Title: General Relativity for Cosmology Lecture - 103123

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Collection: General Relativity for Cosmology

Date: October 31, 2023 - 2:00 PM

URL: https://pirsa.org/23100030

Abstract: Zoom: https://pitp.zoom.us/j/91640855624?pwd=dWVWV2doSnBhUS9JUkhjQVBwY0h0dz09

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$$\left.\frac{dS}{d\lambda}\right|_{\lambda=0} = \frac{1}{2}\int_B T^{\mu\nu}\,\delta g_{\mu\nu}\sqrt{g}\,d^4x$$

(we choose $T^{\mu\nu}$ symmetric because $g_{\mu\nu}$ is symmetric)

- The above is meant when writing:

$$T^{\mu\nu} = \frac{2}{\sqrt{g}}\frac{\delta}{\delta g_{\mu\nu}} S$$

- We found that $T^{\mu\nu}{}_{;\nu} = 0$ always holds.

(Since is consequence of diffeomorphism invariance)

Definition: A space-time (M, g) is called **"stationary"** if it possesses **energy conservation**, i.e., if it possesses a **time-like** Killing vector field, i.e., if it possesses a field ξ which obeys:

$$L_\xi g = 0 \quad \text{and} \quad \xi^\mu \xi_\mu = g(\xi,\xi) < 0$$

(Recall: if $=0$ would be called "null" or light-like; >0 would be called space-like)

⇝ Since ξ is timelike, observers can travel along the integral curves of ξ and set up a coordinate system with their eigen-time as the time coordinate.

In such a "Comoving coordinate system": $\xi = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}$

$\Rightarrow$ $0 = L_\xi g = \xi^\alpha g_{\mu\nu,\alpha} + g_{\alpha\nu}\xi^\alpha{}_{,\mu} + g_{\mu\alpha}\xi^\alpha{}_{,\nu}$ becomes:

$$0 = g_{\mu\nu,0} = \partial_t g_{\mu\nu}, \quad \text{i.e.: } g_{\mu\nu}(x) = \text{constant in time.}$$

$$L_\xi g = 0, \quad \text{i.e., } \xi_{\mu;\nu} = -\xi_{\nu;\mu}$$

Because then: $P^\mu := T^{\mu\nu}\xi_\nu$ obeys $P^\mu{}_{;\mu} = 0$

Thus: $\displaystyle\int_B \underbrace{P^\mu{}_{;\mu}\sqrt{g}\,d^4x}_{\mathrm{div}_\Omega P \,=\, d\, i_P \Omega} \overset{\text{Stokes}}{\underset{(\text{Gauß})}{=}} \underbrace{\int_{\partial B} i_P \Omega = 0}_{\text{a conservation law}}$

Proposition: maximal number of indep. Killing vector fields on spacetime: **10**

Actual spacetime has no Killing vector fields, but realistic simplified models of parts or all of spacetime often do:

$\Rightarrow$ In **"stationary"** spacetimes, one can find a (so-called comoving) coordinate system, in which:

$$\frac{\partial}{\partial x^0} g_{\mu\nu}(x^0, x^1, x^2, x^3) = 0$$

$\Rightarrow$ Comoving observers, i.e., those observers who travel the flow generated by ξ, see no change in the prevailing local curvature.

- **However:** Stationarity does **not** imply that there is a cds in which

$$g = \begin{pmatrix} g_{00}, 0, 0, 0 \\ 0 \quad \\ 0 \quad * \\ 0 \quad \end{pmatrix} \qquad (*)$$

Example: The g of a stationary black hole that is rotating, given by the "Kerr metric".

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⇝ Since ξ is timelike, observers can travel along the integral curves of ξ and set up a coordinate system with their eigentime as the time coordinate.

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$$\Rightarrow 0 = L_\xi g = \xi^\alpha g_{\mu\nu,\alpha} + g_{\alpha\nu}\xi^\alpha{}_{,\mu} + g_{\mu\alpha}\xi^\alpha{}_{,\nu} \quad \text{becomes:}$$

$$0 = g_{\mu\nu,0} = \partial_t g_{\mu\nu} \quad \text{, i.e.: } g_{\mu\nu}(x) = \text{constant in time.}$$

... local curvature.

