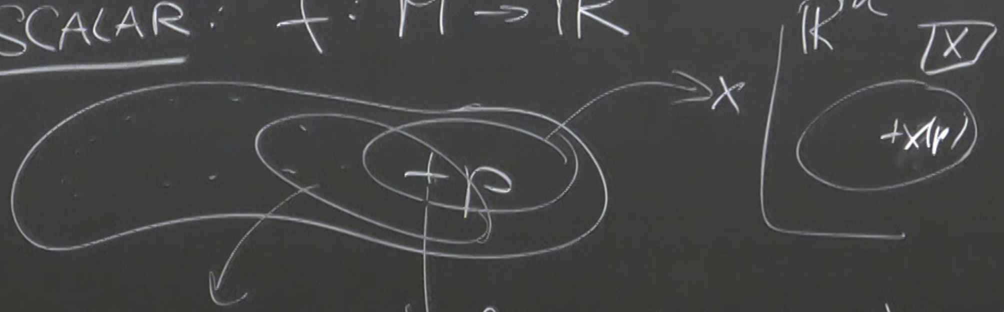

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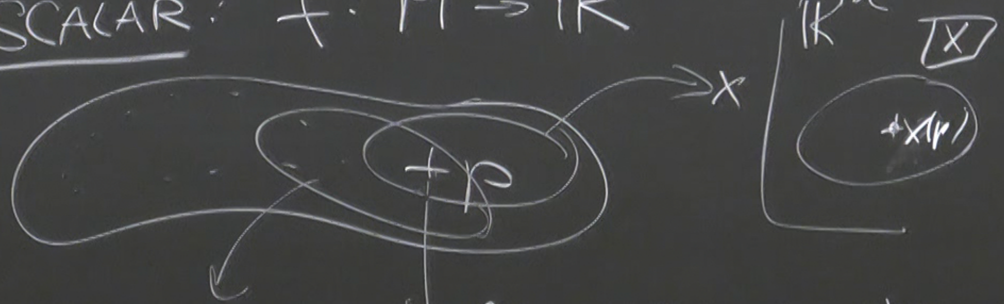


$$f(p) = f(x(p)) = f'(x'(p))$$

$$\boxed{f(x) = f'(x')}$$

AT LIVE ON M^m.

TANGENT VECTOR

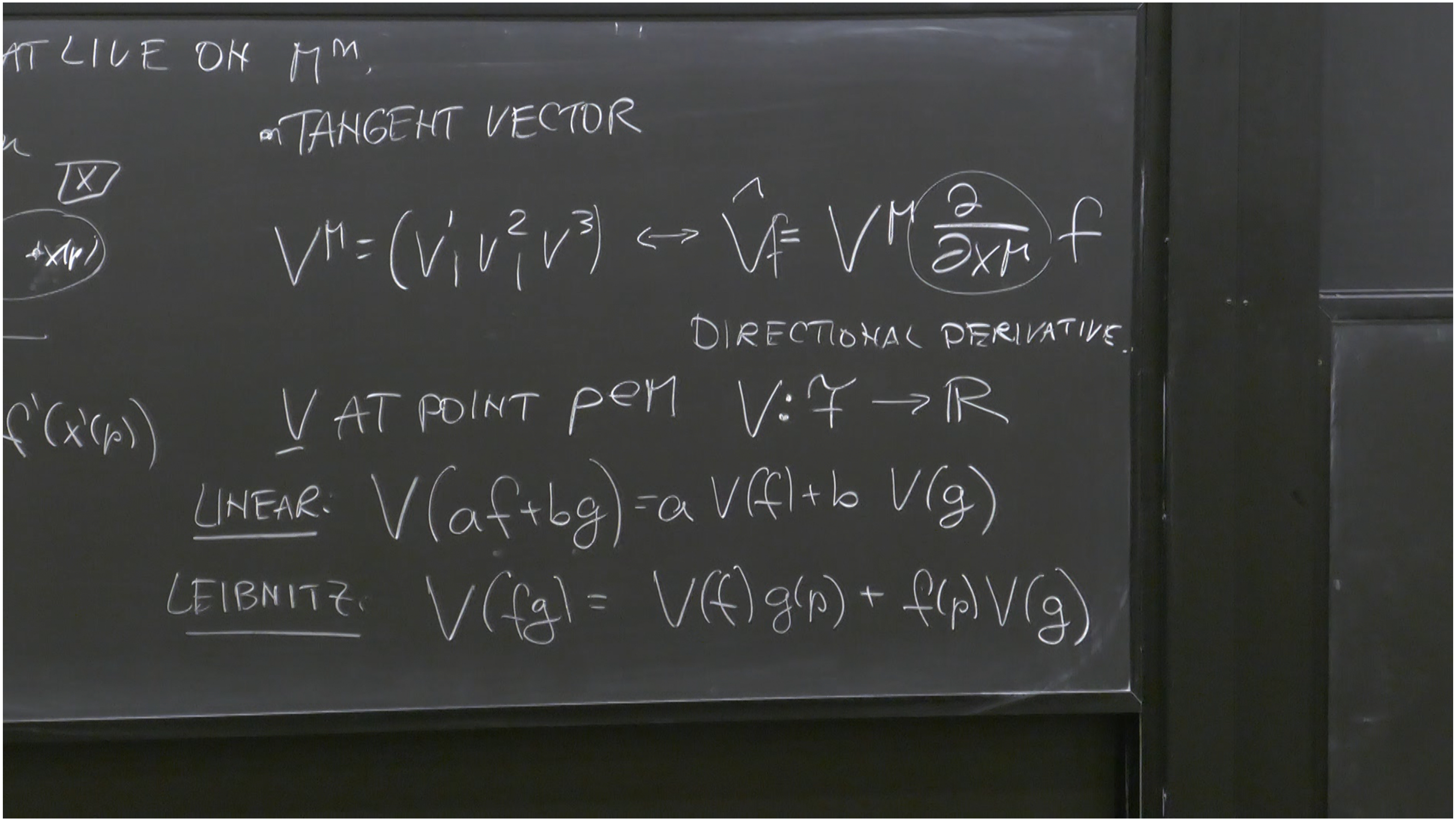
n

$\boxed{X}$

$+x(p)$

$$V^\mu = (V^1, V^2, V^3) \leftrightarrow \hat{V} = V^\mu \frac{\partial}{\partial x^\mu}$$

$f'(x'(p))$



AT LIVE ON M^m.

n $\boxed{X}$

$+x(p)$

- TANGENT VECTOR

$$V^\mu = (V^1, V^2, V^3) \leftrightarrow \hat{V}f = V^\mu \boxed{\frac{\partial}{\partial x^\mu}} f$$

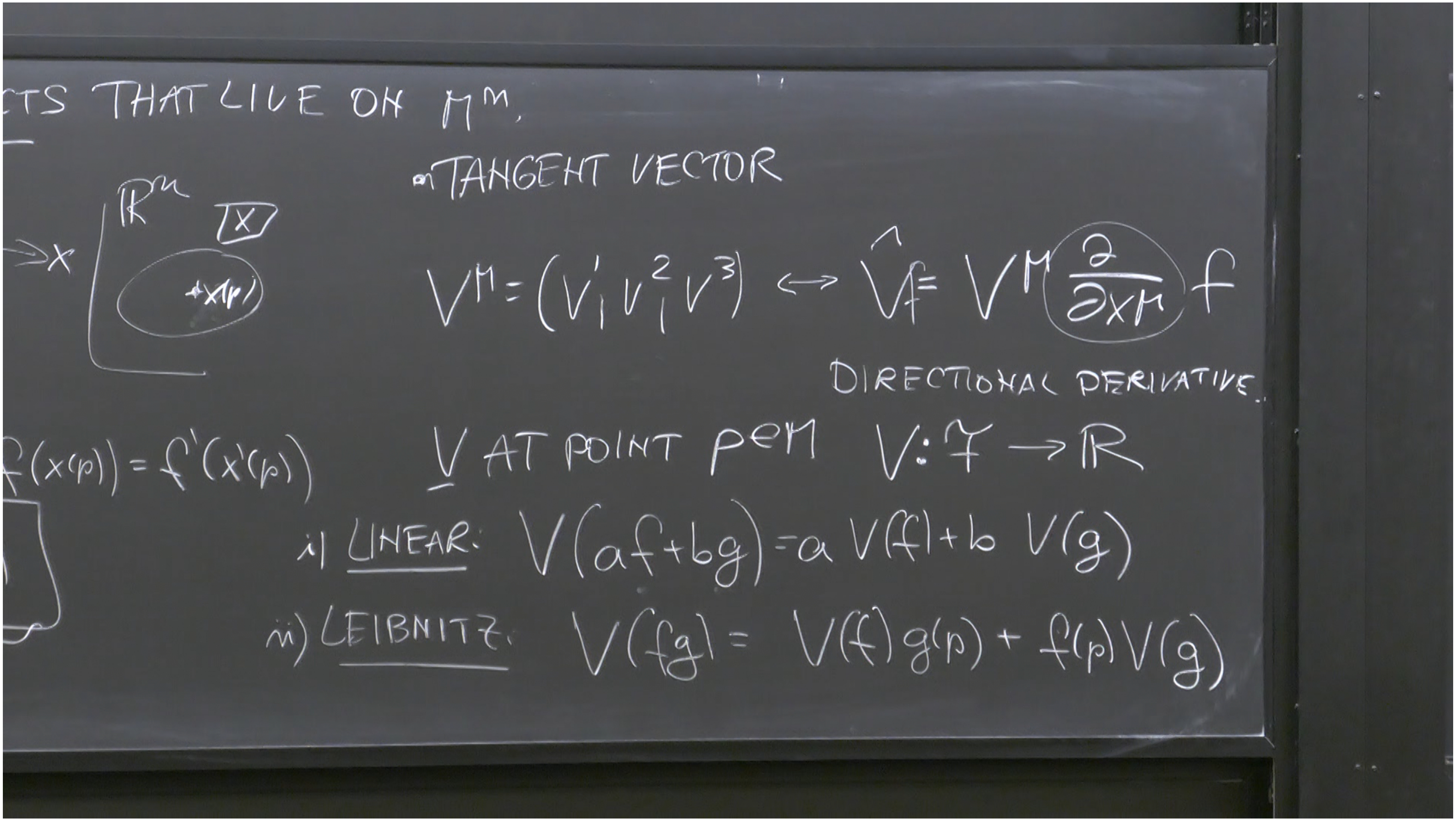
DIRECTIONAL DERIVATIVE.

$f'(x'(p))$

$\underline{V}$ AT POINT $p \in M$ $\quad V: \mathcal{F} \to \mathbb{R}$

LINEAR: $V(af+bg) = aV(f) + bV(g)$

LEIBNITZ: $V(fg) = V(f)g(p) + f(p)V(g)$



CTS THAT LIVE ON M^m.

• TANGENT VECTOR

$$V^\mu = (V^1, V^2, V^3) \leftrightarrow \hat{V}f = V^\mu \frac{\partial}{\partial x^\mu} f$$

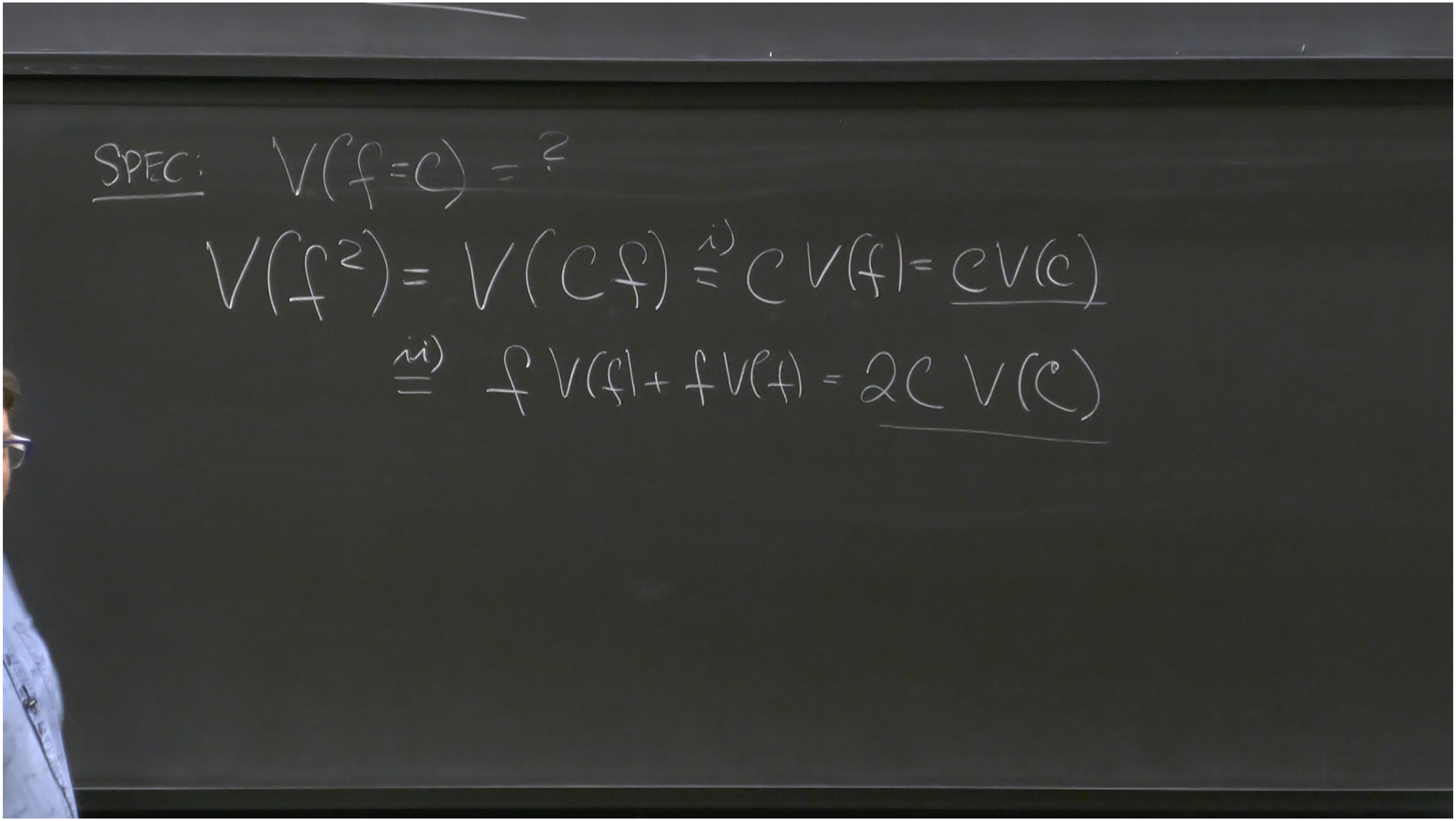
DIRECTIONAL DERIVATIVE.

