Title: General Relativity for Cosmology Lecture - 102623

Speakers: Achim Kempf

Collection: General Relativity for Cosmology

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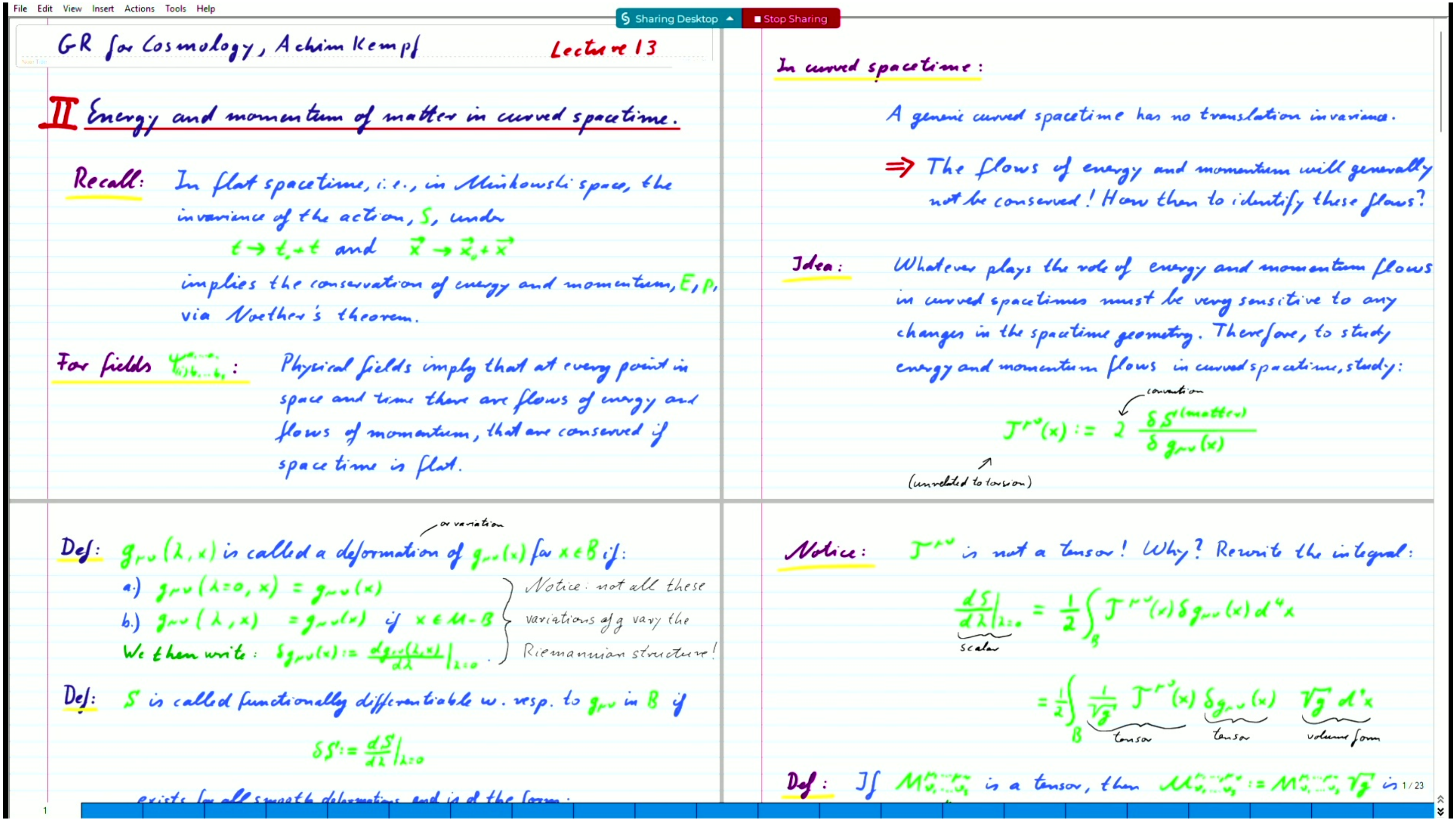
Abstract: Zoom: https://pitp.zoom.us/j/91640855624?pwd=dWVWV2doSnBhUS9JUkhjQVBwY0h0dz09



GR for Cosmology, Achim Kempf

Lecture 13

II Energy and momentum of matter in curved spacetime.

Recall: In flat spacetime, i.e., in Minkowski space, the invariance of the action, S, under

$$t \to t_0 + t \quad \text{and} \quad \vec{x} \to \vec{x}_0 + \vec{x}$$

implies the conservation of energy and momentum, E, p, via Noether's theorem.

For fields $\phi^{\mu_1 \ldots \mu_r}_{\nu_1 \ldots \nu_s}$: Physical fields imply that at every point in space and time there are flows of energy and flows of momentum, that are conserved if spacetime is flat.

In curved spacetime:

A generic curved spacetime has no translation invariance.

⇒ The flows of energy and momentum will generally not be conserved! How then to identify these flows?

Idea: Whatever plays the role of energy and momentum flows in curved spacetimes must be very sensitive to any changes in the spacetime geometry. Therefore, to study energy and momentum flows in curved spacetime, study:

$$T^{\mu\nu}(x) := \overset{\text{convention}}{2}\, \frac{\delta S^{(\text{matter})}}{\delta g_{\mu\nu}(x)}$$

(unrelated to torsion)

or variation

Def: $g_{\mu\nu}(\lambda, x)$ is called a deformation of $g_{\mu\nu}(x)$ for $x \in B$ if:

a.) $g_{\mu\nu}(\lambda = 0, x) = g_{\mu\nu}(x)$

b.) $g_{\mu\nu}(\lambda, x) = g_{\mu\nu}(x)$ if $x \in \mathcal{M} - B$

We then write: $\delta g_{\mu\nu}(x) := \left.\frac{d g_{\mu\nu}(\lambda, x)}{d\lambda}\right|_{\lambda=0}$.

Notice: not all these variations of g vary the Riemannian structure!

Def: S is called functionally differentiable w. resp. to $g_{\mu\nu}$ in B if

$$\delta S := \left.\frac{dS}{d\lambda}\right|_{\lambda=0}$$

exists for all smooth deformations and is of the form:

Notice: $T^{\mu\nu}$ is not a tensor! Why? Rewrite the integral:

$$\underbrace{\left.\frac{dS}{d\lambda}\right|_{\lambda=0}}_{\text{scalar}} = \frac{1}{2}\int_B T^{\mu\nu}(x)\, \delta g_{\mu\nu}(x)\, d^4x$$

$$= \frac{1}{2}\int_B \underbrace{\frac{1}{\sqrt{g}}\, T^{\mu\nu}(x)}_{\text{tensor}}\ \underbrace{\delta g_{\mu\nu}(x)}_{\text{tensor}}\ \underbrace{\sqrt{g}\, d^4x}_{\text{volume form}}$$