□ **However:** Stationarity does not imply that there is a cds in which

$$g = \begin{pmatrix} g_{00},0,0,0 \\ 0 \\ 0 & \times \\ 0 \end{pmatrix} \qquad (\times)$$

Example: The g of a stationary black hole that is rotating, given by the "Kerr metric".

⇝

Definition: A space-time is called "static", if the time-like Killing field ξ, viewed as a 1-form,

$$\xi = \xi_\mu dx^\mu$$

also obeys the "Frobenius condition":

$$\xi \wedge d\xi = 0 \qquad (F)$$

Exercise: write it out in coordinates

Significance? (F) holds $\Leftrightarrow g = \begin{pmatrix} g_{00},0,0,0 \\ 0 \\ 0 & (\times) \\ 0 \end{pmatrix}$ in suitable cds.

- If, in a suitable coordinate system, g is of the form $(\times)$ and the time-like ξ is $\xi = g_{00}\,dt$, then $\xi \wedge d\xi = 0$ trivially.
- One can also show that, conversely, (F) implies existence of cds in which $(\times)$ holds.

Generic properties of $T^{\mu\nu}$:

□ $T^{\mu\nu}$ has contributions from known and also from as yet unknown matter fields (e.g., from dark matter).

□ Thus, in order to draw generic conclusions about, e.g.,

a.) the occurrence of singularities, or (Note: black hole formation stops energy dropping)

b.) the overall positivity of the energy (despite universal attraction!)

one needs plausible conjectures about the full $T_{\mu\nu}$:

The "weak energy condition": (assumption)

$$T_{\mu\nu} v^\mu v^\nu \geq 0 \quad \text{for all timelike } v: \; g(v,v) < 0$$

(by continuity it then also holds for lightlike v)

Why assume it?

(Note: Negative energy would be anti-gravitating, i.e. repulsive.) ⇝ All observers travel with a time-like tangent v. They then see a positive local energy density: $T_{\mu\nu}v^\mu v^\nu \geq 0$

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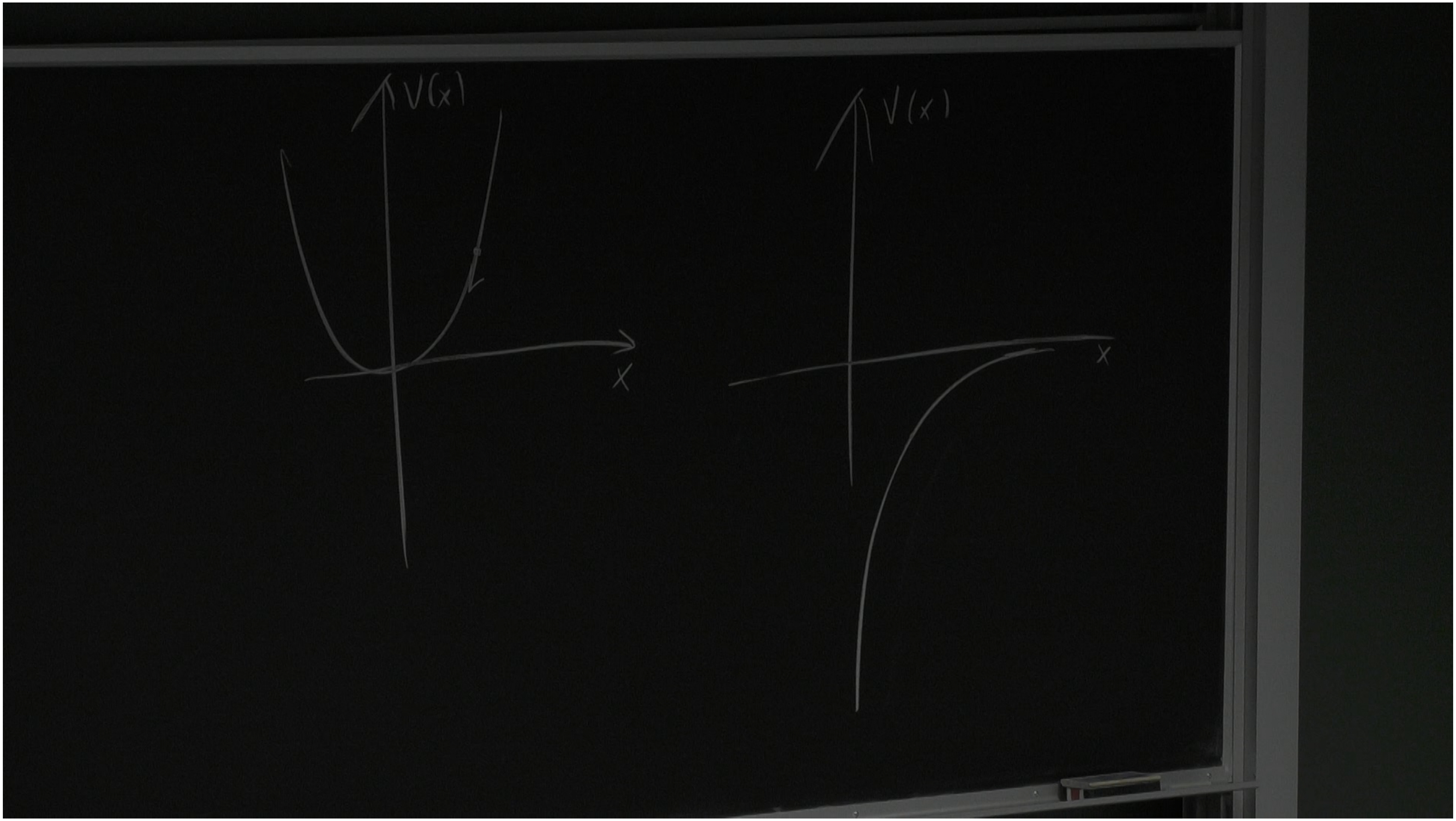
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$\xi \wedge d\xi = 0$ $(\mp)$