$$f(x(p)) = f'(x'(p))$$

$\underline{V}$ AT POINT $p \in M$ $\quad V : \mathcal{F} \to \mathbb{R}$

i) LINEAR: $V(af+bg) = a\,V(f) + b\,V(g)$

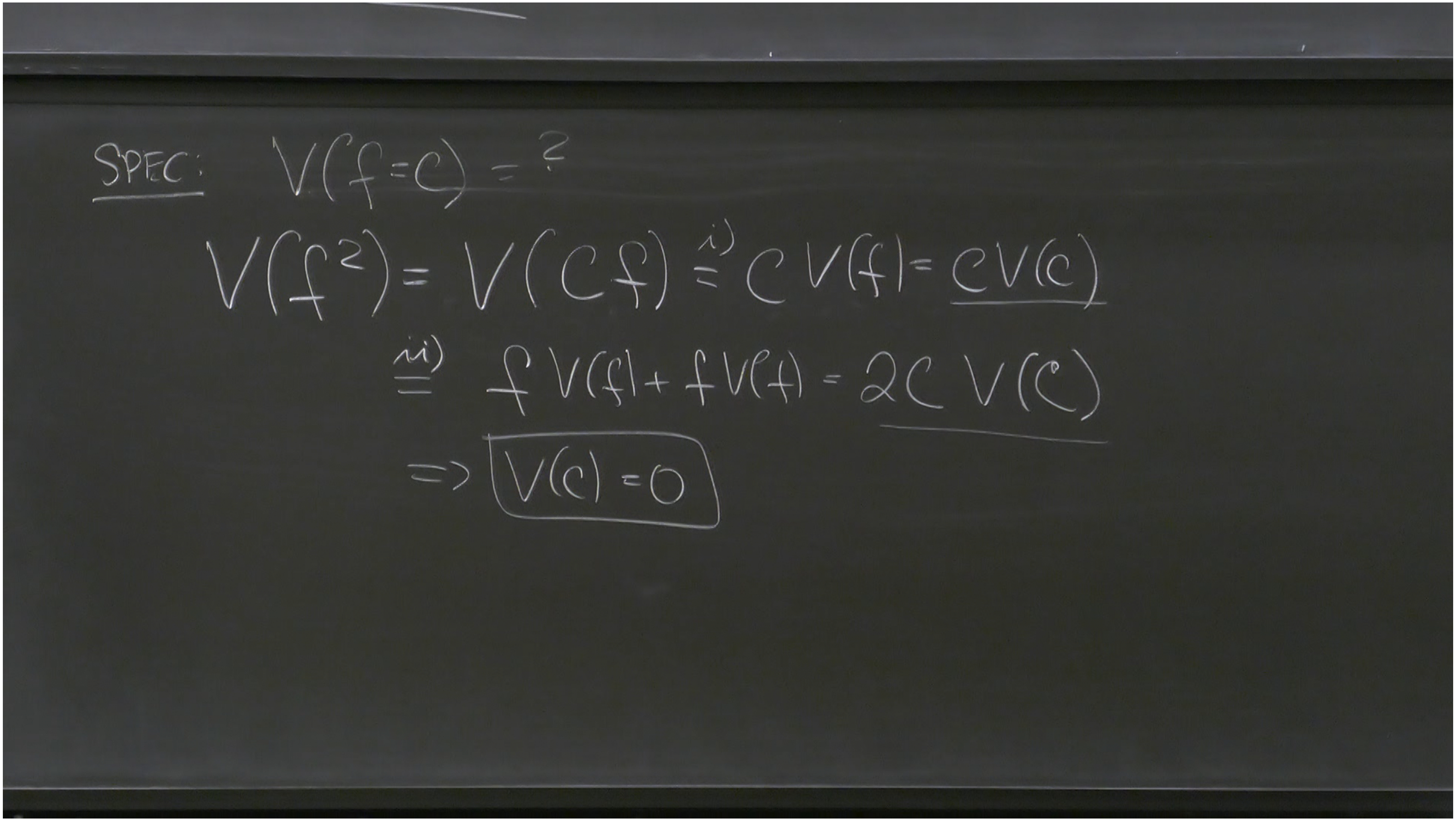
ii) LEIBNITZ: $V(fg) = V(f)\,g(p) + f(p)\,V(g)$



SPEC: $V(f=c) = ?$

$$V(f^2) = V(cf) \overset{i)}{=} cV(f) = \underline{cV(c)}$$

$$\overset{ii)}{=} fV(f) + fV(f) = \underline{2c\,V(c)}$$



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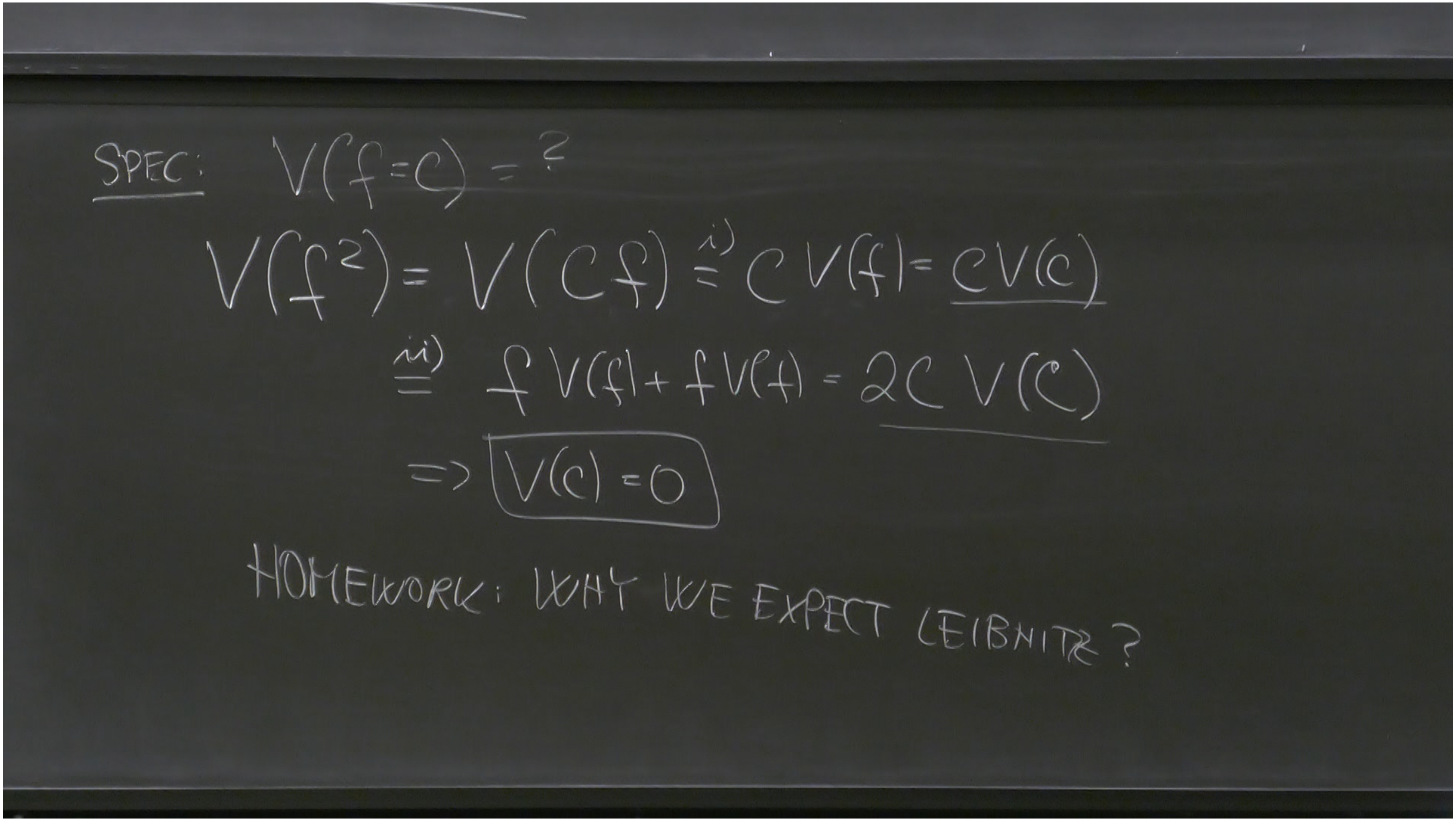
$$\Rightarrow \boxed{V(c) = 0}$$



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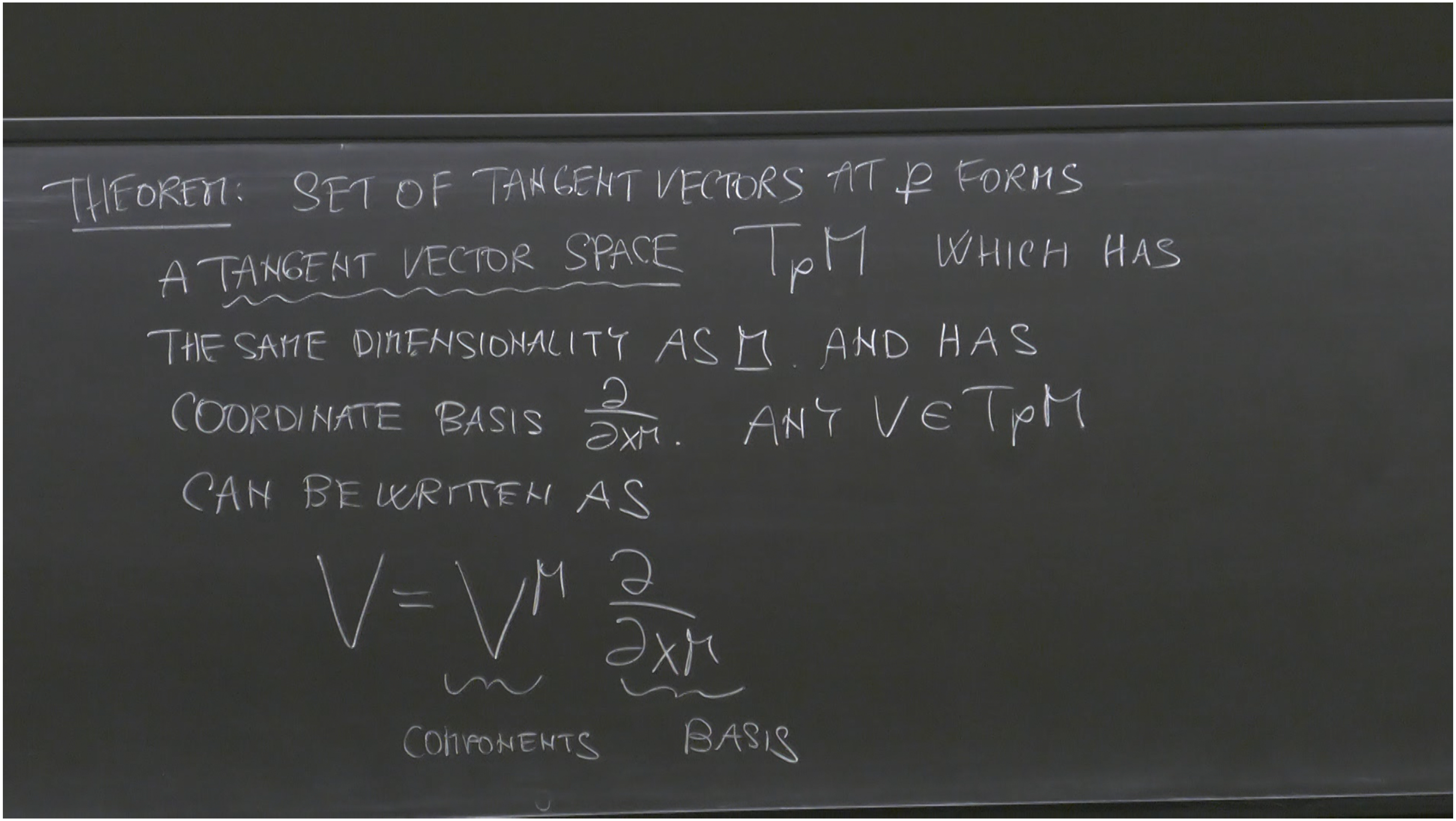
HOMEWORK: WHY WE EXPECT LEIBNITZ?



THEOREM: SET OF TANGENT VECTORS AT p FORMS A TANGENT VECTOR SPACE T_pM WHICH HAS THE SAME DIMENSIONALITY AS M. AND HAS COORDINATE BASIS $\frac{\partial}{\partial x^\mu}$. ANY $V \in T_pM$ CAN BE WRITTEN AS

$$V = \underbrace{V^\mu}_{\text{COMPONENTS}} \underbrace{\frac{\partial}{\partial x^\mu}}_{\text{BASIS}}$$

GENT VECTORS AT p FORMS

SPACE T_pM WHICH HAS

ITY AS M. AND HAS

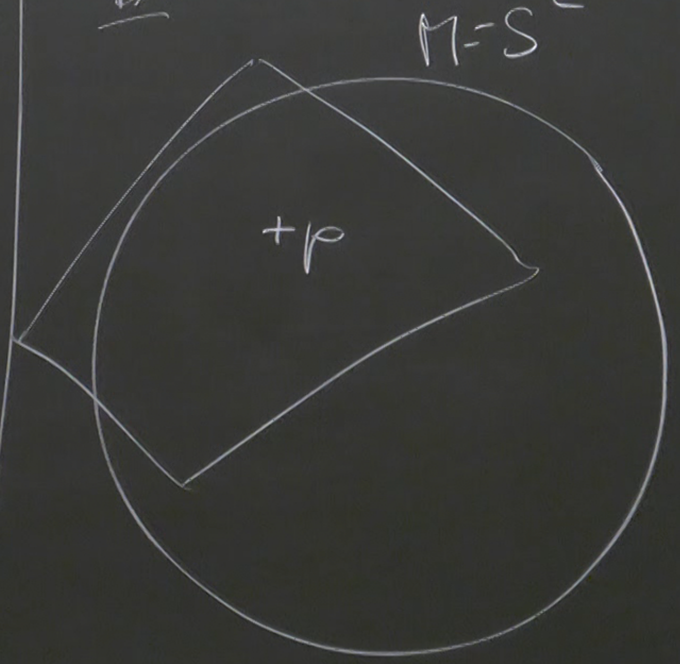
$\frac{\partial}{\partial x^\mu}$. ANY $V \in T_pM$

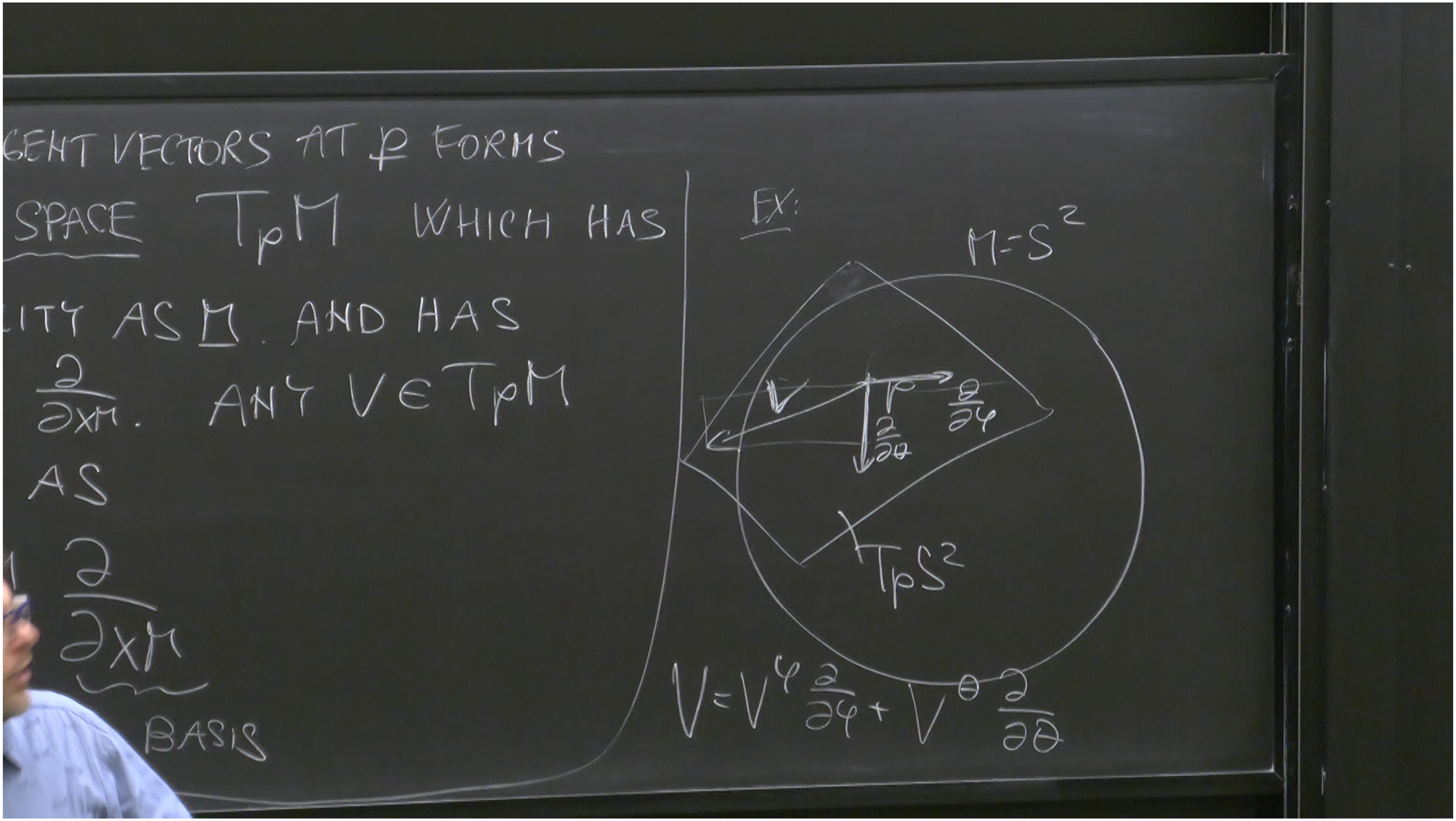
AS

$1 \; \underbrace{\frac{\partial}{\partial x^\mu}}$

TS BASIS

EX:





GENT VECTORS AT p FORMS

SPACE T_pM WHICH HAS

LITY AS M. AND HAS

$\frac{\partial}{\partial x^M}$. ANY $V \in T_pM$

AS

$$M \underbrace{\frac{\partial}{\partial x^M}}_{\text{BASIS}}$$

EX:

$M = S^2$

$$V = V^\varphi \frac{\partial}{\partial \varphi} + V^\theta \frac{\partial}{\partial \theta}$$

COMPON

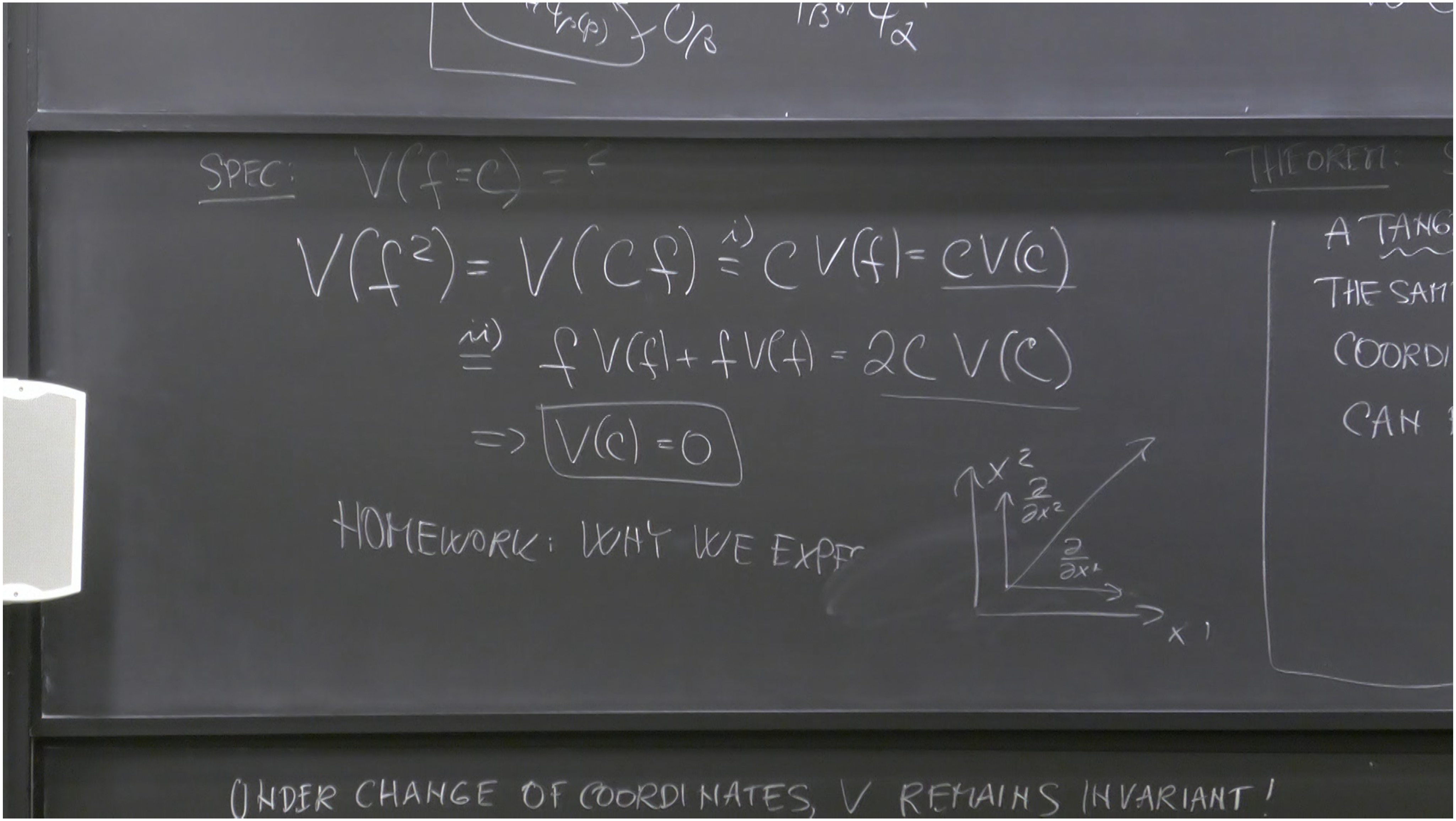
UNDER CHANGE OF COORDINATES, V REMAINS INVARIANT!

$x \to x'(x)$

$$V = V^{\mu}\frac{\partial}{\partial x^{\mu}} \underset{\substack{\uparrow \\ \text{CHAIN RULE}}}{=} \underbrace{V^{\mu}\frac{\partial x^{\nu'}}{\partial x^{\mu}}}_{V^{\nu'}}\frac{\partial}{\partial x^{\nu'}}$$

$$\boxed{V^{\nu'} = \frac{\partial x^{\nu'}}{\partial x^{\mu}} V^{\mu}}$$

TRANSF. OF COMPONENTS!



SPEC: $V(f=c) =$

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HOMEWORK: WHY WE EXPE

THEOREM:

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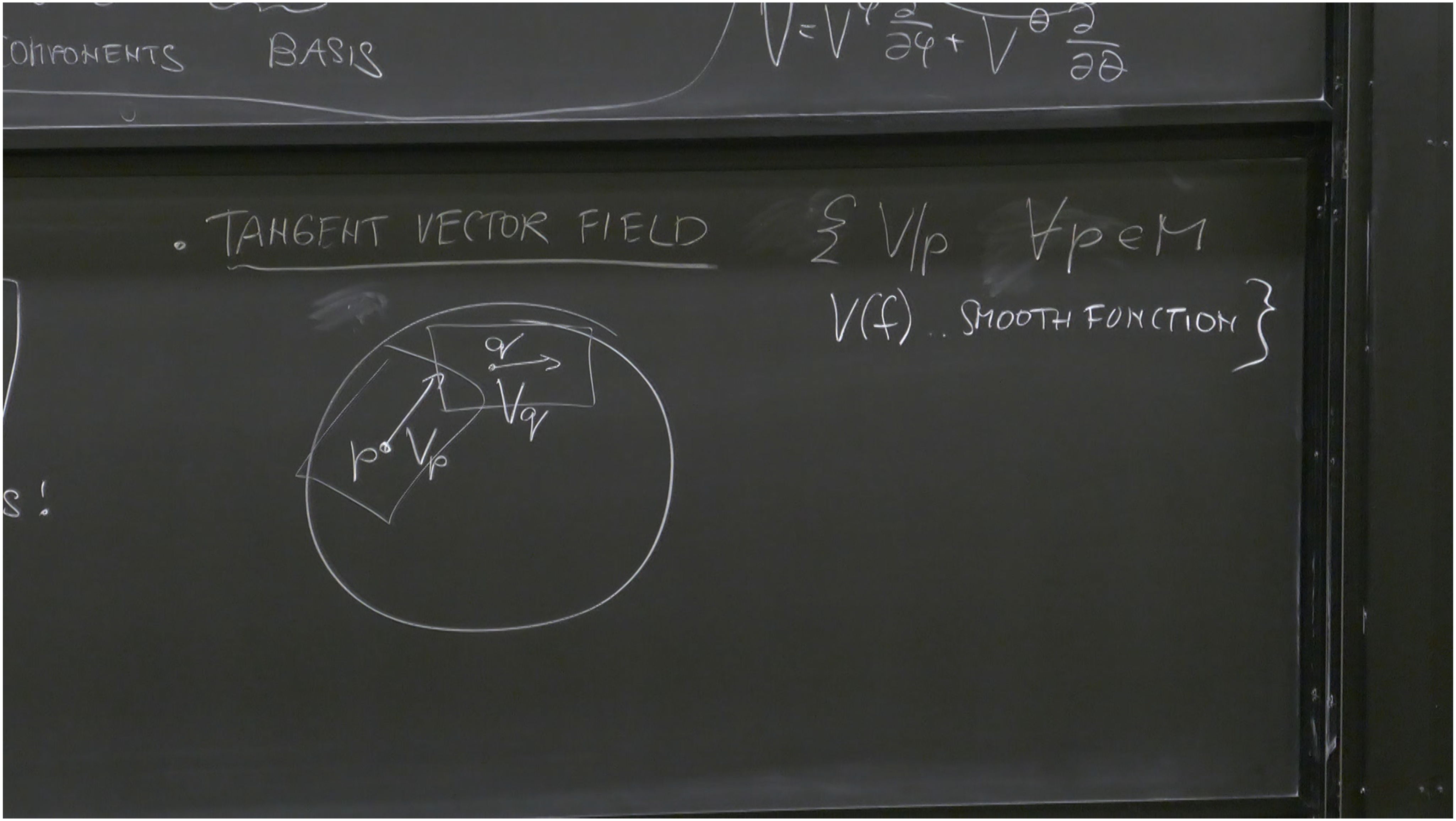

OMPONENTS BASIS

$V = V^{\varphi}\frac{\partial}{\partial\varphi} + V^{\theta}\frac{\partial}{\partial\theta}$

- TANGENT VECTOR FIELD

$\{ V|_p \quad \forall p \in M$

$V(f)$ .. SMOOTH FUNCTION $\}$

s!

s!

- TANGENT VECTOR FIELD $\{ V|_p \ \forall p \in M$

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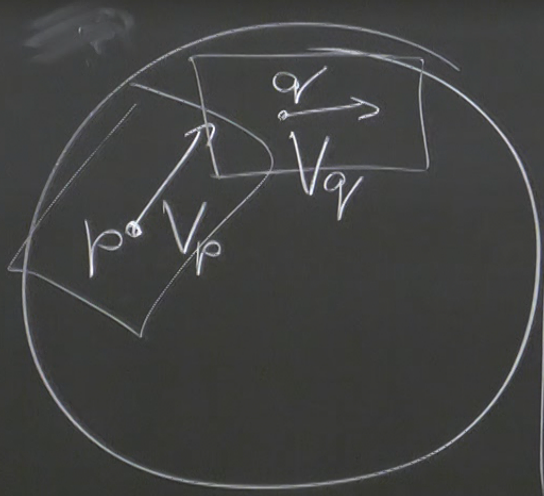
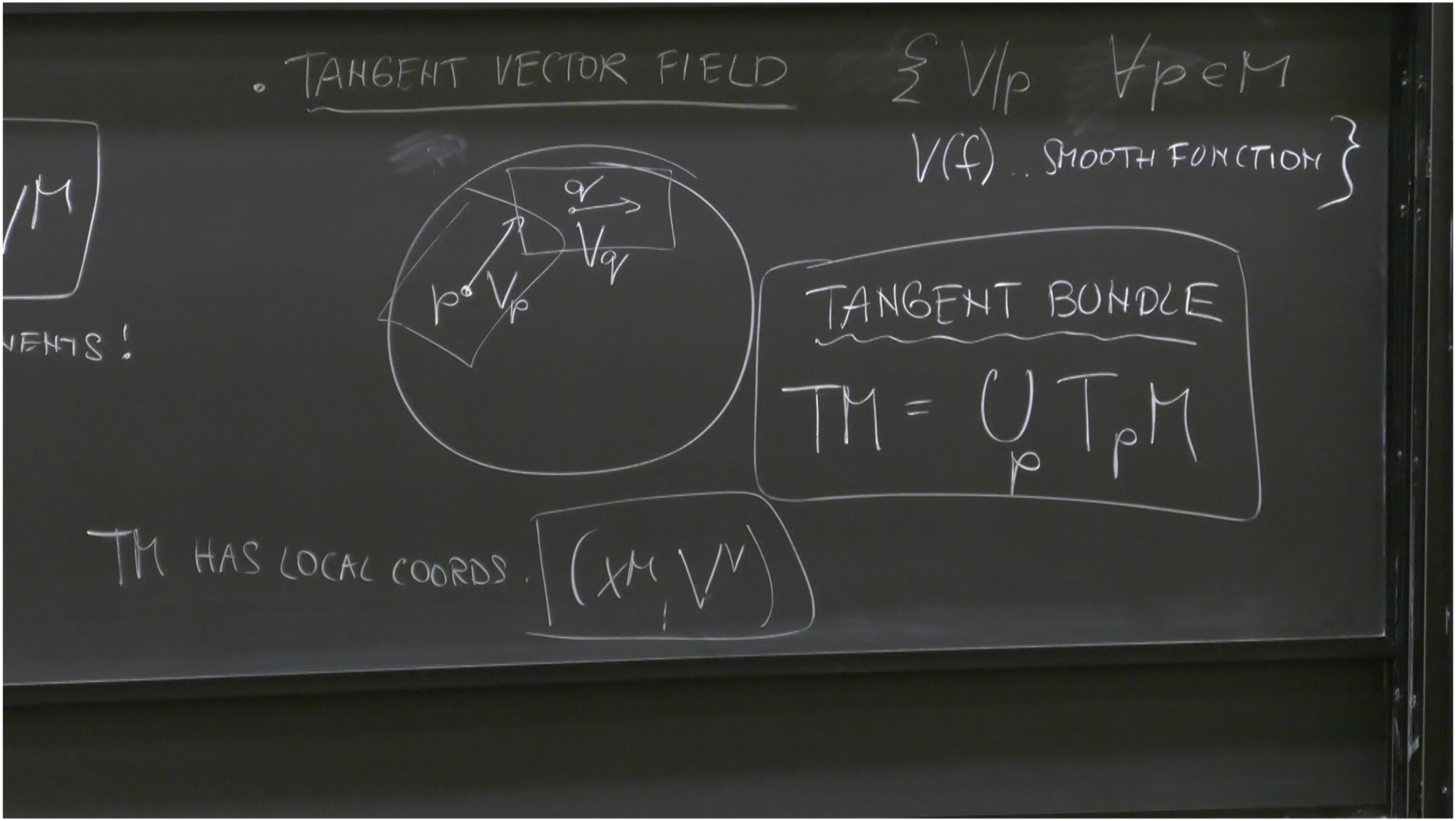


TANGENT BUNDLE

$$\boxed{TM = \bigcup_p T_pM}$$



· TANGENT VECTOR FIELD $\{ V|_p \ \forall p \in M$

$V(f)$ .. SMOOTH FUNCTION $\}$

/M

NENTS !