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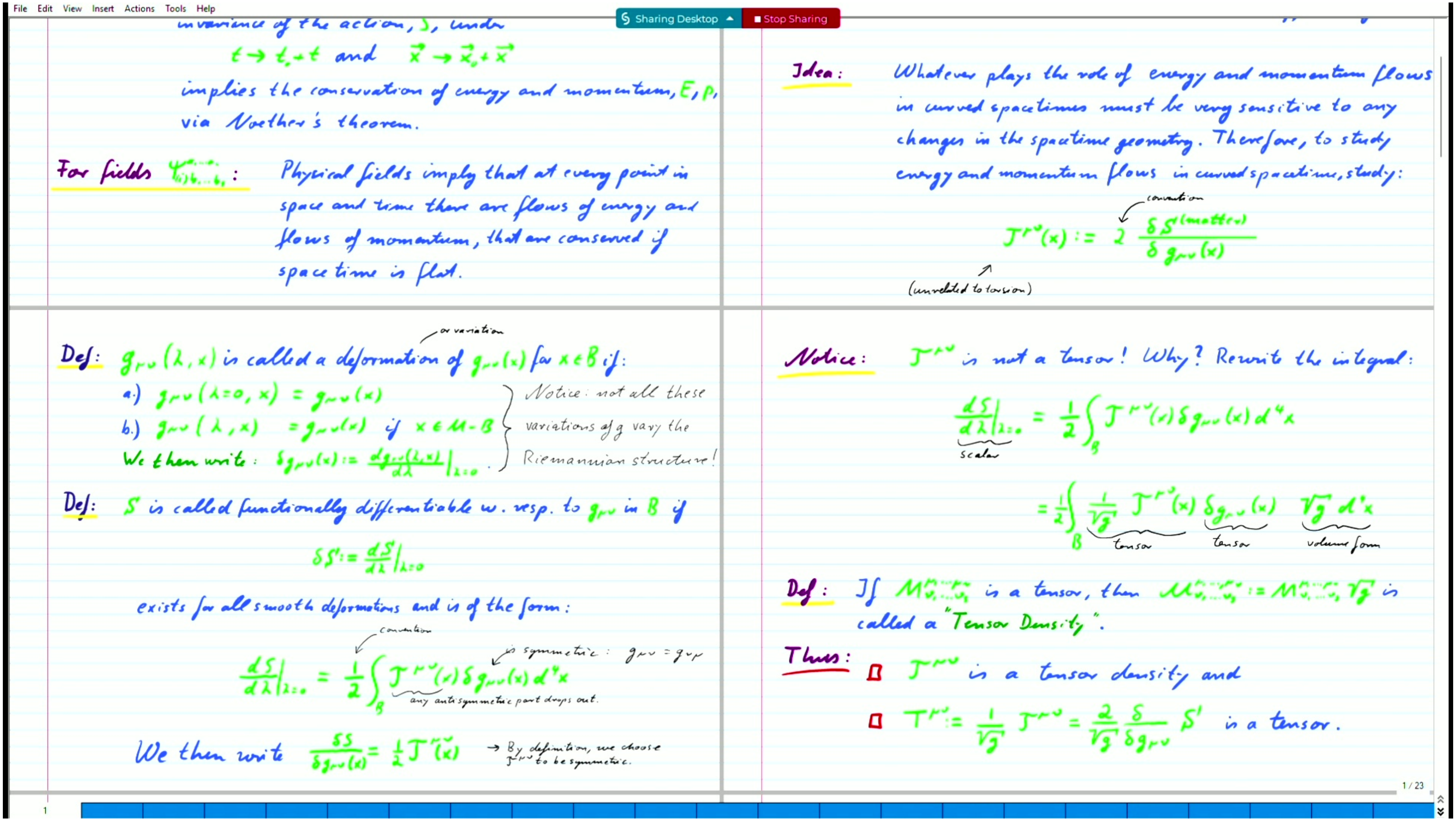

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is symmetric: $g_{\mu\nu} = g_{\nu\mu}$

We then write $\frac{\delta S}{\delta g_{\mu\nu}(x)} = \frac{1}{2} T^{\mu\nu}(x)$ → By definition, we choose $T^{\mu\nu}$ to be symmetric.

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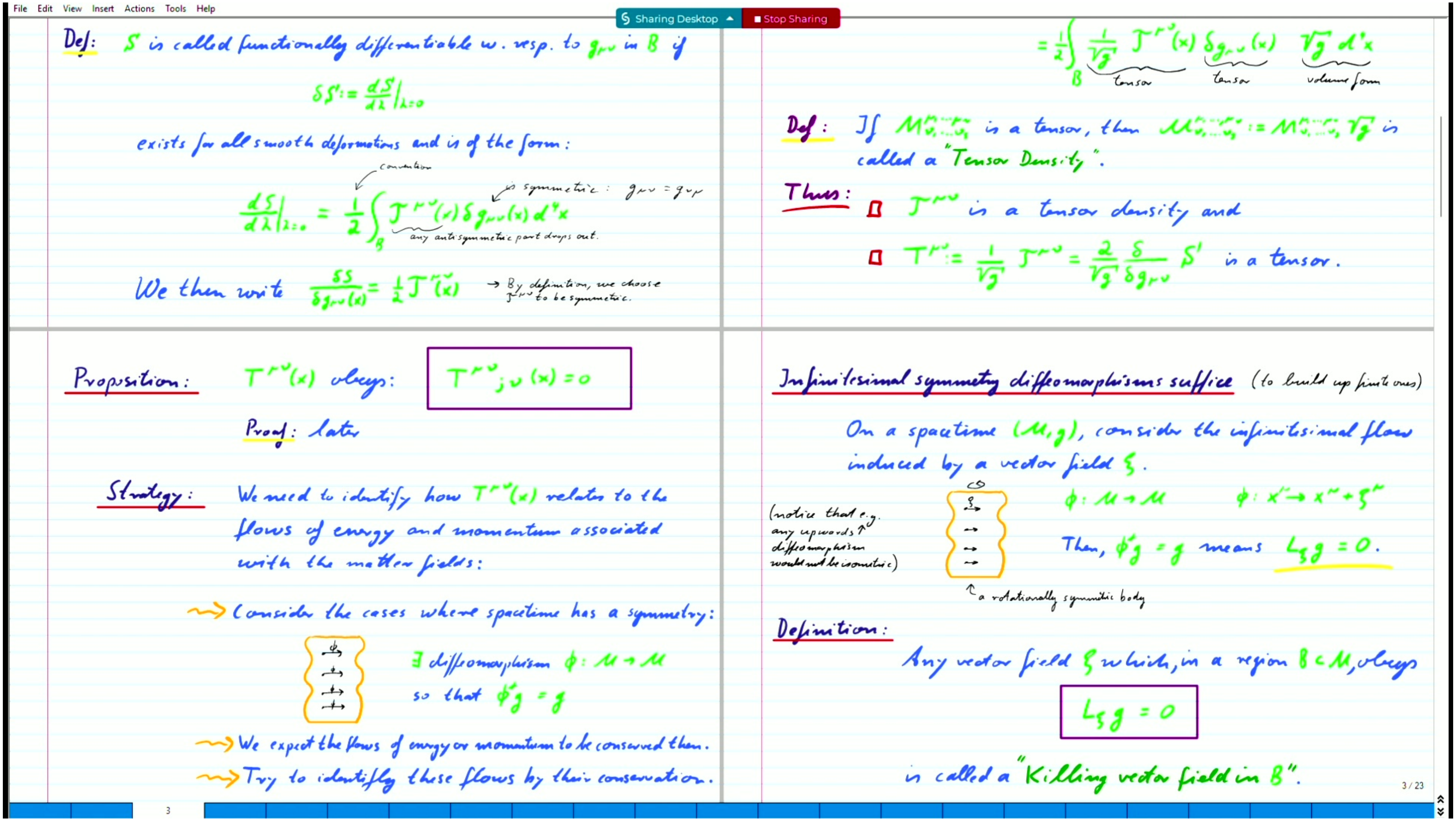
Thus:

- ☐ $T^{\mu\nu}$ is a tensor density and
- ☐ $T^{\mu\nu} := \frac{1}{\sqrt{g}} T^{\mu\nu} = \frac{2}{\sqrt{g}} \frac{\delta}{\delta g_{\mu\nu}} S$ is a tensor.



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- ☐ $T^{\mu\nu} := \frac{1}{\sqrt{g}} J^{\mu\nu} = \frac{2}{\sqrt{g}} \frac{\delta}{\delta g_{\mu\nu}} S'$ is a tensor.

Proposition: $T^{\mu\nu}(x)$ obeys: $\boxed{T^{\mu\nu}{}_{;\nu}(x) = 0}$

Proof: later

Strategy: We need to identify how $T^{\mu\nu}(x)$ relates to the flows of energy and momentum associated with the matter fields:

⇝ Consider the cases where spacetime has a symmetry:

$\exists$ diffeomorphism $\phi: \mathcal{M} \to \mathcal{M}$ so that $\phi^* g = g$

⇝ We expect the flows of energy or momentum to be conserved then.

⇝ Try to identify these flows by their conservation.

Infinitesimal symmetry diffeomorphisms suffice (to build up finite ones)

On a spacetime $(\mathcal{M}, g)$, consider the infinitesimal flow induced by a vector field ξ.

(notice that e.g. any upwards ↑ diffeomorphism would not be isometric)

$\phi: \mathcal{M} \to \mathcal{M}$ $\qquad \phi: x^\mu \to x^\mu + \xi^\mu$

Then, $\phi^* g = g$ means $L_\xi g = 0$.