Exercise: write it out in coordinates

Significance? $(\mp)$ holds $\Leftrightarrow g = \begin{pmatrix} g_{00}, 0,0,0 \\ 0 \\ \vdots & (*) \\ 0 \end{pmatrix}$ in suitable cds.

- If, in a suitable coordinate system, g is of the form **(x)** and the time-like ξ is $\xi = g_{00} dt$, then $\xi \wedge d\xi = 0$ trivially.
- One can also show that, conversely, $(\mp)$ implies existence of cds in which **(x)** holds.

b.) the overall positivity of the energy (despite universal attraction!) one needs plausible conjectures about the full $T_{\mu\nu}$:

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Why assume it?

(Note: Negative energy would be anti-gravitating i.e. repulsive.) ⇝ All observers travel with a time-like tangent v. They then see a positive local energy density: $T_{\mu\nu}v^\mu v^\nu \geq 0$

The "dominant energy condition":

$T_{\mu\nu} v^\mu v^\nu \geq 0$ for all timelike v (i.e., weak energy condition)

and

$K_\mu := T_{\mu\nu} v^\nu$ obeys $K_\mu K^\mu \leq 0$ (i.e., $T_{\mu\nu} v^\nu$ is non-space-like)

Why assume it?

- The local energy-momentum flow vector, K, may not be conserved but should be non-space-like: "All flow should be into the future."
- In an orthonormal basis, the dominant energy condition takes the form:

$$T^{00} \geq |T^{ab}|$$

i.e. "energy dominates over momentum."

(Note: This is all intuition from fluid mechanics analogy. Quantum fields may or may not behave this way.)

The dynamics of space-time!

- Consider the full matter action:

$$S[g,\psi] = \int_M L(g,\psi)\sqrt{g}\, d^4x$$

all matter fields: e^-, photons, quarks, gluons etc.

- The equations of motion of matter fields are

$$\frac{\delta S}{\delta \psi^{a..b}_{(i)\,c..d}} = 0$$

i.e.:

$$\frac{\partial L}{\partial \psi^{a..b}_{(i)\,c..d}} = \left(\frac{\partial L}{\partial \psi^{a..b}_{(i)\,c..d;e}}\right)_{;e}$$

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- Do we obtain suitable equations of motion for $g_{\mu\nu}$ by setting

$$\frac{\delta S'}{\delta g_{\mu\nu}} = 0 \quad ?$$

Apparently not, because it would mean:

$$\frac{\delta S'}{\delta g_{\mu\nu}} = \frac{1}{2} T^{\mu\nu}\sqrt{g} = 0 \; !$$

Thus, the universe would have to be empty of matter (assuming all matter has positive energy).

- **Andrei Sakharov (1968):**

The quantum effects of matter induce suitable extra terms in the action!

fields: e^-, photons, quarks, gluons etc.