TANGENT BUNDLE

$$TM = \bigcup_p T_pM$$

TM HAS LOCAL COORDS. $\boxed{(x^\mu, V^\nu)}$

$\boxed{/M}$

NENTS !

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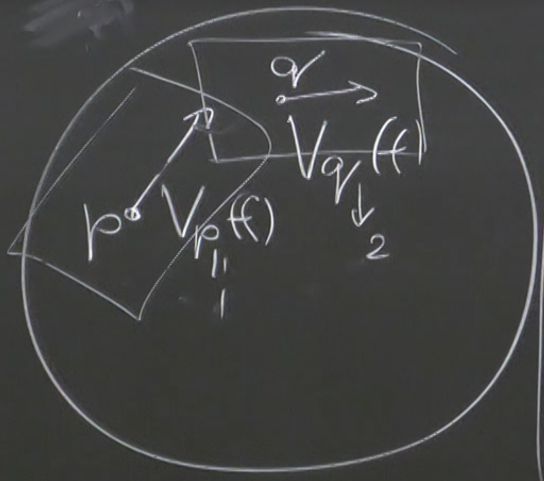


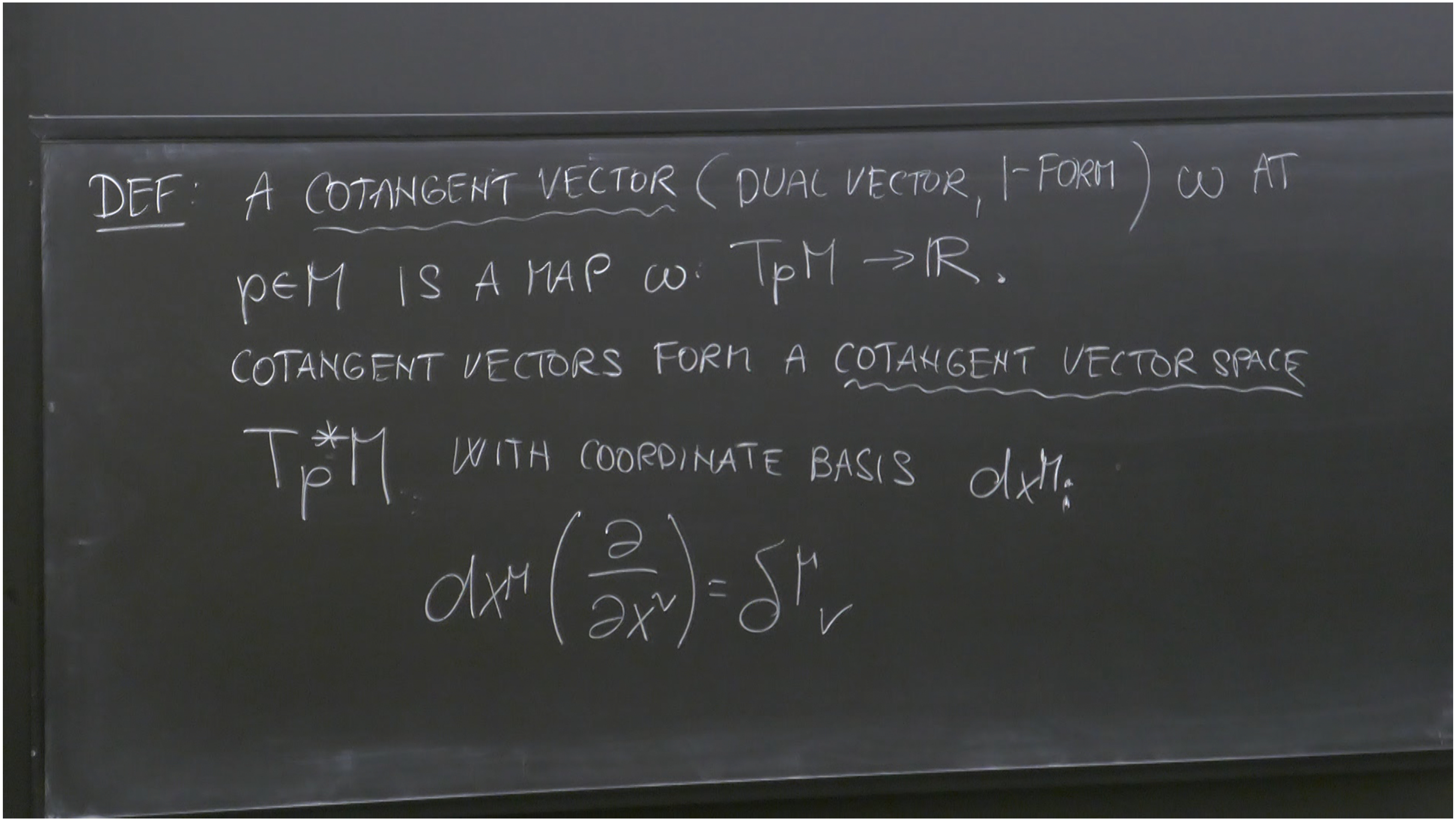
$$\boxed{\text{TANGENT BUNDLE} \quad TM = \bigcup_p T_pM}$$

TM HAS LOCAL COORDS. $\boxed{(x^\mu, V^\nu)}$

DEF: A COTANGENT VECTOR (DUAL VECTOR, 1-FORM) ω AT $p \in M$ IS A MAP $\omega: T_pM \to \mathbb{R}$.

COTANGENT VECTORS FORM A COTANGENT VECTOR SPACE

T_p^*M WITH COORDINATE BASIS dx^μ.



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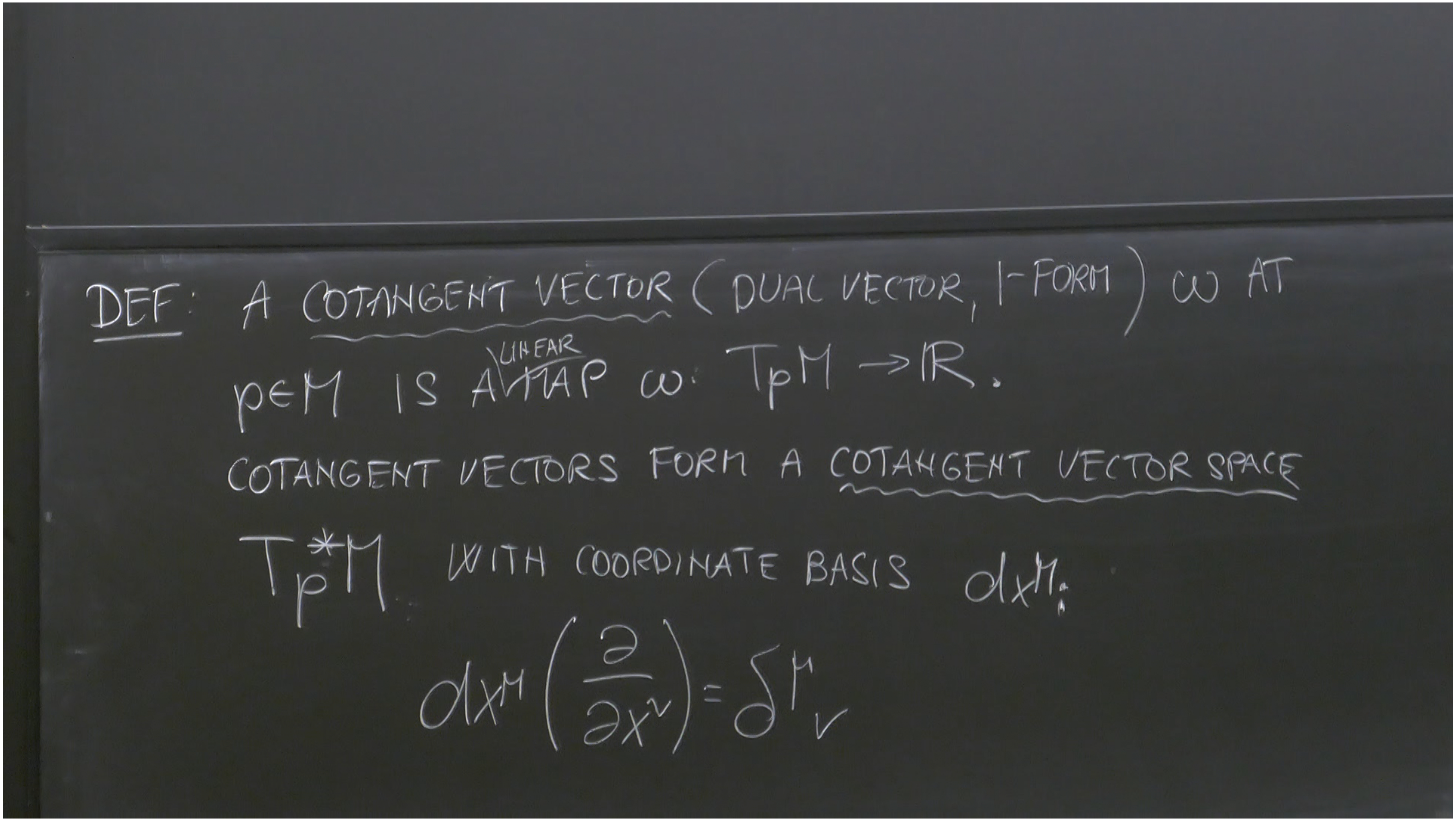
$$dx^\mu\left(\frac{\partial}{\partial x^\nu}\right)=\delta^\mu{}_\nu$$



DEF: A **COTANGENT VECTOR** (DUAL VECTOR, 1-FORM) ω AT $p \in M$ IS A LINEAR MAP $\omega : T_pM \to \mathbb{R}$.

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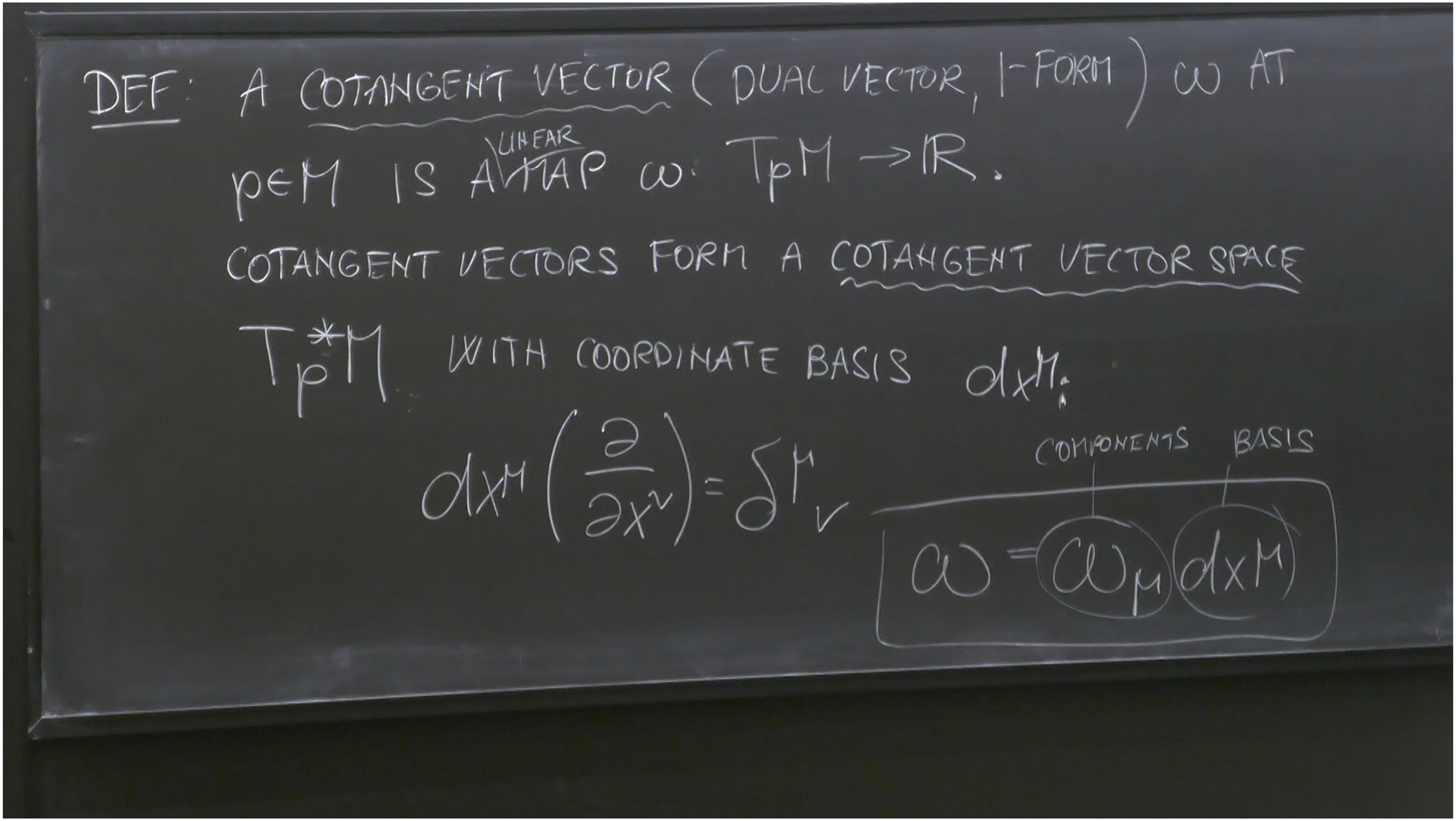
$$dx^\mu\left(\frac{\partial}{\partial x^\nu}\right)=\delta^\mu{}_\nu$$