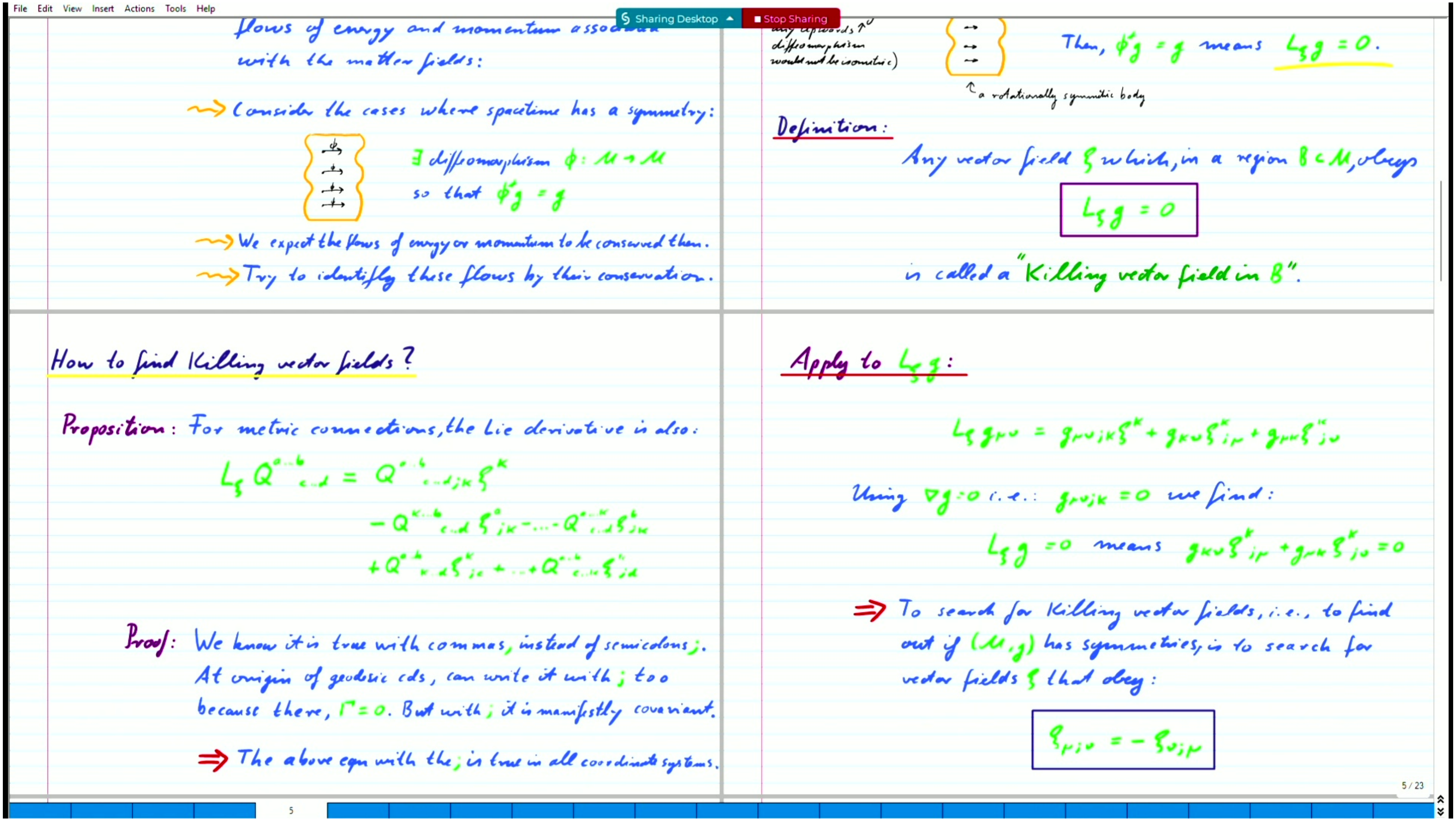
↑ a rotationally symmetric body

Definition:

Any vector field ξ which, in a region $B \subset \mathcal{M}$, obeys

$$\boxed{L_\xi g = 0}$$

is called a "Killing vector field in B".



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How to find Killing vector fields?

Proposition: For metric connections, the Lie derivative is also:

$$L_\xi Q^{a..b}{}_{c..d} = Q^{a..b}{}_{c..d;k}\,\xi^k - Q^{k..b}{}_{c..d}\,\xi^a{}_{;k} - \dots - Q^{a..k}{}_{c..d}\,\xi^b{}_{;k} + Q^{a..b}{}_{k..d}\,\xi^k{}_{;c} + \dots + Q^{a..b}{}_{c..k}\,\xi^k{}_{;d}$$

Proof: We know it is true with commas, instead of semicolons ;. At origin of geodesic cds, can write it with ; too because there, $\Gamma = 0$. But with ; it is manifestly covariant.

⇒ The above eqn with the ; is true in all coordinate systems.

Apply to $L_\xi g$:

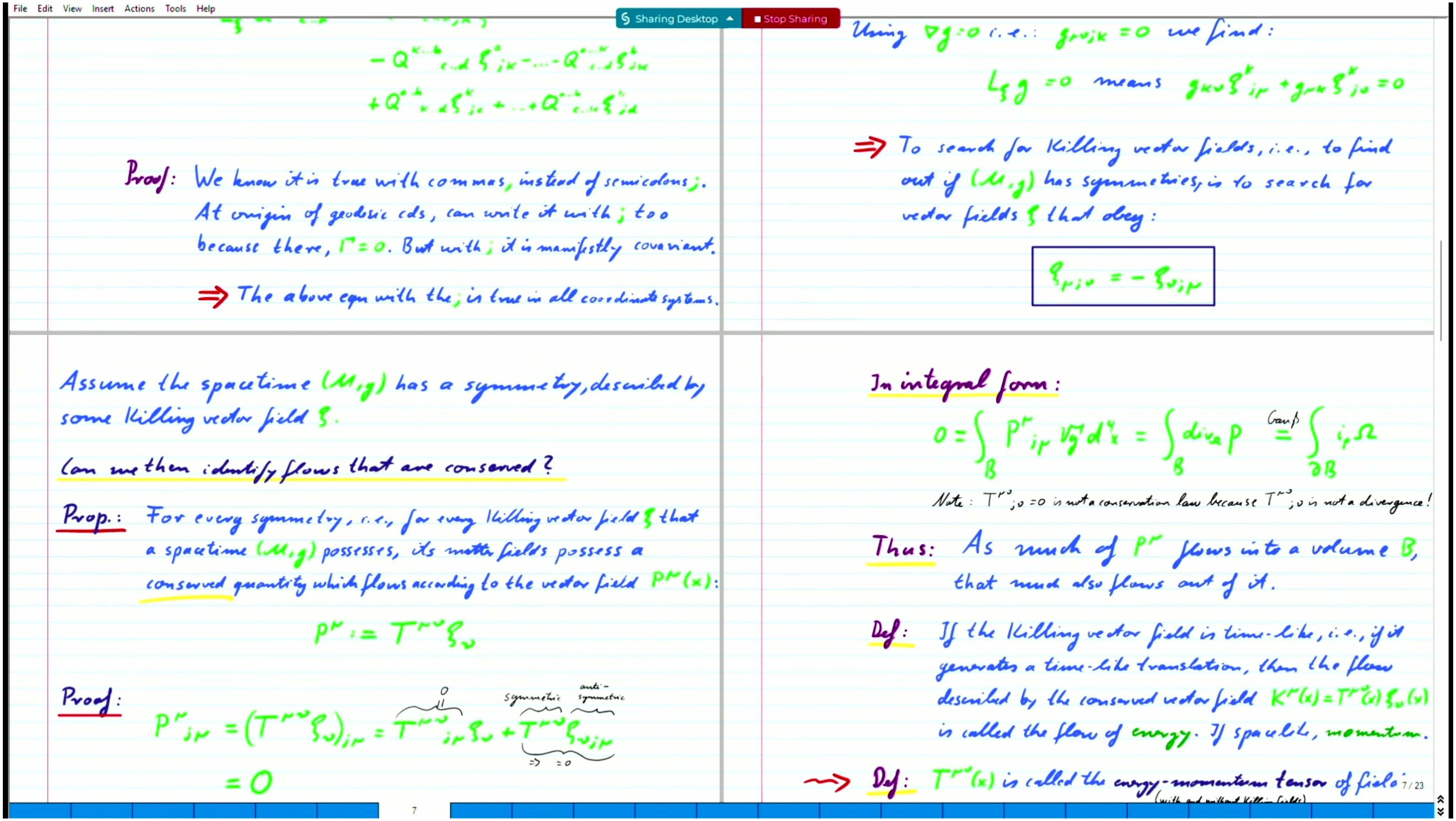
$$L_\xi g_{\mu\nu} = g_{\mu\nu;k}\,\xi^k + g_{k\nu}\,\xi^k{}_{;\mu} + g_{\mu k}\,\xi^k{}_{;\nu}$$

Using $\nabla g = 0$ i.e.: $g_{\mu\nu;k} = 0$ we find:

$L_\xi g = 0$ means $g_{k\nu}\,\xi^k{}_{;\mu} + g_{\mu k}\,\xi^k{}_{;\nu} = 0$

⇒ To search for Killing vector fields, i.e., to find out if (M, g) has symmetries, is to search for vector fields ξ that obey:

$$\boxed{\xi_{\mu;\nu} = -\xi_{\nu;\mu}}$$



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In integral form:

$$0 = \int_B P^\mu{}_{;\mu}\sqrt{g}\,d^4x = \int_B \mathrm{div}_\Omega P \overset{\text{Gauß}}{=} \int_{\partial B} i_P\,\Omega$$

Note: $T^{\mu\nu}{}_{;\nu} = 0$ is not a conservation law because $T^{\mu\nu}{}_{;\nu}$ is not a divergence!