- The equations of motion of matter fields are

$$\frac{\delta S'}{\delta \psi_{(i)\ c..d}^{a..b}} = 0$$

i.e.:

$$\frac{\partial L}{\partial \psi_{(i)\ c..d}^{a..b}} = \left(\frac{\partial L}{\partial \psi_{(i)\ c..d,e}^{a..b}}\right)_{;e}$$

Sakharov's reasoning: (modernized version)

- Classical deterministic evolution of matter obeys:

$$\frac{\delta S}{\delta \psi_{(i)}} = 0$$

- But quantum theory allows every evolution $\psi_{(i)}(x,t)$ to happen "virtually", with probability amplitudes:

a normalization constant.

$$\text{prob.ampl.}[\psi_{(i)}] = N\, e^{\frac{i S'[g,\psi]}{\hbar}}$$

- As usual in quantum theory, the actual or "effective" matter evolution $\langle\psi_{(i)}(x,t)\rangle$ is close

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- But quantum theory allows every evolution $\psi_{(i)}(x,t)$ to happen "virtually", with probability amplitudes:

a normalization constant.

$$\text{prob.ampl.}\left[\psi_{(i)}\right] = N\, e^{\frac{iS'[g,\psi]}{\hbar}}$$

- As usual in quantum theory, the actual or "effective" matter evolution $\langle\psi_{(i)}(x,t)\rangle$ is close to but not identical to the classical matter evolution $\psi_{(i)}(x,t)$.

Why? Path integral picture: The field evolutions with close-to-extremal actions have very similar values $e^{\frac{iS}{\hbar}}$ because for them $\frac{\delta S'}{\delta\psi}\approx 0$ i.e. their prob. amplitudes add up. Other matter evolutions $\psi_{(i)}(x,t)$ have widely different $e^{\frac{iS}{\hbar}}$, so their probabilities average another away.

Heisenberg picture: The (field) operators obey formally the same eqns of motion as do the classical fields. Because of the commutation (and uncertainty) relations, however, the (field) operator expectation values generally do not obey the classical eqns of motion.

- Thus, the effective quantum fields obey equations of motion that are somewhat modified!

$\Rightarrow$ Aim: Calculate the "effective action"

$$S_{eff}[g,\psi]$$

which yields the effective evolution of matter fields when matter quantum effects are taken into account.

- Problems:

  - These calculations are very difficult.

    $\Rightarrow$ Use perturbative methods.

  - There occur integrals that are divergent

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- **But as always in quantum theory**, the effective action will contain terms of all possible forms that are consistent with the symmetries of the theory i.e. here with general covariance (i.e. that are scalars):

The only question is which prefactors these terms will have.

$$S_{eff}[g,\psi] = \int_M \left(L^{matter} + L^{matter}_{quantum\ effects} + c_1 + c_2 R + c_3 O(R^2) \right) \sqrt{g}\, d^4x$$

quantum "vacuum energy" of matter

this is the local change of the vacuum energy due to curvature deforming the quantum harmonic oscillators of the field modes.

- the order of perturbation
- the value of the short-distance cutoff:

The λ_i are unitless numbers that are roughly of order one, depending on the precise matter lagrangian

$c_1 = \lambda_1 \ell_c^{-4}$ (must make up for $[\text{length}]^4$ from d^4x)

$c_2 = \lambda_2 \ell_c^{-2}$ (because R has units $[\text{length}]^{-2}$)

$c_3 = \lambda_3 \ell_c^{0}$ (terms R^2 or $R^{\mu\nu}R_{\mu\nu}$ etc have units $[\text{length}]^{-4}$)

$c_4 = \lambda_4 \ell_c^{2}$ (higher powers in R have prefactors $\sim \ell_c^{\text{positive power}}$)

- $\Rightarrow$ For small ℓ_c, we have:

$$c_1 \gg c_2 \gg c_3 \gg c_4 \gg \dots$$

- Consider the lowest order terms:

$$S_g'[g,\psi] = \int_M \left(L + c_1 + c_2 R \right) \sqrt{g}\, d^4x$$

total effective matter lagrangian; $c_1 = -\Lambda$ cosmological constant; R: Einstein action

and postulate now that the equations of motion for the metric follow from the action principle:

$$\frac{\delta S_g[g,\psi]}{\delta g_{\mu\nu}} = 0$$

- Einstein had postulated the same action principle!

We note that:

Every (effective) quantum field theory with minimum length induces Einstein gravity.