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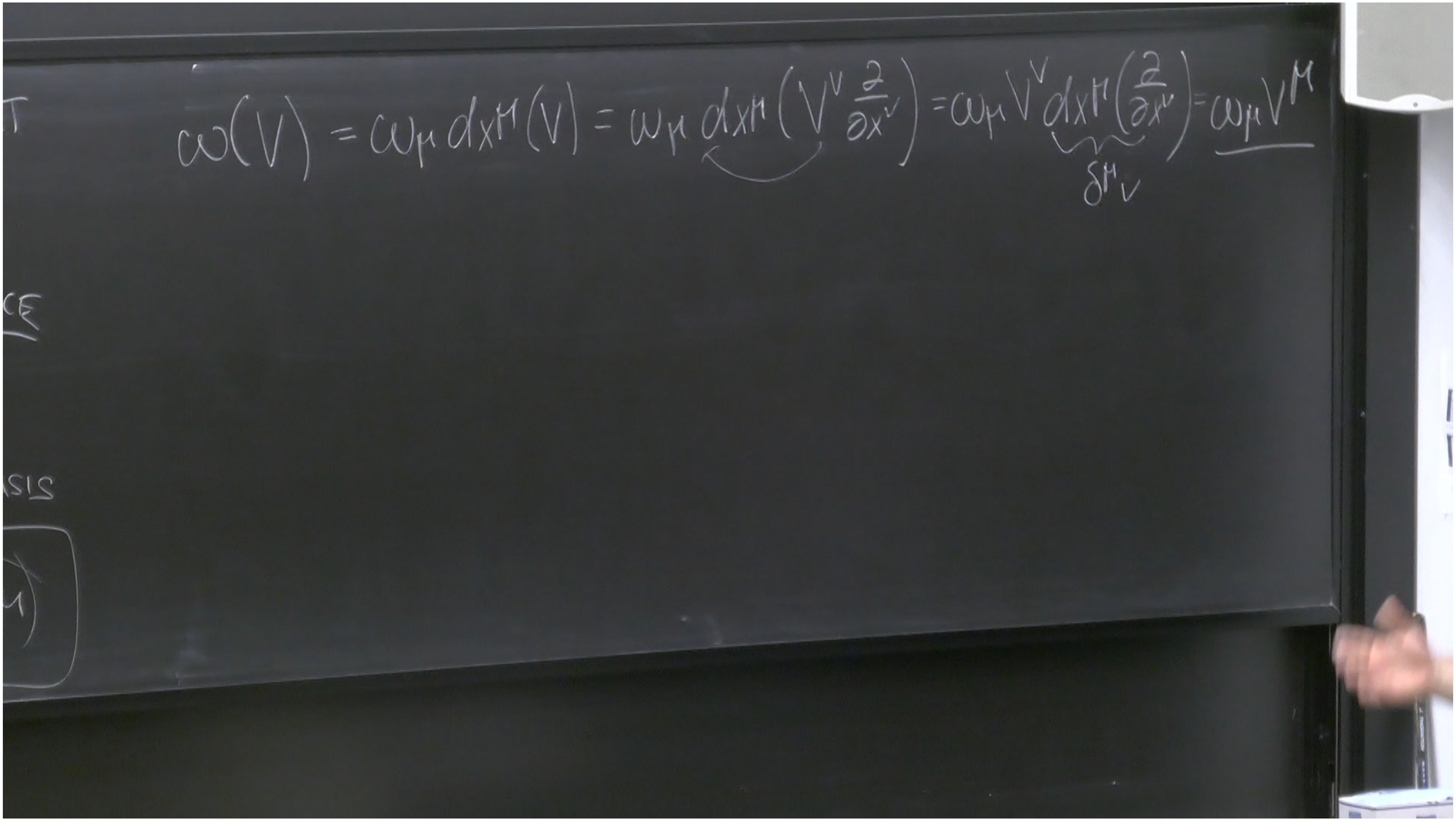
$$\boxed{\omega = \overbrace{\omega_\mu}^{\text{COMPONENTS}} \overbrace{dx^\mu}^{\text{BASIS}}}$$

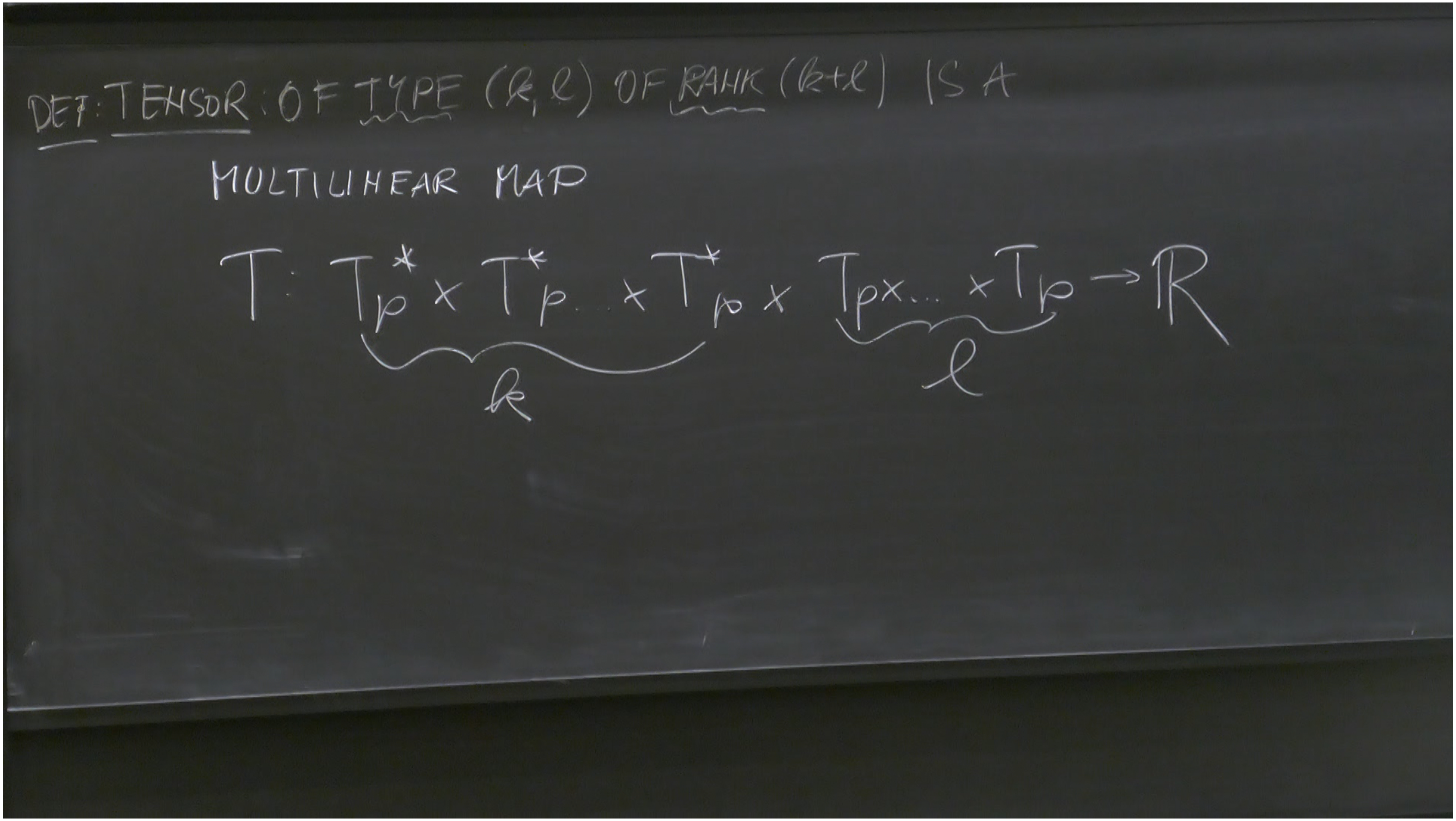


$$\omega(V) = \omega_\mu\, dx^\mu(V) = \omega_\mu\, dx^\mu\left(V^\nu \frac{\partial}{\partial x^\nu}\right) = \omega_\mu V^\nu \underbrace{dx^\mu\left(\frac{\partial}{\partial x^\nu}\right)}_{\delta^\mu{}_\nu} = \underline{\omega_\mu V^\mu}$$

$\omega(V) = \omega_\mu dx^\mu(V)$ … $\delta^\mu{}_\nu$

COTANGENT BUNDLE $T^*M = \bigcup_p T^*_p M$

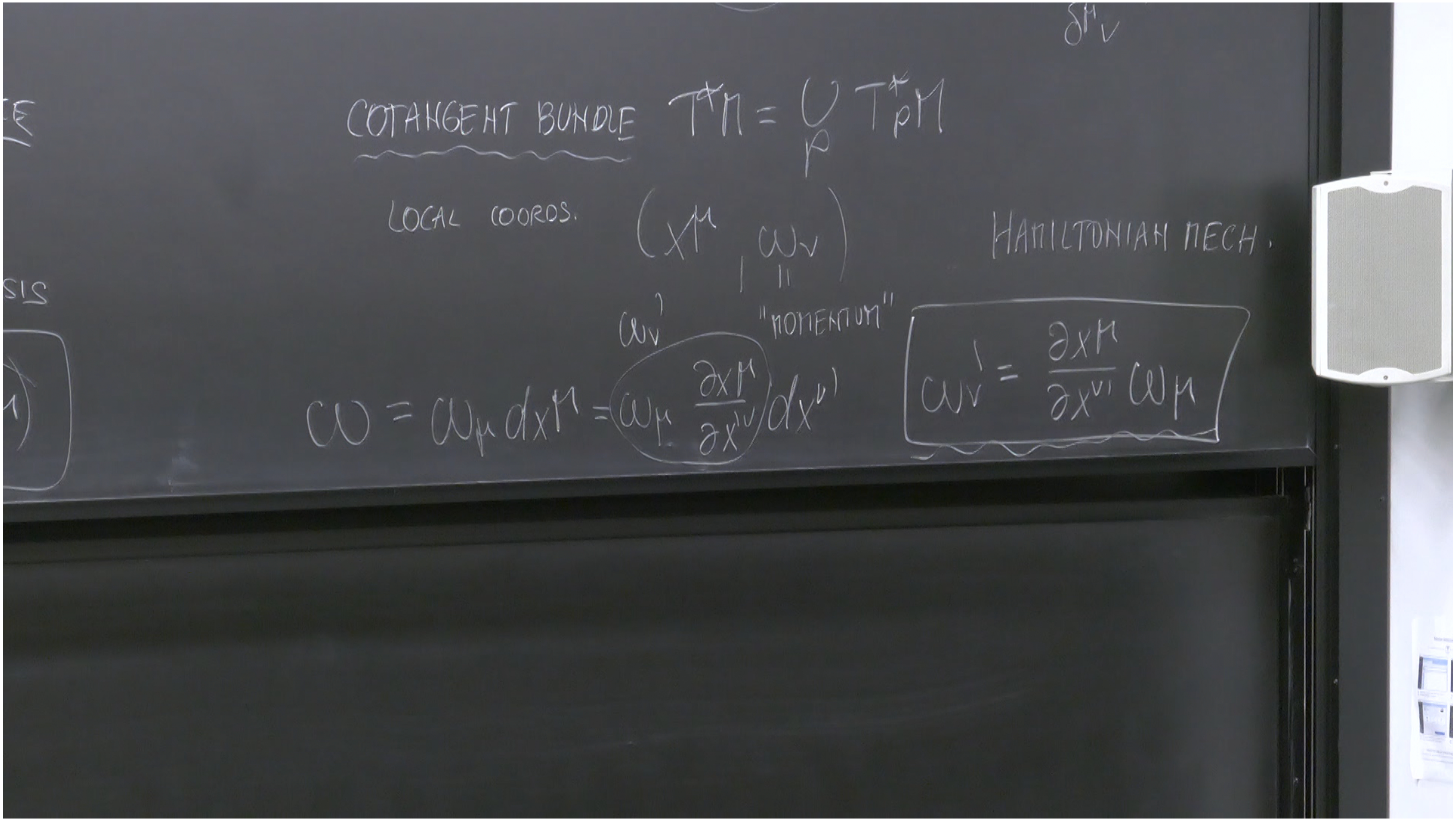
LOCAL COORDS. (x^μ, ω_ν)



DEF: TENSOR OF TYPE (k, l) OF RANK $(k+l)$ IS A

MULTILINEAR MAP

$$T: \underbrace{T_p^* \times T_p^* \cdots \times T_p^*}_{k} \times \underbrace{T_p \times \cdots \times T_p}_{l} \to \mathbb{R}$$



$\delta^\mu{}_\nu$

E

COTANGENT BUNDLE $T^*M = \bigcup_p T^*_pM$

LOCAL COORDS. (x^μ, ω_ν)

"MOMENTUM"

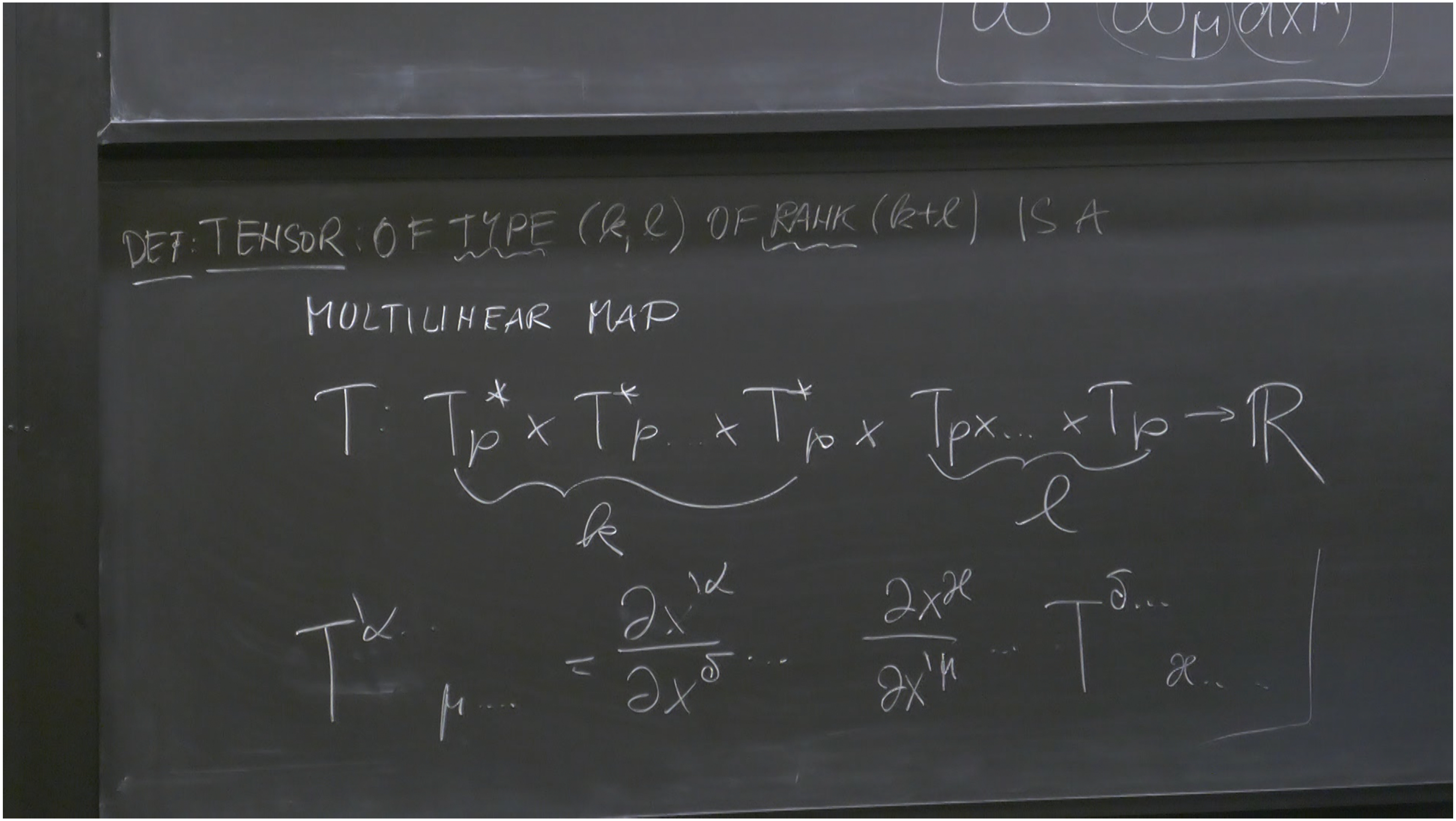
HAMILTONIAN MECH.