Thus: As much of P^μ flows into a volume B, that much also flows out of it.

Def: If the Killing vector field is time-like, i.e., if it generates a time-like translation, then the flow described by the conserved vector field $K^\mu(x) = T^{\mu\nu}(x)\,\xi_\nu(x)$ is called the flow of energy. If spacelike, momentum.

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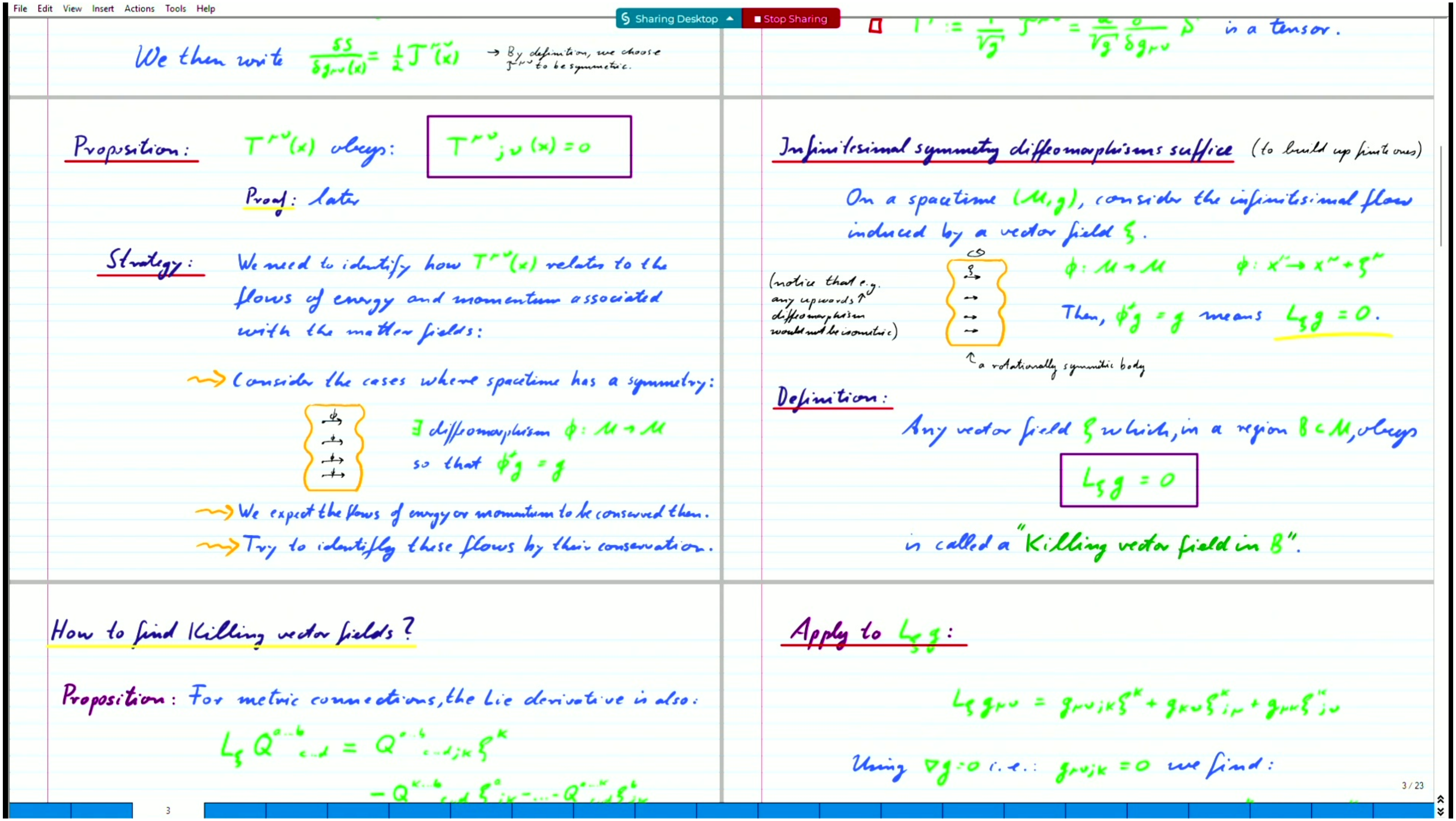

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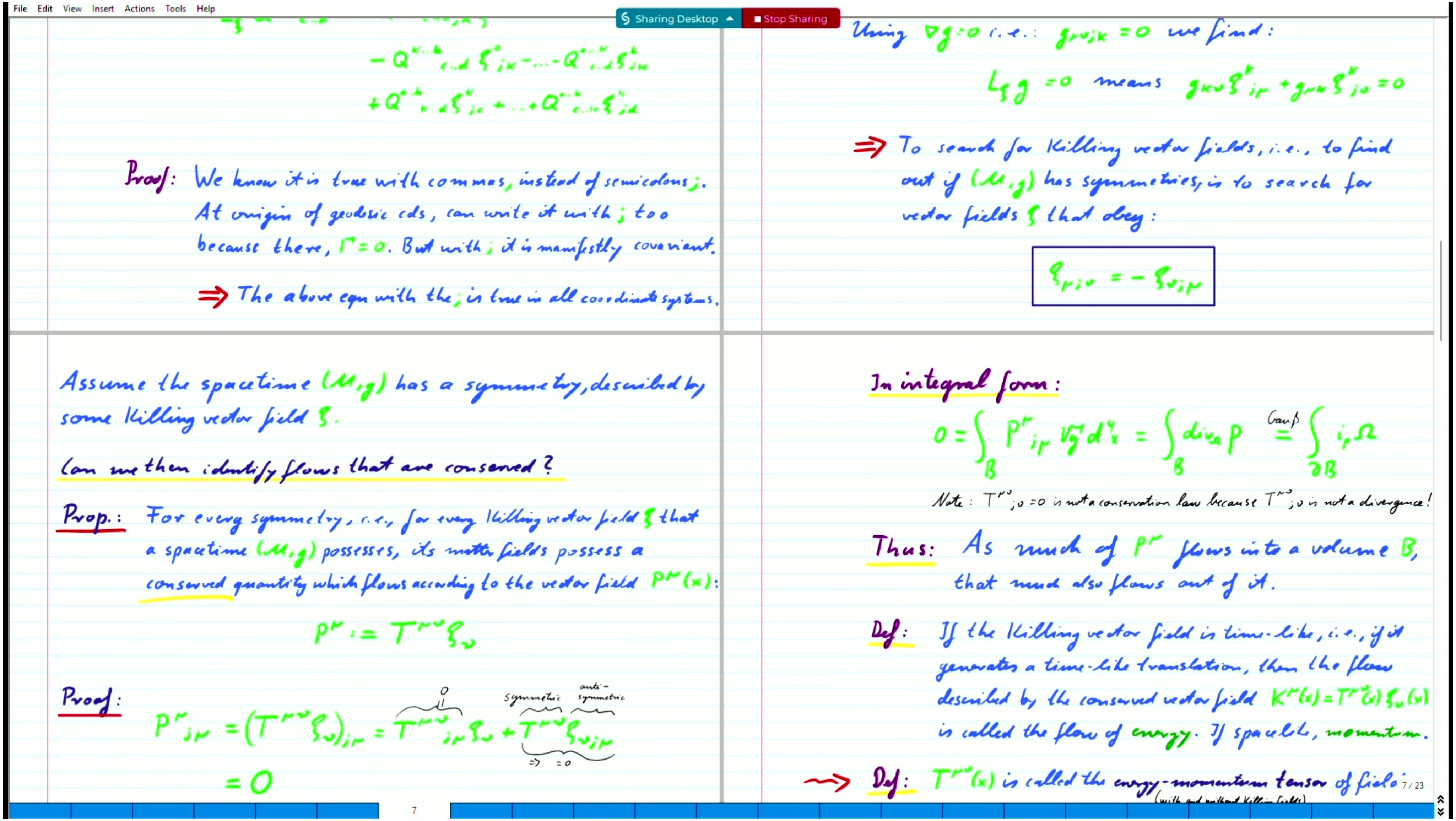