See, e.g., review: gr-qc/0204062

The equations of motion for g:

The action principle, $\frac{\delta S_{eff}}{\delta g_{\mu\nu}} = 0$, yields:

$$\frac{\delta}{\delta g_{\mu\nu}} \int_B \left(c_1 + c_2 R_{\mu\nu} g^{\mu\nu} \right) \sqrt{g}\, d^4x = -\frac{1}{2}\sqrt{g}\, T^{\mu\nu}$$

in principle, it is the effective quantum expectation value

Evaluate the left hand side:

a) $$\delta \int_B c_1 \sqrt{g}\, d^4x = \int_B c_1 \frac{\delta \sqrt{g}}{\delta g_{\mu\nu}} \delta g_{\mu\nu}\, d^4x$$

recall: $\frac{\delta\sqrt{g}}{\delta g_{\mu\nu}} = \frac{1}{2} g^{\mu\nu}\sqrt{g}$

$$= \int_B c_1 \frac{1}{2} g^{\mu\nu} \sqrt{g}\, \delta g_{\mu\nu}\, d^4x$$

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Evaluate the left hand side:

a) $$\delta \int_B c_1 \sqrt{g}\, d^4x = \int_B c_1 \overbrace{\frac{\delta\sqrt{g}}{\delta g_{\mu\nu}}}^{\text{recall: } = \frac{1}{2} g^{\mu\nu}\sqrt{g}} \delta g_{\mu\nu}\, d^4x$$

$$= \int_B c_1 \frac{1}{2} g^{\mu\nu} \sqrt{g}\, \delta g_{\mu\nu}\, d^4x$$

b) $$\delta \int_B c_2 R_{\mu\nu} g^{\mu\nu} \sqrt{g}\, d^4x$$

$$= \underbrace{\int_B c_2 (\delta R_{\mu\nu}) g^{\mu\nu} \sqrt{g}\, d^4x}_{\text{Term I}} + \underbrace{\int_B c_2 R_{\mu\nu}\, \delta(g^{\mu\nu}\sqrt{g})\, d^4x}_{\text{Term II}}$$

Proposition: Term I = 0

Proof: Choose origin of geodesic coordinate system

$$R_{\mu\nu} = \Gamma^{\alpha}{}_{\mu\nu,\alpha} - \Gamma^{\alpha}{}_{\mu\alpha,\nu} + \underbrace{\Gamma^{\cdot}_{\cdot\cdot}\Gamma^{\cdot}_{\cdot\cdot} + \Gamma^{\cdot}_{\cdot\cdot}\Gamma^{\cdot}_{\cdot\cdot}}_{\text{vanish at origin because } \Gamma = 0}$$

Thus: $$\delta R_{\mu\nu} = (\delta\Gamma^{\alpha}{}_{\mu\nu})_{,\alpha} - (\delta\Gamma^{\alpha}{}_{\mu\alpha})_{,\nu}$$

$$\Rightarrow \quad g^{\mu\nu}\delta R_{\mu\nu} = g^{\mu\nu}(\delta\Gamma^{\alpha}{}_{\mu\nu})_{,\alpha} - g^{\mu\nu}(\delta\Gamma^{\alpha}{}_{\mu\alpha})_{,\nu}$$

$$= w^{\alpha}{}_{,\alpha} \quad \text{for} \quad w^{\alpha} = g^{\mu\nu}\delta\Gamma^{\alpha}_{\mu\nu} - g^{\mu\alpha}\delta\Gamma^{\nu}_{\mu\nu}$$

recall: $g^{\mu\nu}{}_{;\alpha} = 0$ here.

Thus, in arbitrary coordinate system:

$$g^{\mu\nu}\delta R_{\mu\nu} = w^{\alpha}{}_{;\alpha}$$

$$\Rightarrow \int_B c_2 g^{\mu\nu}\delta R_{\mu\nu} \sqrt{g}\, d^4x = \int_B w^{\alpha}{}_{;\alpha} \sqrt{g}\, d^4x$$

$$= \int_B \mathrm{div}_\omega \Omega$$

$$\overset{\text{Gauss}}{=} \int_{\partial B} i_{\omega} \Omega$$

$= 0$ on ∂B, assuming $\delta g_{\mu\nu}$ and $\delta g_{\mu\nu,\sigma} = 0$ on ∂B.