SIS

ω'_ν

$$\omega = \omega_\mu dx^\mu = \omega_\mu \frac{\partial x^\mu}{\partial x'^\nu} dx'^\nu$$

$$\boxed{\omega'_\nu = \frac{\partial x^\mu}{\partial x'^\nu} \omega_\mu}$$

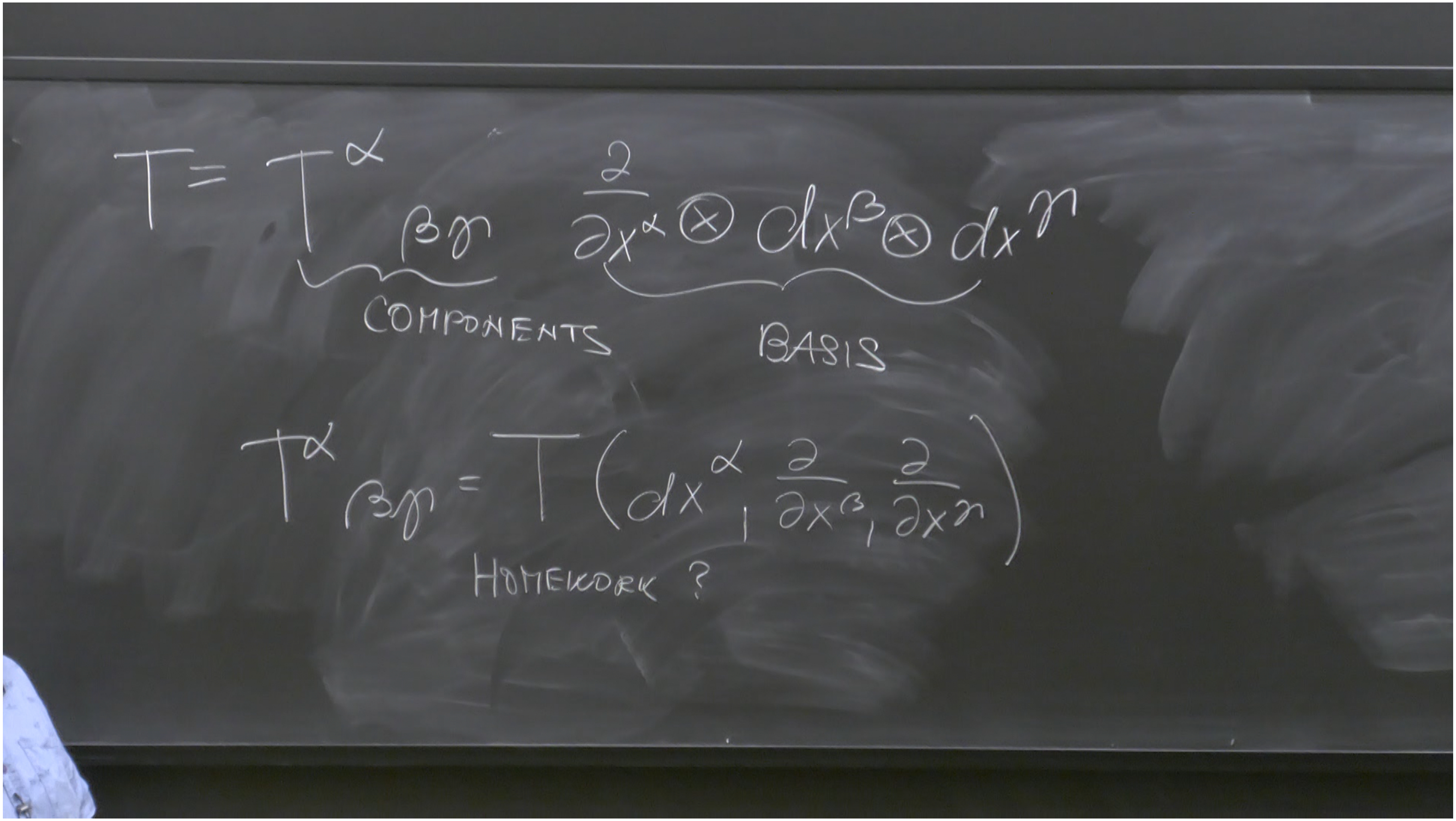


$\omega \quad \omega_\mu \, dx^\mu$

DEF: TENSOR OF TYPE (k, l) OF RANK $(k+l)$ IS A MULTILINEAR MAP

$$T: \underbrace{T_p^* \times T_p^* \ldots \times T_p^*}_{k} \times \underbrace{T_p \times \ldots \times T_p}_{l} \to \mathbb{R}$$

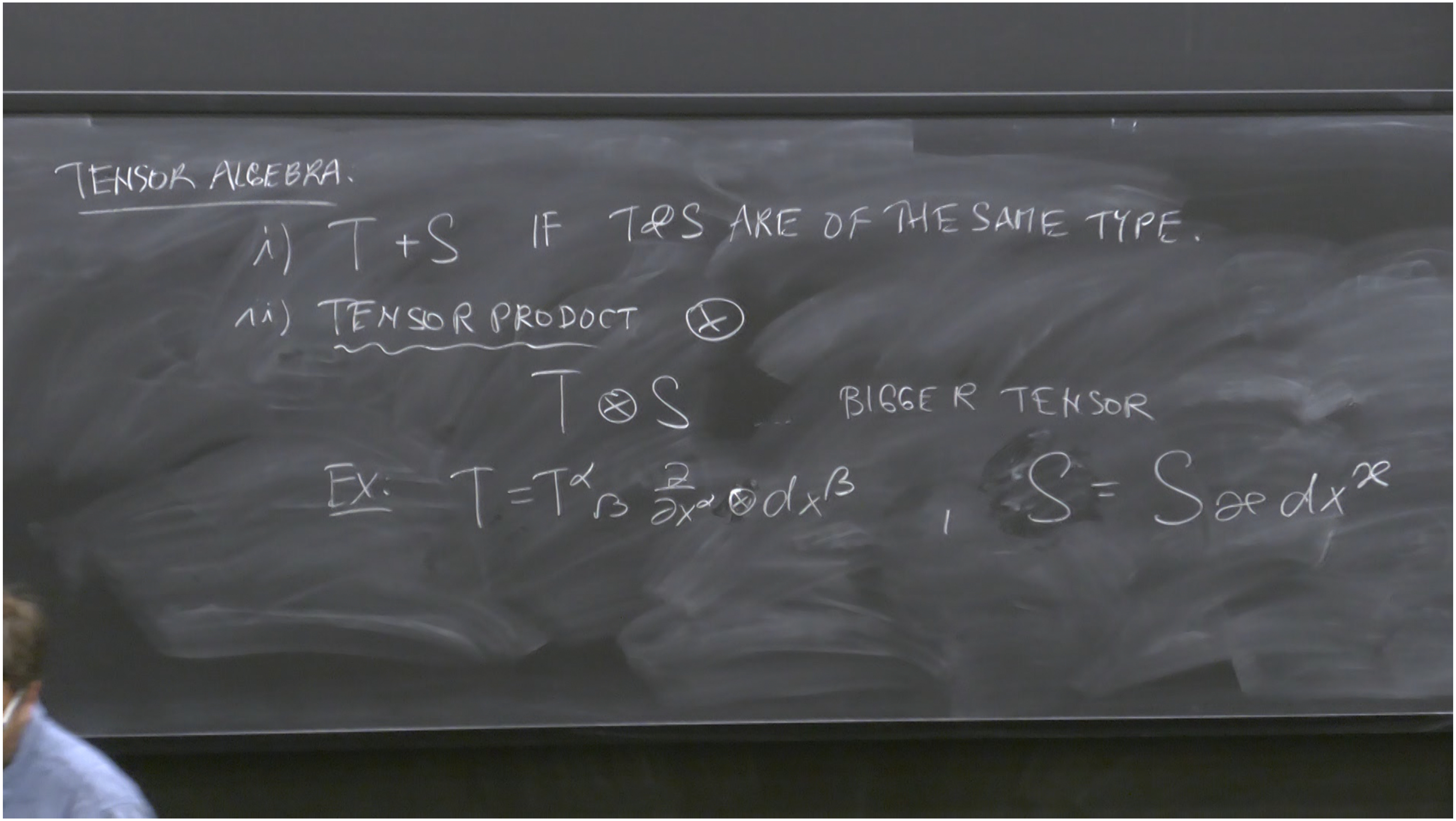
$$T'^{\alpha \ldots}{}_{\mu \ldots} = \frac{\partial x'^{\alpha}}{\partial x^{\delta}} \cdots \frac{\partial x^{\varkappa}}{\partial x'^{\mu}} \cdots T^{\delta \ldots}{}_{\varkappa \ldots}$$



$$T = \underbrace{T^{\alpha}{}_{\beta\gamma}}_{\text{COMPONENTS}} \underbrace{\frac{\partial}{\partial x^{\alpha}} \otimes dx^{\beta} \otimes dx^{\gamma}}_{\text{BASIS}}$$

$$T^{\alpha}{}_{\beta\gamma} = T\left(dx^{\alpha}, \frac{\partial}{\partial x^{\beta}}, \frac{\partial}{\partial x^{\gamma}}\right)$$

HOMEWORK ?



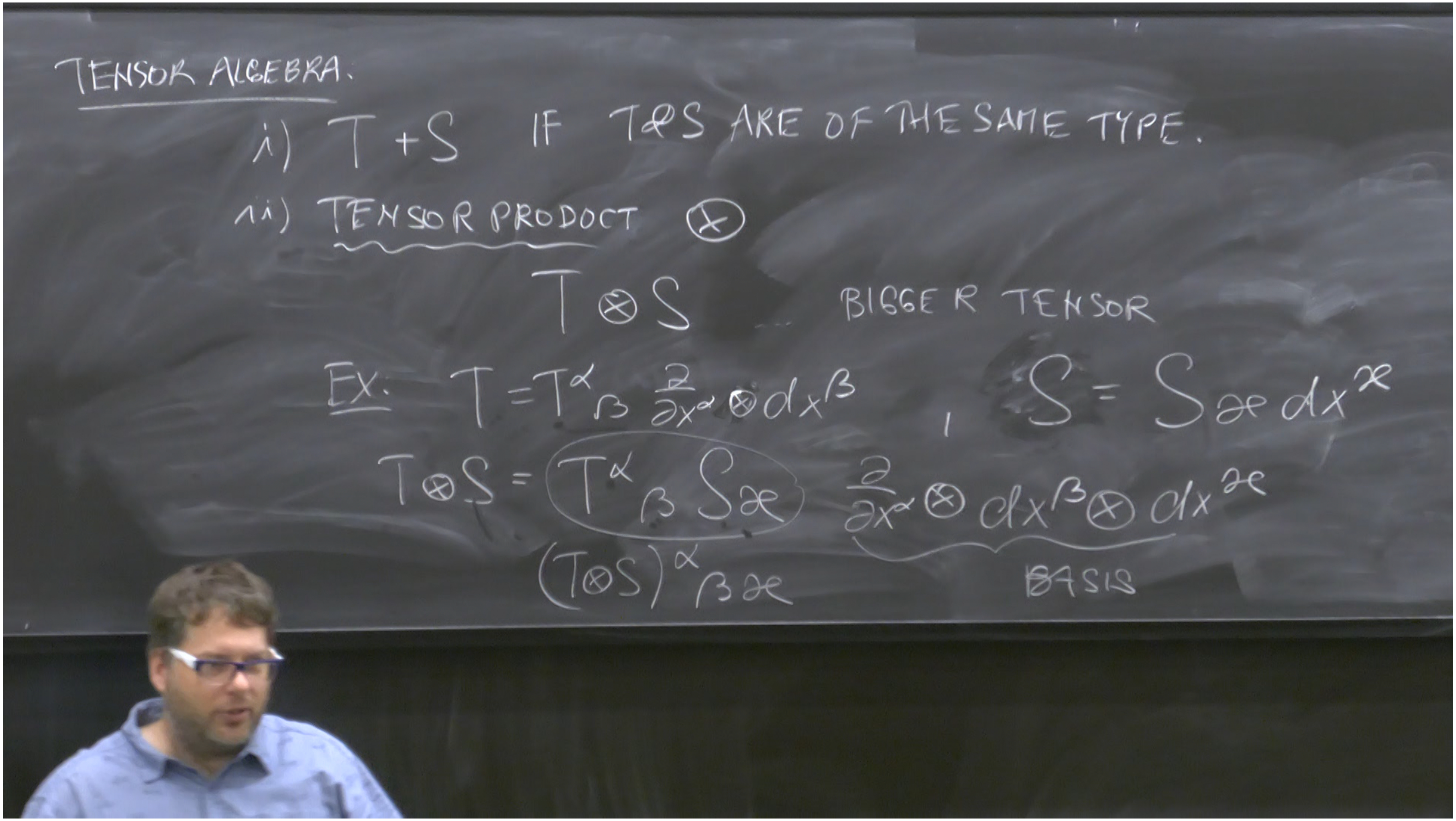
TENSOR ALGEBRA.

i) $T + S$ IF T & S ARE OF THE SAME TYPE.

ii) TENSOR PRODUCT $\otimes$

$T \otimes S$ ... BIGGER TENSOR

Ex: $T = T^{\alpha}{}_{\beta} \frac{\partial}{\partial x^{\alpha}} \otimes dx^{\beta}$, $S = S_{\varkappa} dx^{\varkappa}$