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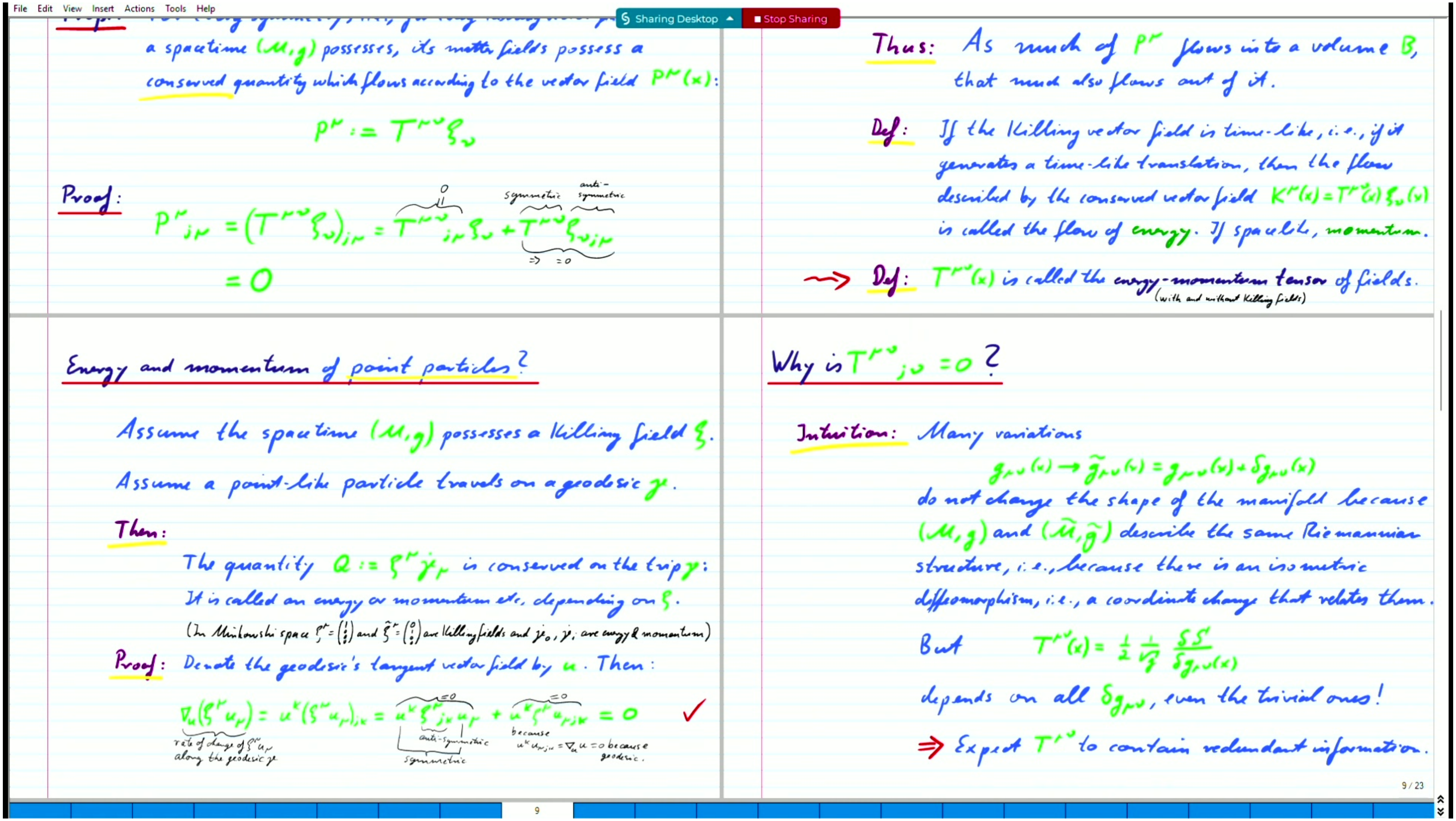

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Energy and momentum of point particles?

Assume the spacetime $(\mathcal{M},g)$ possesses a Killing field ξ.

Assume a point-like particle travels on a geodesic γ.

Then:

The quantity $Q := \xi^\mu\dot\gamma_\mu$ is conserved on the trip γ:
It is called an energy or momentum etc, depending on ξ.
(In Minkowski space $\xi^\mu = \begin{pmatrix}1\\ \vdots\\ \vdots\end{pmatrix}$ and $\tilde\xi^\mu = \begin{pmatrix}0\\ \vdots\\ \vdots\end{pmatrix}$ are Killing fields and $\dot\gamma_0, \dot\gamma_i$ are energy & momentum)

Proof: Denote the geodesic's tangent vector field by u. Then:

$$\underbrace{\nabla_u(\xi^\mu u_\mu)}_{\substack{\text{rate of change of } \xi^\mu u_\mu\\ \text{along the geodesic } \gamma}} = u^\kappa(\xi^\mu u_\mu)_{;\kappa} = \underbrace{\overbrace{u^\kappa \underbrace{\xi^\mu{}_{;\kappa}}_{\text{anti-symmetric}} u_\mu}^{=0}}_{\text{symmetric}} + \underbrace{\overbrace{u^\kappa\xi^\mu u_{\mu;\kappa}}^{=0}}_{\substack{\text{because}\\ u^\kappa u_{\mu;\kappa} = \nabla_u u = 0 \text{ because}\\ \text{geodesic.}}} = 0 \quad ✓$$

Why is $T^{\mu\nu}{}_{;\nu} = 0$?

Intuition: Many variations

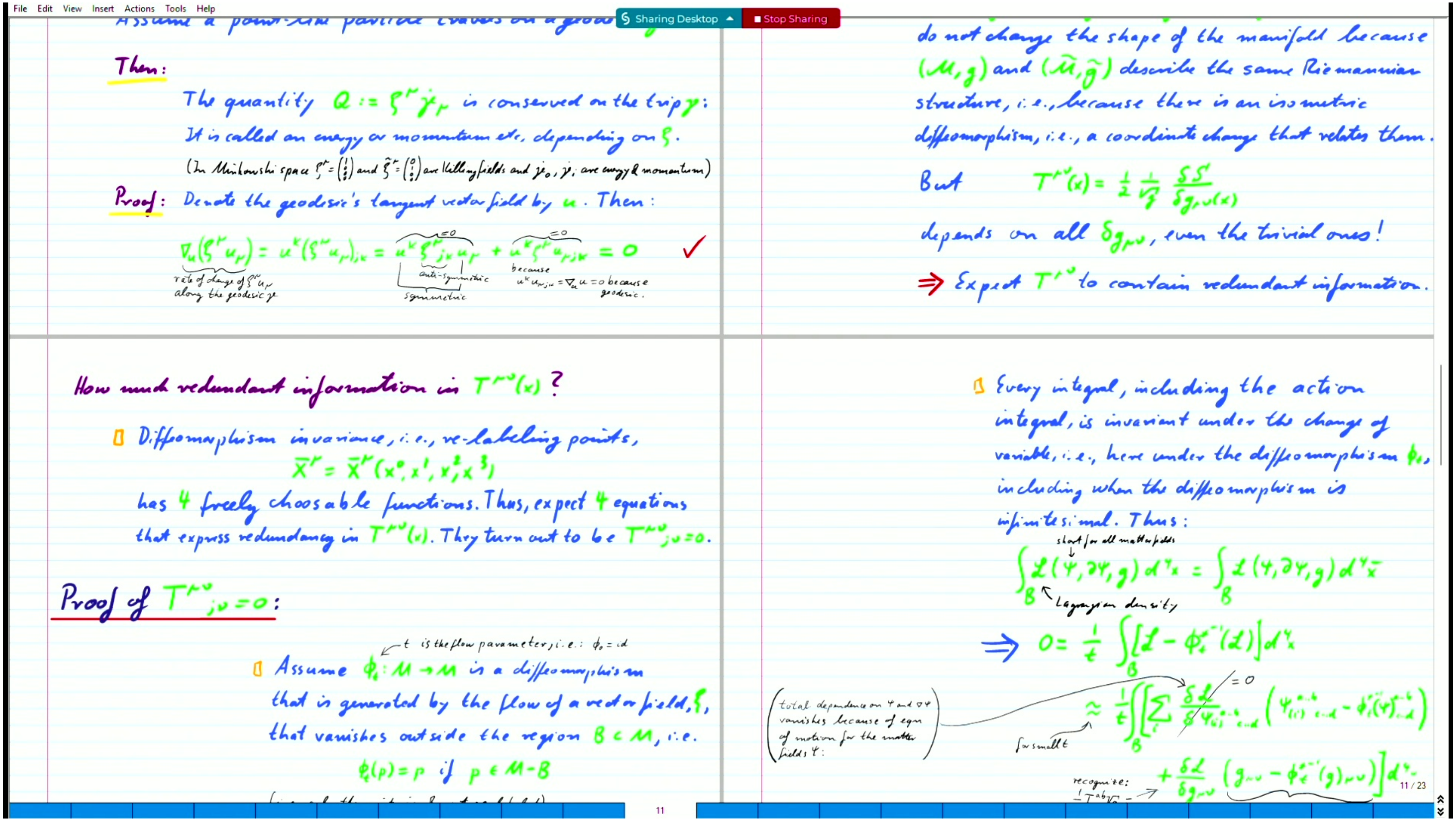
$$g_{\mu\nu}(x) \rightarrow \tilde g_{\mu\nu}(x) = g_{\mu\nu}(x) + \delta g_{\mu\nu}(x)$$