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See, e.g., review: gr-qc/0204062

b) $\delta \int_B c_2 R_{\mu\nu} g^{\mu\nu} \sqrt{g}\, d^4x$

$$= \underbrace{\int_B c_2 (\delta R_{\mu\nu}) g^{\mu\nu} \sqrt{g}\, d^4x}_{\text{Term I}} + \underbrace{\int_B c_2 R_{\mu\nu} \delta(g^{\mu\nu}\sqrt{g})\, d^4x}_{\text{Term II}}$$

<u>Proposition</u>: Term I $= 0$

<u>Proof</u>: Choose origin of geodesic coordinate system

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Thus: $\delta R_{\mu\nu} = (\delta\Gamma^\alpha{}_{\mu\nu})_{,\alpha} - (\delta\Gamma^\alpha{}_{\mu\alpha})_{,\nu}$

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$$= w^\alpha{}_{,\alpha} \text{ for } w^\alpha = g^{\mu\nu}\delta\Gamma^\alpha{}_{\mu\nu} - g^{\mu\alpha}\delta\Gamma^\nu{}_{\mu\nu}$$

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Thus, in arbitrary coordinate system:

$$g^{\mu\nu}\delta R_{\mu\nu} = w^\alpha{}_{;\alpha}$$

$$\Rightarrow \int_B c_2 g^{\mu\nu}\delta R_{\mu\nu}\sqrt{g}\, d^4x = \int_B w^\alpha{}_{;\alpha}\sqrt{g}\, d^4x$$

$$= \int_B \mathrm{div}_w \Omega$$

$$\overset{\text{Gauß}}{=} \int_{\partial B} i_w \Omega$$

$= 0$ on ∂B, assuming $\delta g_{\cdot\cdot}$ and $\delta g_{\mu\nu,\beta} = 0$ on ∂B.

$$= 0$$

c) Evaluate term II:

$$\int_B c_2 R_{\mu\nu} \delta(g^{\mu\nu}\sqrt{g})\, d^4x = \,?$$

We have:

$$\delta(g^{\mu\nu}\sqrt{g}) = (\delta g^{\mu\nu})\sqrt{g} + g^{\mu\nu}\frac{\partial\sqrt{g}}{\partial g_{ab}}\delta g_{ab}$$

$$\delta \int_B (c_1 + c_2 R_{\mu\nu} g^{\mu\nu})\sqrt{g}\, d^4x$$

$$= \int_B \underbrace{\left(c_1 \tfrac{1}{2} g^{\mu\nu} - c_2 G^{\mu\nu}\right)}_{\substack{\text{as in the case of the } T^{\mu\nu} \text{ calculation,} \\ \text{one could add an antisymmetric part} \\ \text{here and it would drop from the integrand.}}} \sqrt{g}\, \underbrace{\delta g_{ab}}_{\text{symmetric}}\, d^4x$$

$$\Rightarrow \quad \delta \int \ldots$$

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Proposition: Term I $= 0$

Proof: Choose origin of geodesic coordinate system

$$R_{\mu\nu} = \Gamma^{\alpha}{}_{\mu\nu,\alpha} - \Gamma^{\alpha}{}_{\mu\alpha,\nu} + \underbrace{\Gamma^{\cdot}_{\cdot\cdot}\Gamma^{\cdot}_{\cdot\cdot} + \Gamma^{\cdot}_{\cdot\cdot}\Gamma^{\cdot}_{\cdot\cdot}}_{\text{vanish at origin because } \Gamma = 0}$$

Thus:

$$\delta R_{\mu\nu} = (\delta\Gamma^{\alpha}{}_{\mu\nu})_{,\alpha} - (\delta\Gamma^{\alpha}{}_{\mu\alpha})_{,\nu}$$

$$\Rightarrow \int_B c_2\, g^{\mu\nu}\delta R_{\mu\nu}\sqrt{g}\, d^4x = \int_B w^{\alpha}{}_{;\alpha}\sqrt{g}\, d^4x$$

$$= \int_B \operatorname{div}_w \Omega$$

$$\overset{\text{Gauss}}{=} \int_{\partial B} i_w \Omega$$

$= 0$ on ∂B, assuming $\delta g_{\cdot\cdot}$ and $\delta g_{\mu\nu,\beta} = 0$ on ∂B.