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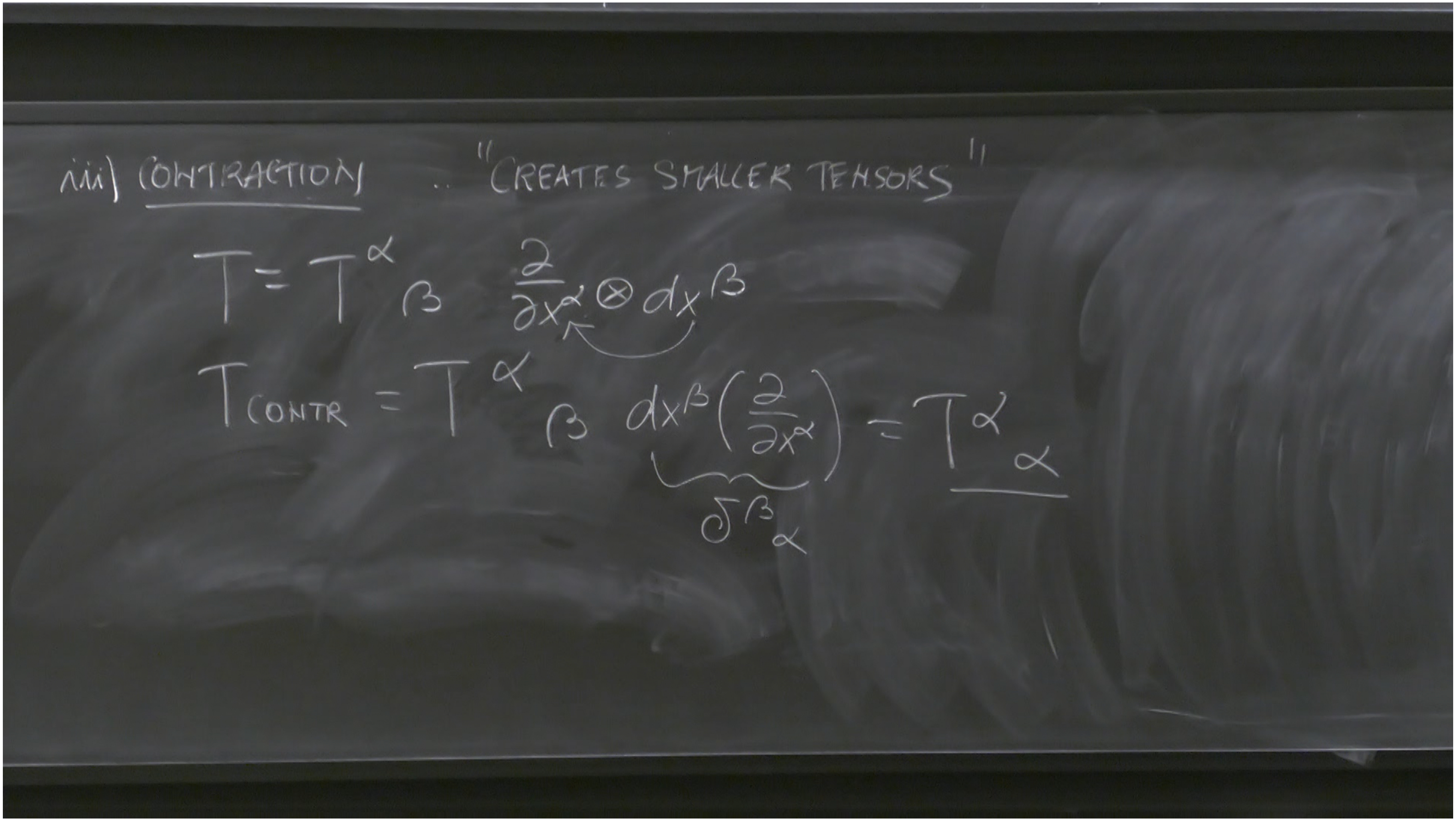
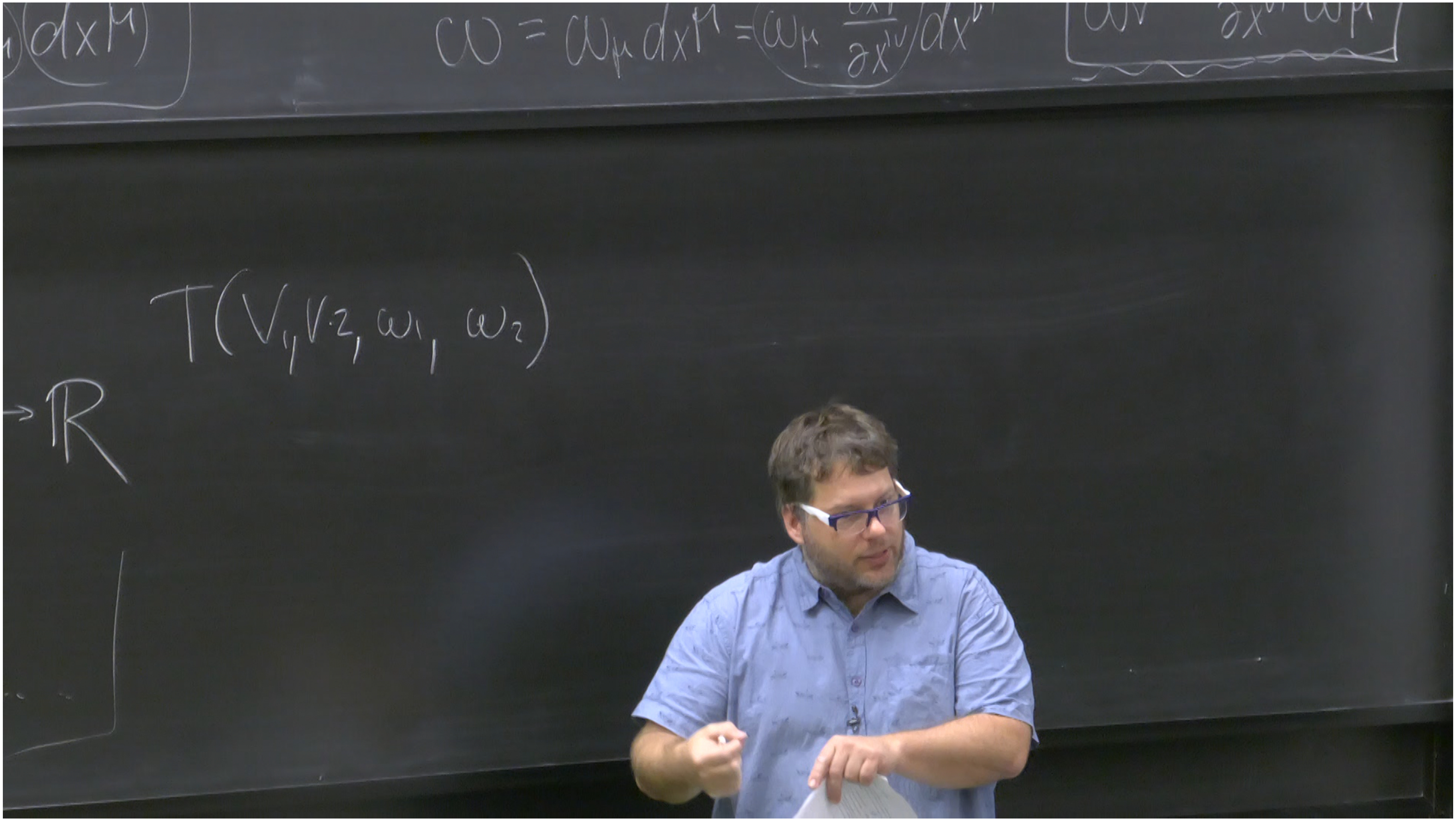
$$T \otimes S = \underset{(T\otimes S)^{\alpha}{}_{\beta\varkappa}}{\boxed{T^{\alpha}{}_{\beta} S_{\varkappa}}} \underbrace{\frac{\partial}{\partial x^{\alpha}} \otimes dx^{\beta} \otimes dx^{\varkappa}}_{\text{BASIS}}$$

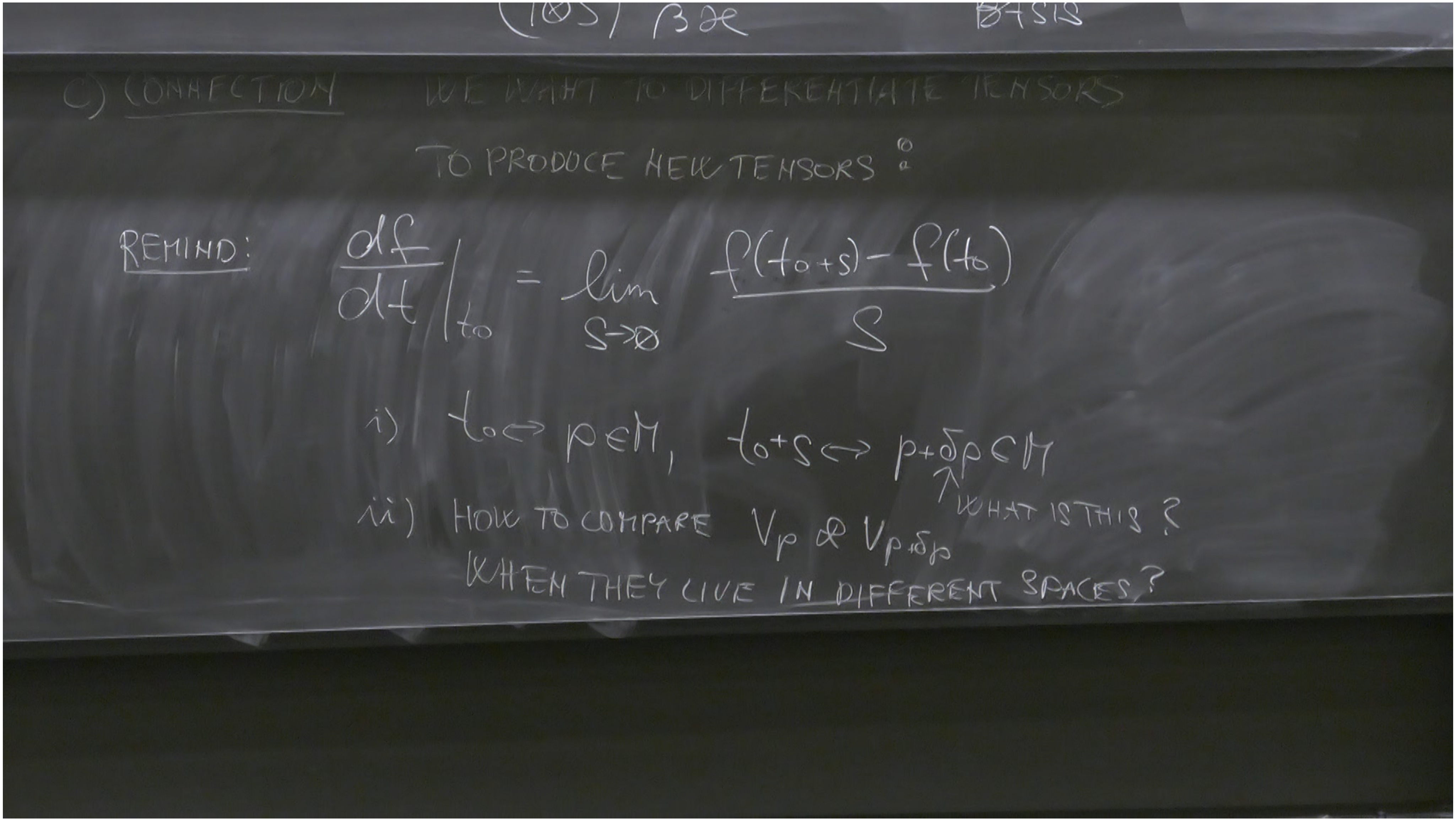


iii) CONTRACTION .. "CREATES SMALLER TENSORS"

$$T = T^{\alpha}{}_{\beta} \, \frac{\partial}{\partial x^{\alpha}} \otimes dx^{\beta}$$

$$T_{\text{CONTR}} = T^{\alpha}{}_{\beta} \, \underbrace{dx^{\beta}\left(\frac{\partial}{\partial x^{\alpha}}\right)}_{\delta^{\beta}{}_{\alpha}} = \underline{T^{\alpha}{}_{\alpha}}$$


(dx^μ)
ω = ω_μ dx^μ = ω_μ ∂x^μ/∂x'^ν dx'^ν
ω_ν' = ∂x^μ/∂x'^ν ω_μ
T(V_1, V_2, ω_1, ω_2)
→ ℝ



$(T\otimes S)$ $\beta\partial e$ BASIS

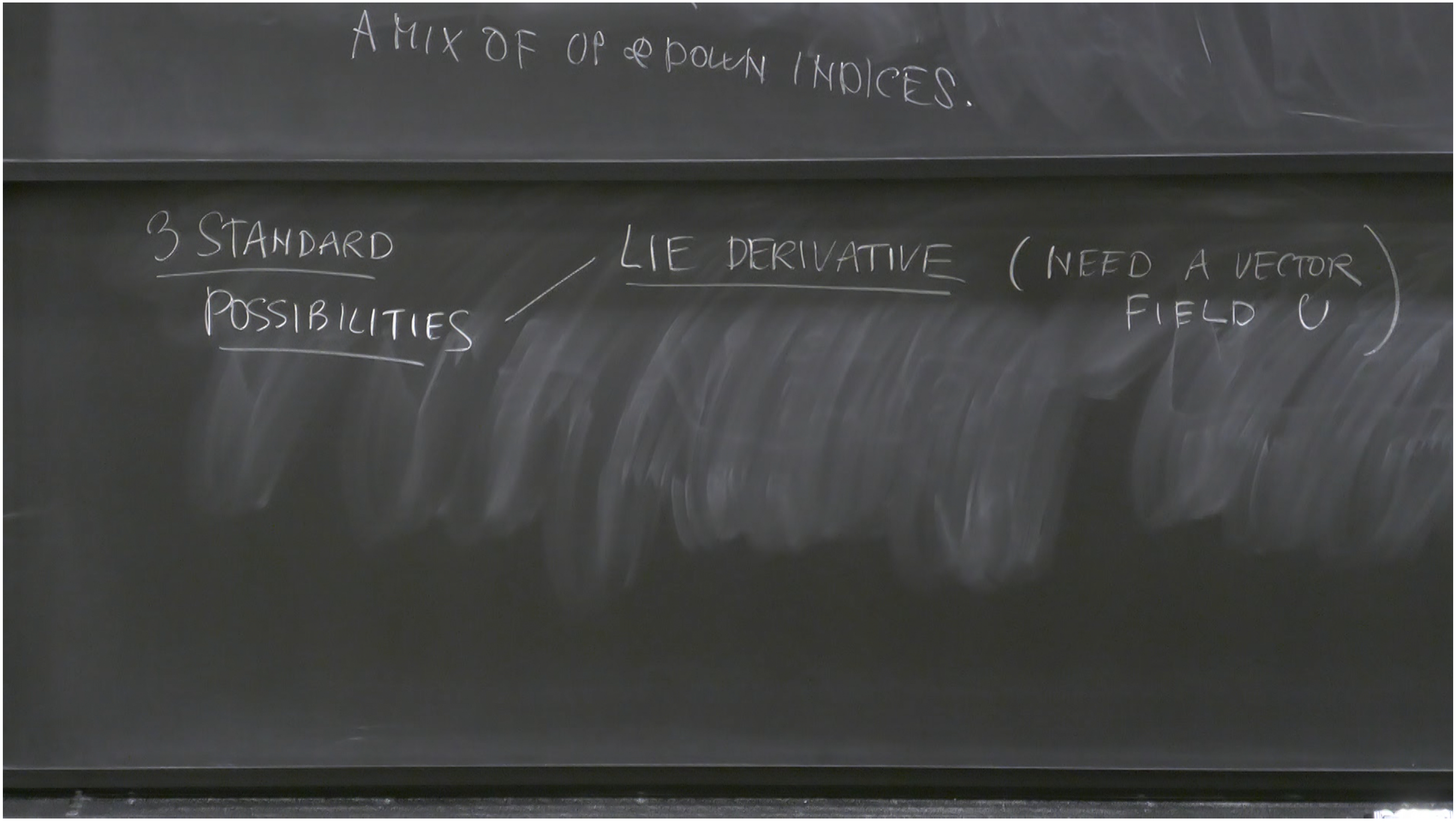
C) CONNECTION WE WANT TO DIFFERENTIATE TENSORS

TO PRODUCE NEW TENSORS :

REMIND: $$\left.\frac{df}{dt}\right|_{t_0} = \lim_{s\to 0}\frac{f(t_0+s)-f(t_0)}{s}$$