do not change the shape of the manifold because $(\mathcal{M},g)$ and $(\tilde{\mathcal{M}},\tilde g)$ describe the same Riemannian structure, i.e., because there is an isometric diffeomorphism, i.e., a coordinate change that relates them.

But $$T^{\mu\nu}(x) = \frac{1}{2}\frac{1}{\sqrt{g}}\frac{\delta S}{\delta g_{\mu\nu}(x)}$$

depends on all $\delta g_{\mu\nu}$, even the trivial ones!

⇒ Expect $T^{\mu\nu}$ to contain redundant information.



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anti-symmetric

symmetric

because $u^\kappa u_{\mu;\kappa} = \nabla_u u = 0$ because geodesic.

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But $$T^{\mu\nu}(x) = \frac{1}{2}\frac{1}{\sqrt{g}}\frac{\delta S}{\delta g_{\mu\nu}(x)}$$

depends on all $\delta g_{\mu\nu}$, even the trivial ones!

$\Rightarrow$ Expect $T^{\mu\nu}$ to contain redundant information.

How much redundant information in $T^{\mu\nu}(x)$?

- Diffeomorphism invariance, i.e., re-labeling points,
$$\bar{x}^\mu = \bar{x}^\mu(x^0, x^1, x^2, x^3)$$
has 4 freely choosable functions. Thus, expect 4 equations that express redundancy in $T^{\mu\nu}(x)$. They turn out to be $T^{\mu\nu}{}_{;\nu} = 0$.

Proof of $T^{\mu\nu}{}_{;\nu} = 0$:

- Assume $\phi_t : \mathcal{M} \to \mathcal{M}$ is a diffeomorphism that is generated by the flow of a vector field, ξ, that vanishes outside the region $B \subset \mathcal{M}$, i.e.

t is the flow parameter, i.e.: $\phi_0 = id$

$$\phi_t(p) = p \text{ if } p \in \mathcal{M} - B$$

- Every integral, including the action integral, is invariant under the change of variable, i.e., here under the diffeomorphism ϕ_t, including when the diffeomorphism is infinitesimal. Thus:

short for all matter fields

$$\int_B \mathcal{L}(\psi, \partial\psi, g)\, d^4x = \int_B \mathcal{L}(\psi, \partial\psi, g)\, d^4\bar{x}$$

Lagrangian density

$$\Rightarrow 0 = \frac{1}{t}\int_B \left[\mathcal{L} - \phi_t^{*-1}(\mathcal{L})\right] d^4x$$

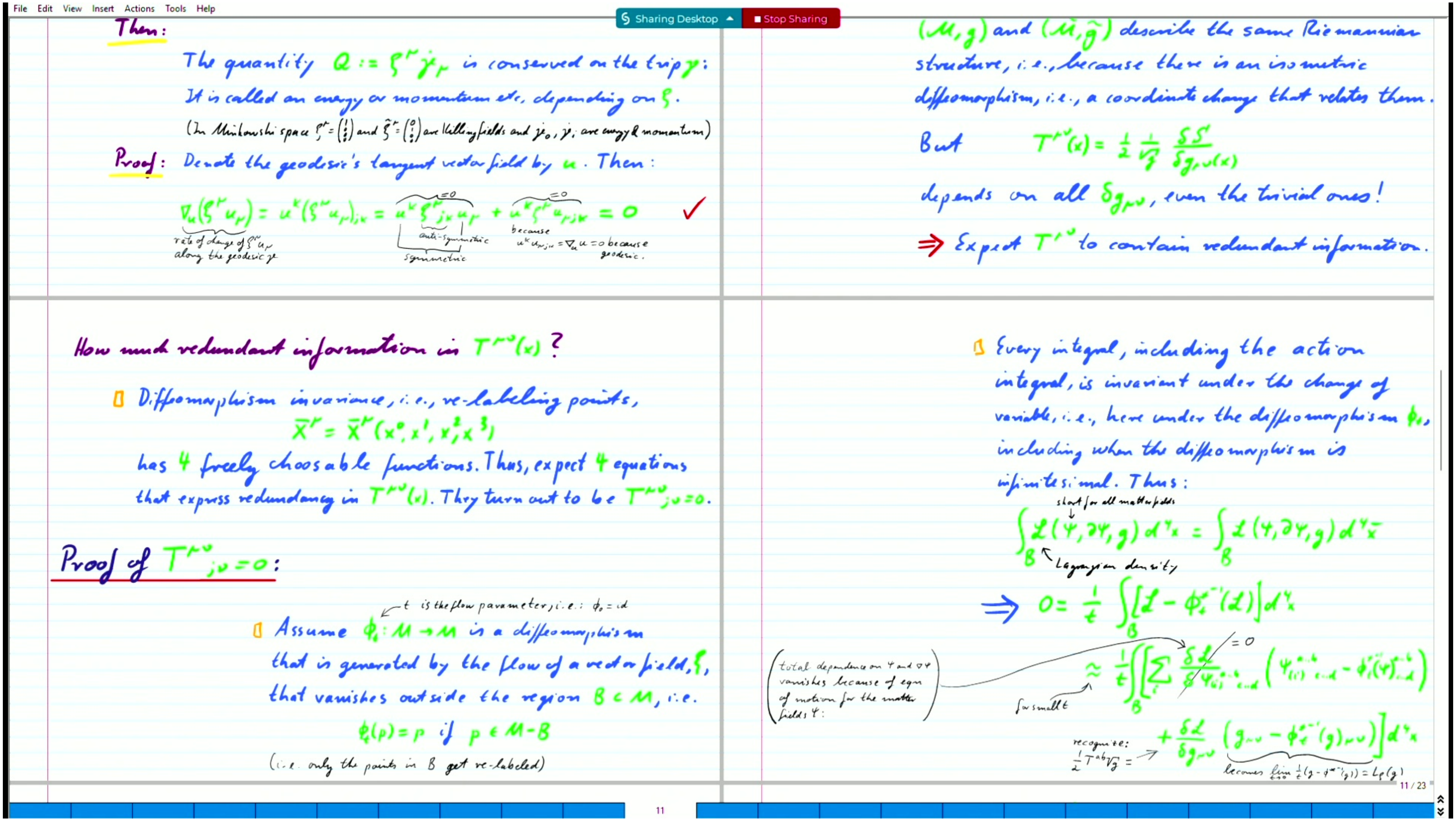
$$\approx \frac{1}{t}\int_B \Bigg[\sum_i \cancel{\frac{\delta \mathcal{L}}{\delta \psi_{(i)}{}^{a..b}{}_{c..d}}}^{=0} \left(\psi_{(i)}{}^{a..b}{}_{c..d} - \phi_t^{*-1}(\psi)^{a..b}{}_{c..d}\right)$$

$$+ \frac{\delta \mathcal{L}}{\delta g_{\mu\nu}} \underbrace{\left(g_{\mu\nu} - \phi_t^{*-1}(g)_{\mu\nu}\right)}\Bigg] d^4x$$

for small t

(total dependence on ψ and $\nabla\psi$ vanishes because of eqn of motion for the matter fields ψ:

recognize: $\frac{1}{2}T^{ab}\sqrt{g}$



Then:

The quantity $Q := \xi^\mu \dot{\gamma}_\mu$ is conserved on the trip γ:
It is called an energy or momentum etc, depending on ξ.
(In Minkowski space $\xi^\mu = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}$ and $\tilde{\xi}^\mu = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}$ are Killing fields and $\dot{\gamma}_0, \dot{\gamma}_i$ are energy & momentum)

Proof: Denote the geodesic's tangent vector field by u. Then:

$$\underbrace{\nabla_u(\xi^\mu u_\mu)}_{\text{rate of change of } \xi^\mu u_\mu \text{ along the geodesic } \gamma} = u^\kappa(\xi^\mu u_\mu)_{;\kappa} = \overbrace{u^\kappa \xi^\mu{}_{;\kappa} u_\mu}^{=0} + \overbrace{u^\kappa \xi^\mu u_{\mu;\kappa}}^{=0} = 0 \quad ✓$$

anti-symmetric

symmetric

because $u^\kappa u_{\mu;\kappa} = \nabla_u u = 0$ because geodesic.

$(\mathcal{M}, g)$ and $(\tilde{\mathcal{M}}, \tilde{g})$ describe the same Riemannian structure, i.e., because there is an isometric diffeomorphism, i.e., a coordinate change that relates them.

But $$T^{\mu\nu}(x) = \frac{1}{2}\frac{1}{\sqrt{g}}\frac{\delta S}{\delta g_{\mu\nu}(x)}$$

depends on all $\delta g_{\mu\nu}$, even the trivial ones!

⇒ Expect $T^{\mu\nu}$ to contain redundant information.

How much redundant information in $T^{\mu\nu}(x)$?

- Diffeomorphism invariance, i.e., re-labeling points,
  $$\bar{x}^\mu = \bar{x}^\mu(x^0, x^1, x^2, x^3)$$
  has 4 freely choosable functions. Thus, expect 4 equations that express redundancy in $T^{\mu\nu}(x)$. They turn out to be $T^{\mu\nu}{}_{;\nu} = 0$.

Proof of $T^{\mu\nu}{}_{;\nu} = 0$:

- Assume $\phi_t : \mathcal{M} \to \mathcal{M}$ is a diffeomorphism that is generated by the flow of a vector field, ξ, that vanishes outside the region $B \subset \mathcal{M}$, i.e.

  t is the flow parameter, i.e.: $\phi_0 = id$

  $$\phi_t(p) = p \text{ if } p \in \mathcal{M} - B$$

  (i.e. only the points in B get re-labeled)

- Every integral, including the action integral, is invariant under the change of variable, i.e., here under the diffeomorphism ϕ_t, including when the diffeomorphism is infinitesimal. Thus:

  short for all matter fields

  $$\int_B \mathcal{L}(\Psi, \partial\Psi, g)\, d^4x = \int_B \mathcal{L}(\Psi, \partial\Psi, g)\, d^4\bar{x}$$

  Lagrangian density

  $$\Rightarrow 0 = \frac{1}{t}\int_B \left[\mathcal{L} - \phi_t^{*-1}(\mathcal{L})\right] d^4x$$

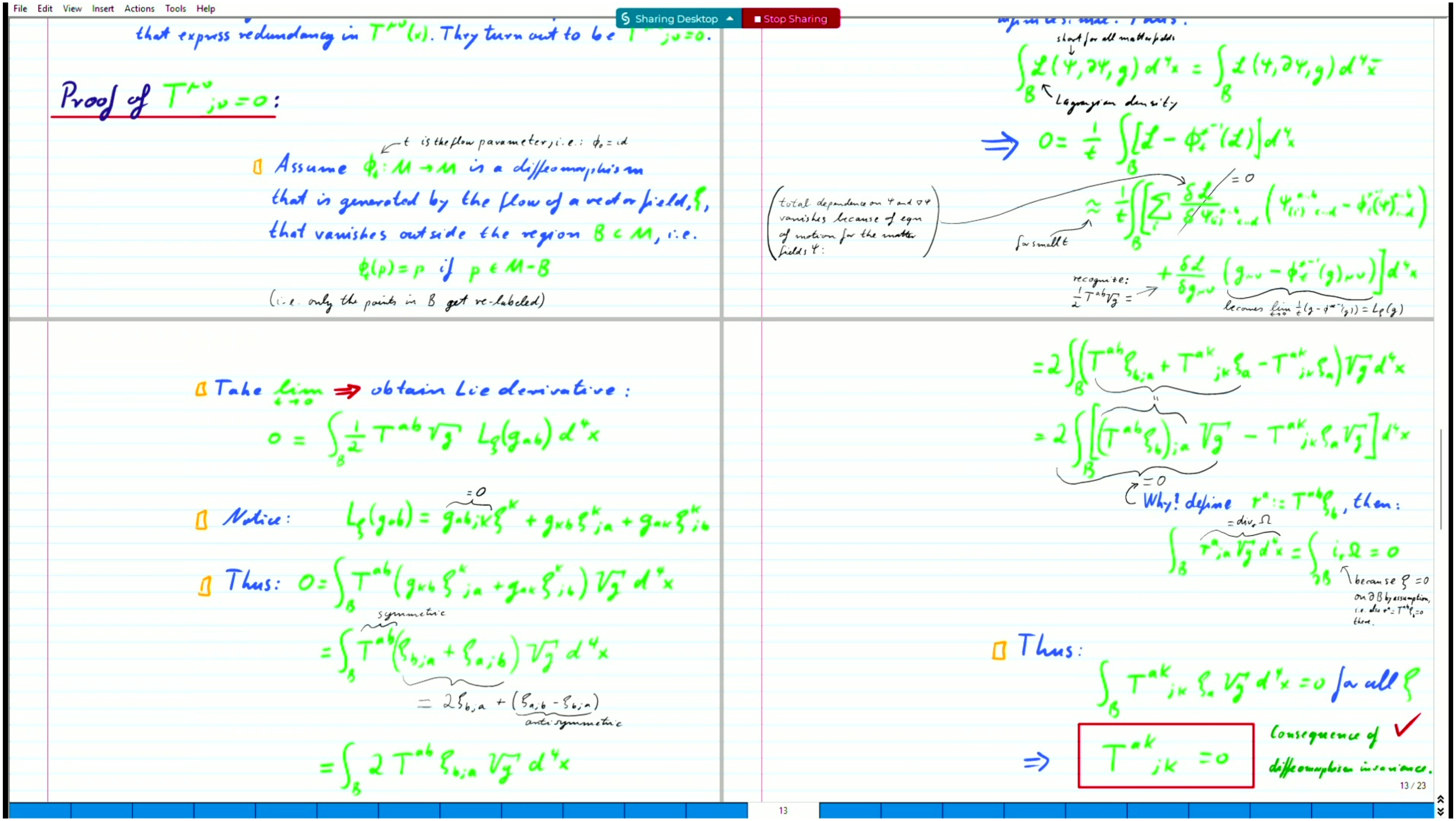
  (total dependence on Ψ and $\nabla\Psi$ vanishes because of eqn of motion for the matter fields Ψ:)

  $$\approx \frac{1}{t}\int_B \Big[\sum_i \overset{=0}{\cancel{\frac{\delta\mathcal{L}}{\delta\Psi_{(i)}{}^{a\dots b}{}_{c\dots d}}}} \left(\Psi_{(i)}{}^{a\dots b}{}_{c\dots d} - \phi_t^{*-1}(\Psi)^{a\dots b}{}_{c\dots d}\right)$$

  for small t

  $$+ \frac{\delta\mathcal{L}}{\delta g_{\mu\nu}} \underbrace{\left(g_{\mu\nu} - \phi_t^{*-1}(g)_{\mu\nu}\right)}_{\text{becomes } \lim_{t\to 0}\frac{1}{t}(g - \phi^{*-1}(g)) = L_\xi(g)}\Big] d^4x$$

  recognize: $\frac{1}{2}T^{ab}\sqrt{g} =$



that express redundancy in $T^{\mu\nu}(x)$. They turn out to be $T^{\mu\nu}{}_{;\nu}=0$.