$$= 0$$

c) Evaluate term II:

$$\int_B c_2 R_{\mu\nu}\,\delta(g^{\mu\nu}\sqrt{g})\, d^4x = \ ?$$

We have:

$$\delta(g^{\mu\nu}\sqrt{g}) = (\delta g^{\mu\nu})\sqrt{g} + g^{\mu\nu}\frac{\partial\sqrt{g}}{\partial g_{ab}}\delta g_{ab}$$

$$= -g^{\mu a}g^{\nu b}\delta g_{ab}\sqrt{g} + g^{\mu\nu}\tfrac{1}{2}g^{ab}\sqrt{g}\,\delta g_{ab}$$

$\Rightarrow$

$$\int_B c_2 R_{\mu\nu}\delta(g^{\mu\nu}\sqrt{g})\, d^4x = -\int_B c_2 \underbrace{\left(+R^{ab} - \tfrac{1}{2}g^{ab}R\right)}_{\text{recall: } = G^{ab}}\sqrt{g}\,\delta g_{ab}\, d^4x$$

"Einstein tensor"

(up to anti-symm. components, which we set to zero)

Bringing together a) + b) + c) $\Rightarrow$

$$\delta \int_B (c_1 + c_2 R_{\mu\nu}g^{\mu\nu})\sqrt{g}\, d^4x$$

$$= \int_B \underbrace{\left(c_1 \tfrac{1}{2}g^{\mu\nu} - c_2 G^{\mu\nu}\right)}_{\text{as in the case of the } T^{\mu\nu} \text{ calculation, one could add an antisymmetric part here and it would drop from the integrand.}}\sqrt{g}\,\underbrace{\delta g_{ab}}_{\text{symmetric}}\, d^4x$$

$$\Rightarrow \frac{\delta}{\delta g_{\mu\nu}}\int_B (c_1 + c_2 R_{\mu\nu}g^{\mu\nu})\sqrt{g}\, d^4x$$

$$= \left(c_1 \tfrac{1}{2}g^{\mu\nu} - c_2 G^{\mu\nu}\right)\sqrt{g}$$

Remark: Choosing to add an antisymmetric part would make $G^{\mu\nu}$ and $R^{\mu\nu}$ non-symmetric, in violation of $g_{\mu\nu;\beta} = 0$.

(Notice: the "reduced Bianchi identity" $(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R)_{;\nu} = 0$ ...)

Notice: The symmetry of R ...

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$$\delta(g^{\mu\nu}\sqrt{g}) = (\delta g^{\mu\nu})\sqrt{g} + g^{\mu\nu}\frac{\partial\sqrt{g}}{\partial g_{ab}}\delta g_{ab}$$

$$= -g^{\mu a}g^{\nu b}\delta g_{ab}\sqrt{g} + g^{\mu\nu}\frac{1}{2}g^{ab}\sqrt{g}\,\delta g_{ab}$$

$\Rightarrow$

$$\int_B c_2 R_{\mu\nu}\,\delta(g^{\mu\nu}\sqrt{g})\,d^4x = -\int_B c_2 \underbrace{\left(+R^{ab} - \frac{1}{2}g^{ab}R\right)}_{\overset{\text{recall}}{=} G^{ab}} \sqrt{g}\,\delta g_{ab}\,d^4x$$

"Einstein tensor"

Bringing together a) + b) + c) $\Rightarrow$

have and Λ would drop from the integrand.

$$\Rightarrow \quad \frac{\delta}{\delta g_{\mu\nu}}\int_B \left(c_1 + c_2 R_{\mu\nu}g^{\mu\nu}\right)\sqrt{g}\,d^4x$$

(up to anti-sym. components, which we set to zero)

$$= \left(c_1\frac{1}{2}g^{\mu\nu} - c_2 G^{\mu\nu}\right)\sqrt{g}$$

Remark: Choosing to add an antisymmetric part would make $G^{\mu\nu}$ and $R^{\mu\nu}$ non-symmetric, in violation of $g_{\mu\nu;s} = 0$.

Finally, we conclude:

$$\frac{\delta S'_{tot}}{\delta g_{\mu\nu}} = 0$$

leads to this equation of motion for g:

$$\left(\frac{1}{2}c_1 g^{\mu\nu} - c_2 G^{\mu\nu}\right)\sqrt{g} = -\frac{1}{2}\sqrt{g}\,T^{\mu\nu}$$

i.e.:

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R - \frac{c_1}{2c_2}g^{\mu\nu} = \frac{1}{2c_2}T^{\mu\nu}$$

As is well-known, comparison with experiment requires:

$$c_2 = \frac{1}{16\pi G}$$

Newton's constant.