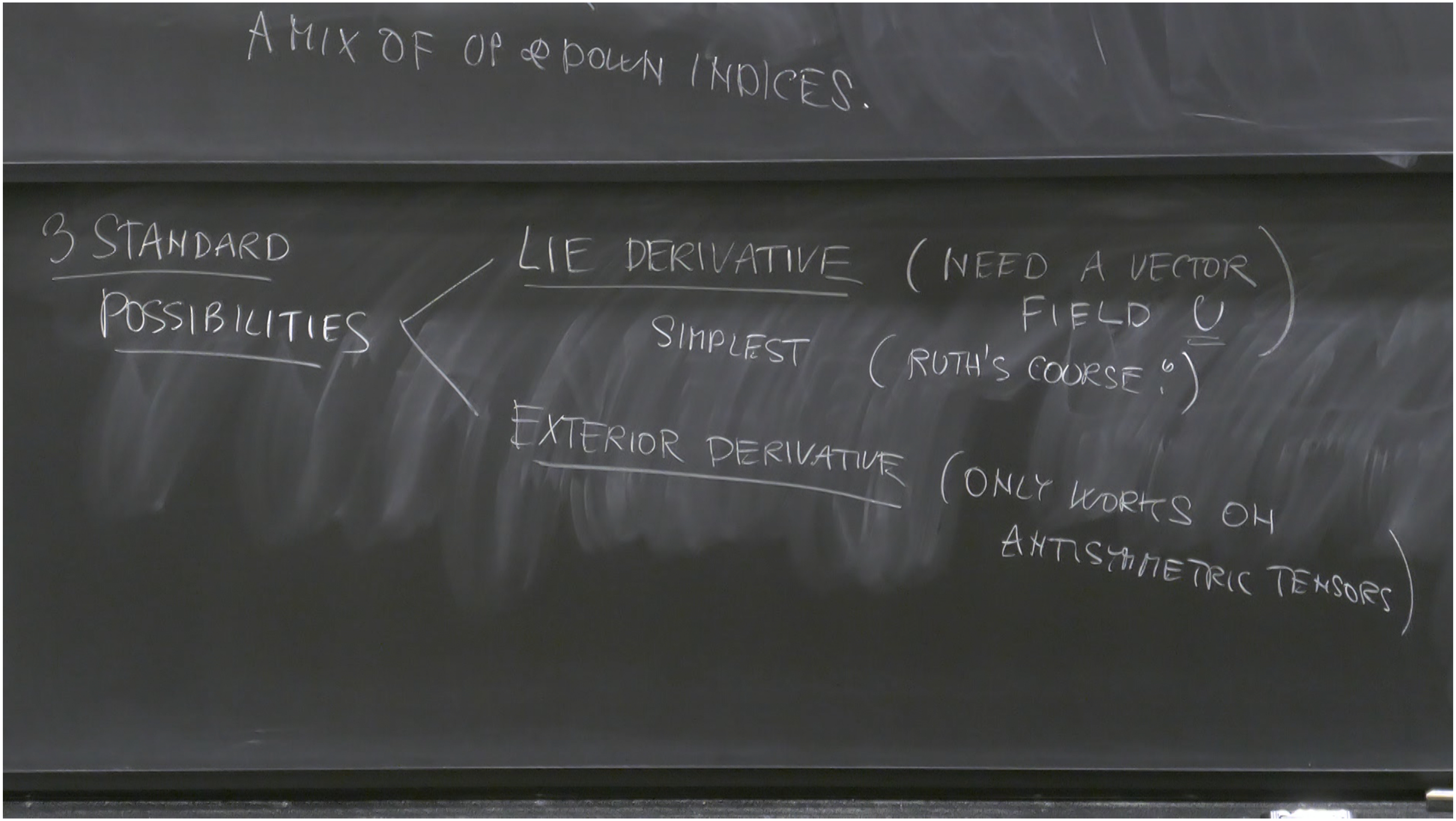
i) $t_0 \leftrightarrow p\in M$, $t_0+s \leftrightarrow p+\underset{\uparrow}{\delta p}\in M$ WHAT IS THIS ?

ii) HOW TO COMPARE V_p & $V_{p+\delta p}$ WHEN THEY LIVE IN DIFFERENT SPACES ?



A MIX OF UP & DOWN INDICES.

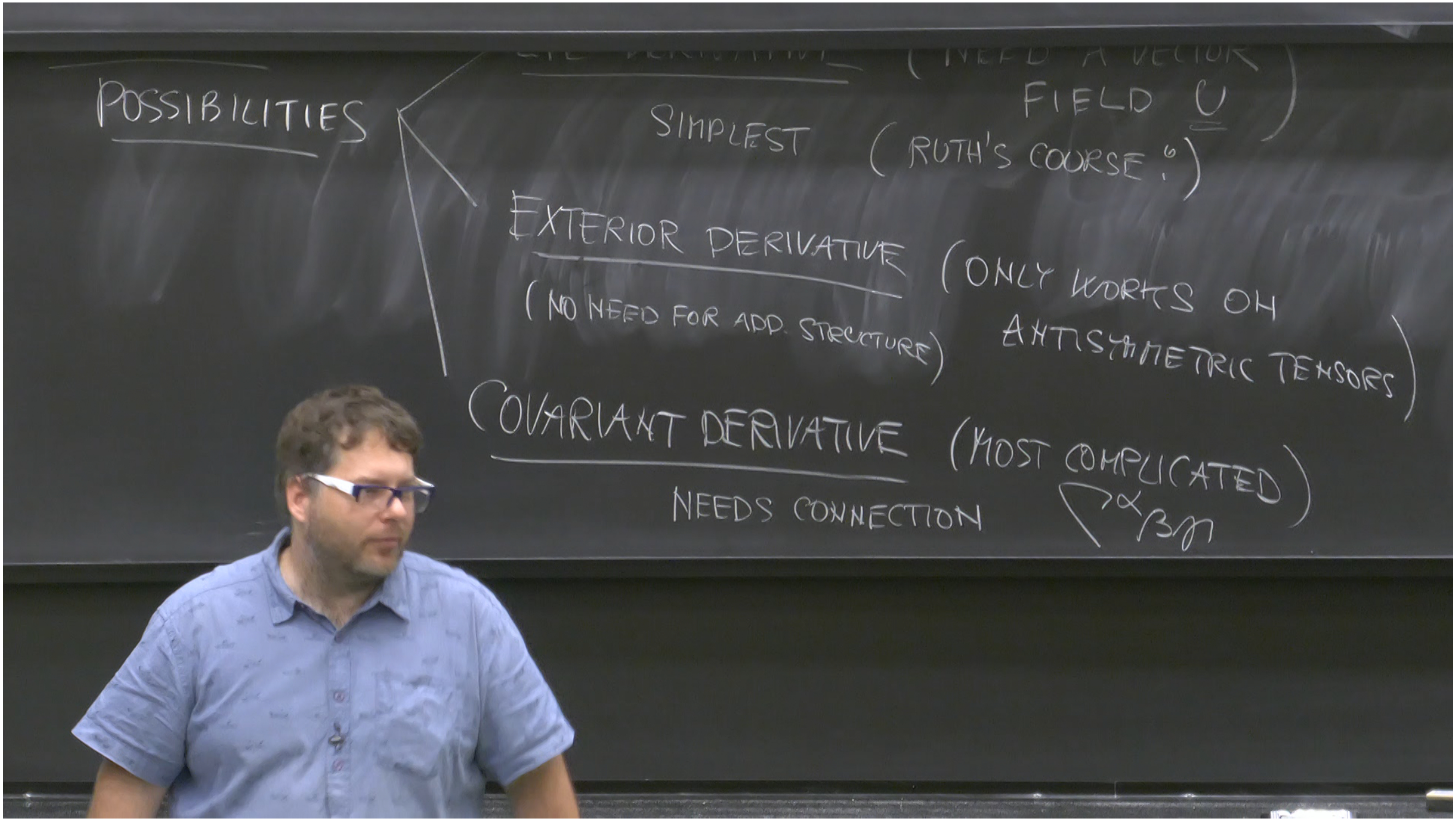
3 STANDARD POSSIBILITIES — LIE DERIVATIVE (NEED A VECTOR FIELD v)



A MIX OF UP & DOWN INDICES.

3 STANDARD POSSIBILITIES

- LIE DERIVATIVE (NEED A VECTOR FIELD $\underline{U}$)
  SIMPLEST (RUTH'S COURSE ?)
- EXTERIOR DERIVATIVE (ONLY WORKS ON ANTISYMMETRIC TENSORS)



POSSIBILITIES

LIE DERIVATIVE (NEED A VECTOR FIELD $\underline{U}$)

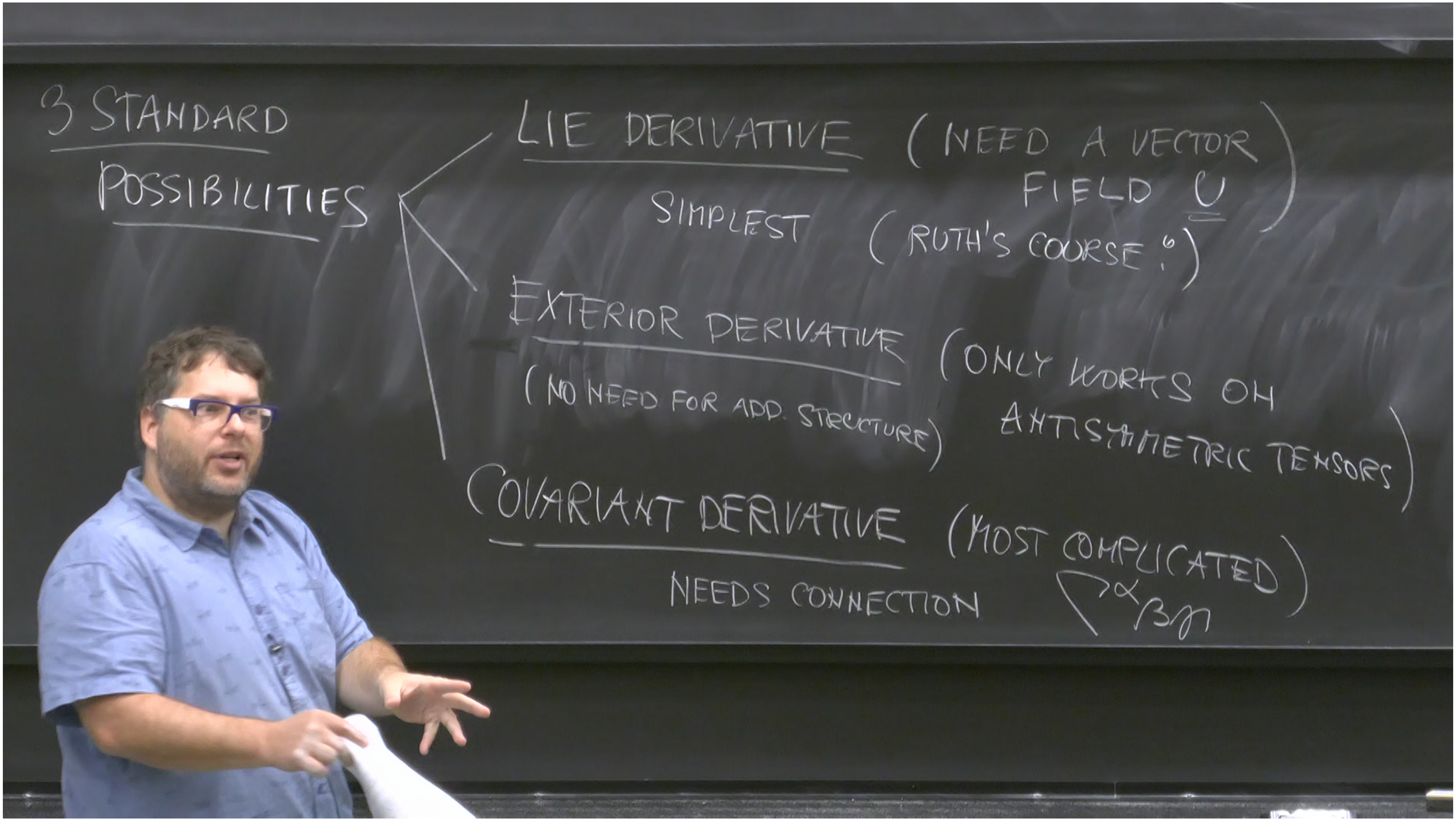
SIMPLEST (RUTH'S COURSE ☺)

EXTERIOR DERIVATIVE (ONLY WORKS ON ANTISYMMETRIC TENSORS)

(NO NEED FOR ADD. STRUCTURE)

COVARIANT DERIVATIVE (MOST COMPLICATED)

NEEDS CONNECTION $\Gamma^{\alpha}{}_{\beta\gamma}$



3 STANDARD POSSIBILITIES

- LIE DERIVATIVE (NEED A VECTOR FIELD $\underline{U}$)
  SIMPLEST (RUTH'S COURSE :))
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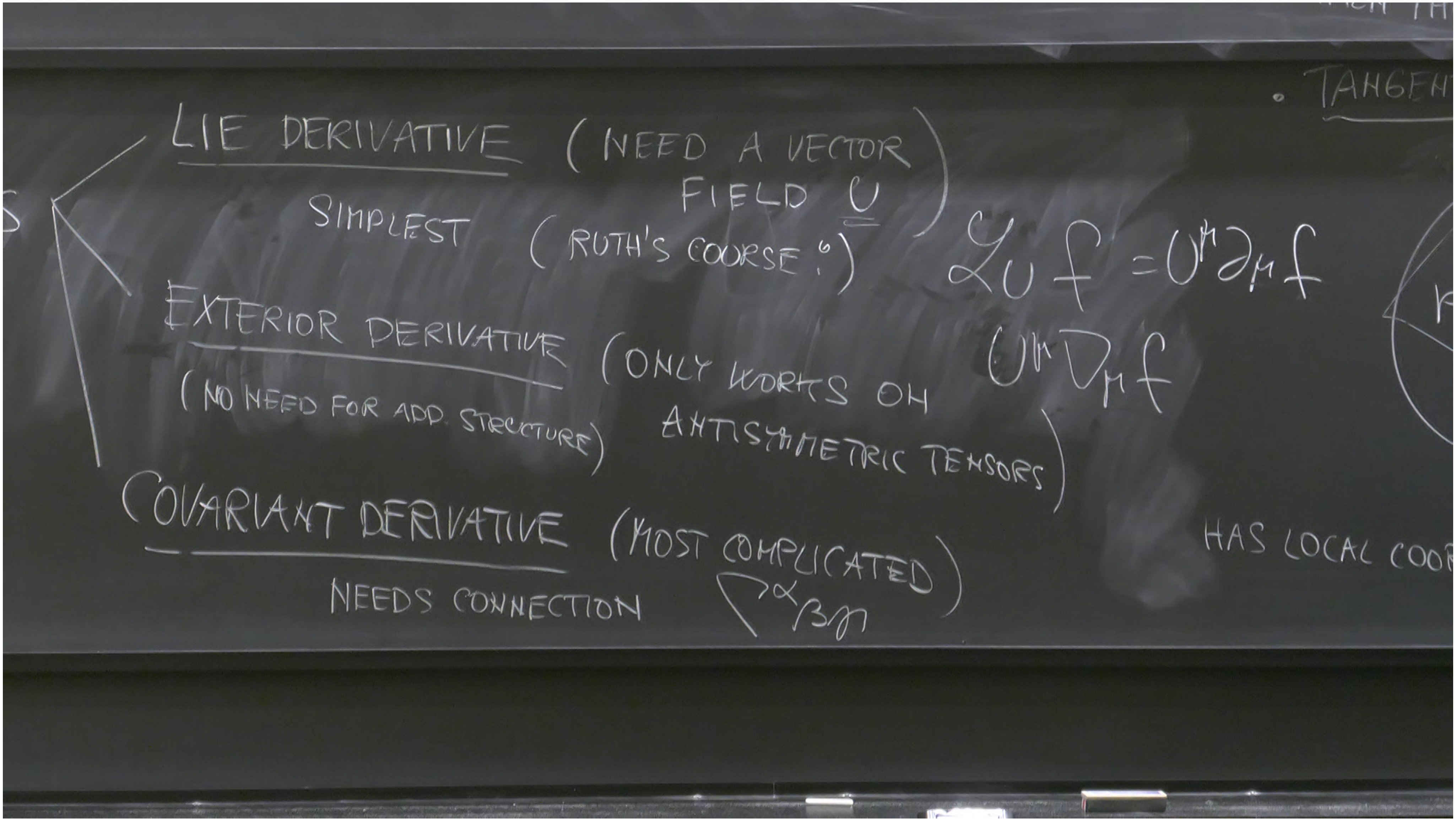

S

LIE DERIVATIVE (NEED A VECTOR FIELD $\underline{U}$)

SIMPLEST (RUTH'S COURSE ☺)

$$\mathcal{L}_U f = U^\mu \partial_\mu f$$

$$U^\mu \nabla_\mu f$$

EXTERIOR DERIVATIVE (ONLY WORKS ON ANTISYMMETRIC TENSORS)

(NO NEED FOR ADD. STRUCTURE)

COVARIANT DERIVATIVE (MOST COMPLICATED)

NEEDS CONNECTION $\Gamma^\alpha{}_{\beta\gamma}$

• TANGEN

HAS LOCAL COOR