$\Rightarrow$ Einstein equation:

(Notice: the "reduced Bianchi identity" $(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R)_{;\nu} = 0$ now is equivalent to the statement $T^{\mu\nu}{}_{;\nu} = 0$)

Notice: The symmetry of $R^{\mu\nu}$ (due to $g_{\mu\nu;s} = 0$) enforces the symmetry of $T^{\mu\nu}$.

$$\boxed{R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R \underbrace{- 8\pi G c_1}_{= \Lambda \text{ "cosmol. constant"}} g^{\mu\nu} = 8\pi G\,T^{\mu\nu}}$$

What is Λ?

Recall: The λ_i are constants prefactor of order one which depends on the matter Lagrangian.

First find ℓ_c:

(I.e. also: $G \approx \ell_c^2$, which we'll need on the next slide)

Given that $\frac{1}{16\pi G} = c_2 = \lambda_2 \ell_c^{-2}$

undoing $\hbar = 1$ and $c = 1$ so that get $1/g$ m, s:

we obtain: $\ell_c = \lambda_2\sqrt{16\pi G} = \sqrt{16\pi\frac{\hbar G}{c}}$

$$= \lambda_2 \cdot 4\sqrt{\pi}\cdot \underbrace{1.616\times 10^{-35}\,m}_{\text{the "Planck length"}}$$

Recall: The λ_i are some numbers of order $O(1)$, depending on details which sort of particles are in the matter action, and their quantum effects

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$$\left(\tfrac{1}{2} c_1 g^{\mu\nu} - c_2 G^{\mu\nu}\right)\sqrt{g} = -\tfrac{1}{2}\sqrt{g}\, T^{\mu\nu}$$

i.e.: $$R^{\mu\nu} - \tfrac{1}{2} g^{\mu\nu} R - \frac{c_1}{2c_2} g^{\mu\nu} = \frac{1}{2c_2} T^{\mu\nu}$$

As is well-known, comparison with experiment requires:

$$c_2 = \frac{1}{16\pi G}$$

Newton's constant.

What is Λ?

Recall: The λ_i are constants prefactor of order one which depends on the matter Lagrangian.

First find ℓ_c:

(I.e. also: $G \approx \ell_c^2$, which we'll need on the next slide)

Given that $\frac{1}{16\pi G} = c_2 = \lambda_2 \ell_c^{-2}$

we obtain: $$\ell_c = \lambda_2 \sqrt{16\pi G} = \sqrt{16\pi \frac{\hbar G}{c}}$$

undoing $\hbar = 1$ and $c = 1$ so that get kg, m, s:

$$= \lambda_2 \cdot 4\sqrt{\pi} \cdot \underbrace{1.616 \times 10^{-35}\,m}_{\text{the "Planck length"}}$$

Recall: The λ_i are some numbers of order $O(1)$, depending on details which sort of particles are in the matter action, and their quantum effects

We note, therefore: When quantum field theories are cut off at about the Planck length they induce gravity with the correct eqns of motion and coupling strength!

Now find c_1:

Given that $\ell_c \approx 10^{-35}\,m$, quantum field theories generate a value of c_1, i.e., a cosmological constant of about:

$$c_1 = \lambda_1 \ell_c^{-4}$$

of $O(1)$

Finally, find Λ:

$$\Lambda_{theory} = -8\pi G c_1 = -8\pi G \lambda_1 \ell_c^{-4}$$
$$= G \ell_c^{-4} \approx \ell_c^{-2} \quad (\text{using } \lambda_1 \approx 1 \text{ and } G \approx \ell_c^2)$$
$$\approx 10^{70}\, m^{-2}$$

This is the worst physical prediction ever:

Experiment:

Based on cosmic microwave background data and on supernova brightness versus redshift data:

$$\Lambda_{experiment} \approx 10^{-52}\, m^{-2} \quad \text{i.e. } \frac{\Lambda_{th}}{\Lambda_{exp}} \approx 10^{122}$$

$\rightsquigarrow$ For some unknown reason, the constant part, c_1, of the vacuum energy of quantum field theories does essentially not gravitate – while its disturbance through curvature, $c_2 R$, is real: it induces regular gravity.

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