Proof of $T^{\mu\nu}{}_{;\nu}=0$:

t is the flow parameter, i.e.: $\phi_0 = id$

☐ Assume $\phi_t : M \to M$ is a diffeomorphism that is generated by the flow of a vector field, ξ, that vanishes outside the region $B \subset M$, i.e.

$$\phi_t(p) = p \text{ if } p \in M - B$$

(i.e. only the points in B get re-labeled)

short for all matter fields

$$\int_B \mathcal{L}(\Psi, \partial\Psi, g)\, d^4x = \int_B \mathcal{L}(\Psi, \partial\Psi, g)\, d^4\bar{x}$$

Lagrangian density

$$\Rightarrow \quad 0 = \frac{1}{t}\int_B \left[\mathcal{L} - \phi_t^{*-1}(\mathcal{L})\right] d^4x$$

(total dependence on Ψ and $\nabla\Psi$ vanishes because of eqn of motion for the matter fields Ψ:)

for small t

$$\approx \frac{1}{t}\int_B \Bigg[\sum_i \overset{=0}{\cancel{\frac{\delta\mathcal{L}}{\delta\Psi_{(i)}{}^{a..b}{}_{c..d}}}}\left(\Psi_{(i)}{}^{a..b}{}_{c..d} - \phi_t^{*-1}(\Psi)^{a..b}{}_{c..d}\right)$$

$$+ \frac{\delta\mathcal{L}}{\delta g_{\mu\nu}} \underbrace{\left(g_{\mu\nu} - \phi_t^{*-1}(g)_{\mu\nu}\right)}_{\text{becomes } \lim_{t\to 0}\frac{1}{t}(g - \phi^{*-1}(g)) = L_\xi(g)}\Bigg] d^4x$$

recognize: $\frac{1}{2}T^{ab}\sqrt{g} =$

☐ Take $\lim_{t\to 0}$ ⇒ obtain Lie derivative:

$$0 = \int_B \frac{1}{2} T^{ab}\sqrt{g}\, L_\xi(g_{ab})\, d^4x$$

☐ Notice: $L_\xi(g_{ab}) = \overbrace{g_{ab;k}}^{=0}\xi^k + g_{kb}\xi^k{}_{;a} + g_{ak}\xi^k{}_{;b}$

☐ Thus: $0 = \int_B T^{ab}\left(g_{kb}\xi^k{}_{;a} + g_{ak}\xi^k{}_{;b}\right)\sqrt{g}\, d^4x$

$$= \int_B \overbrace{T^{ab}}^{\text{symmetric}} \underbrace{\left(\xi_{b;a} + \xi_{a;b}\right)}_{= 2\xi_{b;a} + \underbrace{(\xi_{a;b} - \xi_{b;a})}_{\text{antisymmetric}}} \sqrt{g}\, d^4x$$

$$= \int_B 2\, T^{ab}\xi_{b;a}\sqrt{g}\, d^4x$$

$$= 2\int_B \Big(\underbrace{T^{ab}\xi_{b;a} + T^{ak}{}_{;k}\xi_a}_{=} - T^{ak}{}_{;k}\xi_a\Big)\sqrt{g}\, d^4x$$

$$= 2\underbrace{\int_B \Big[\overbrace{(T^{ab}\xi_b)_{;a}}\sqrt{g}}_{=0} - T^{ak}{}_{;k}\xi_a\sqrt{g}\Big] d^4x$$

Why? define $r^a := T^{ab}\xi_b$, then:

$$\int_B \overbrace{r^a{}_{;a}\sqrt{g}\, d^4x}^{=\mathrm{div}_r\Omega} = \int_{\partial B} i_r\Omega = 0$$

because $\xi = 0$ on ∂B by assumption, i.e. also $r^a = T^{ab}\xi_b = 0$ there.

☐ Thus:

$$\int_B T^{ak}{}_{;k}\xi_a\sqrt{g}\, d^4x = 0 \text{ for all } \xi$$

$$\Rightarrow \quad \boxed{T^{ak}{}_{;k} = 0}$$

Consequence of ✓ diffeomorphism invariance.

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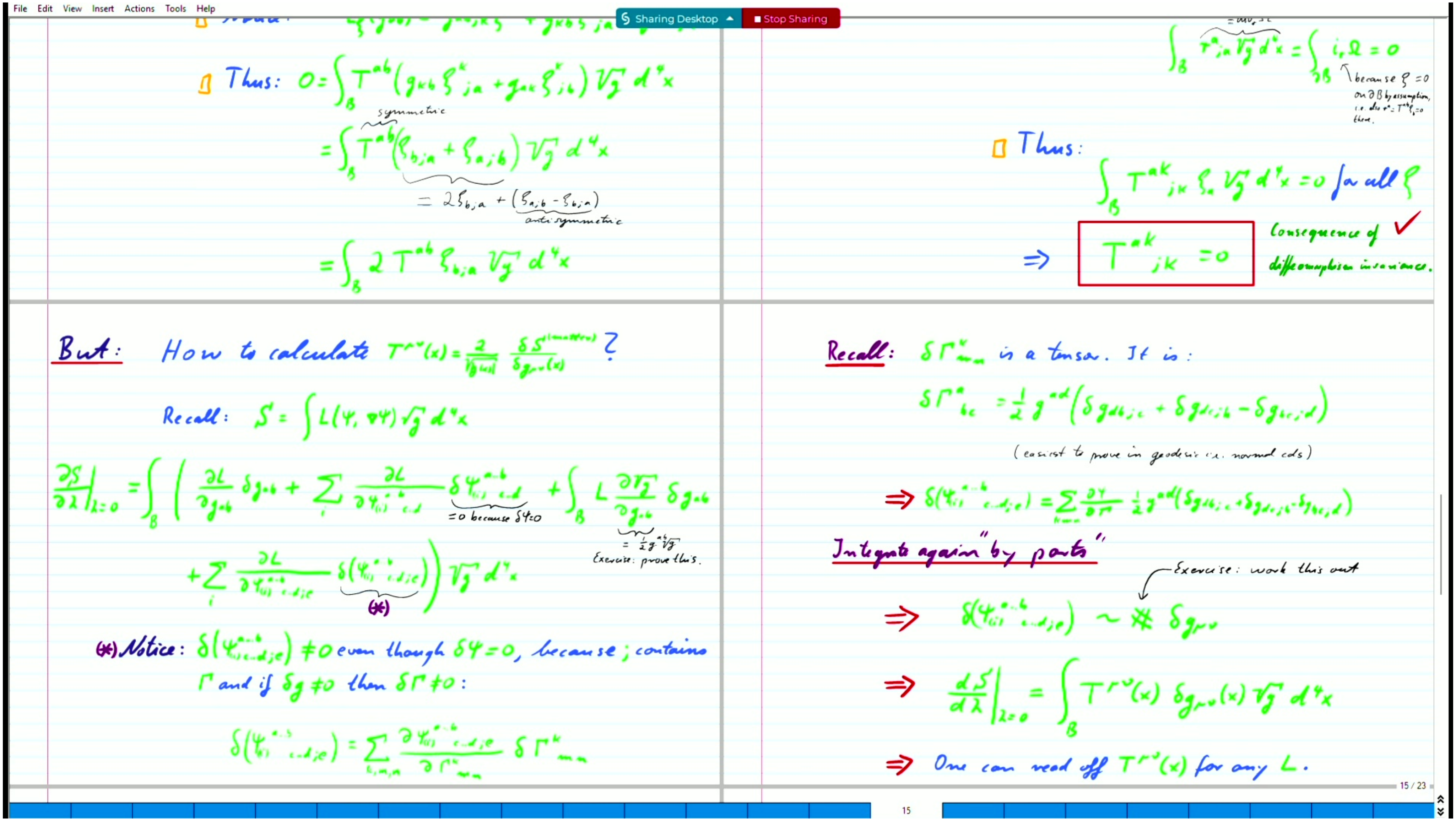


Thus: $$0=\int_B T^{ab}\left(g_{kb}\xi^k{}_{;a}+g_{ak}\xi^k{}_{;b}\right)\sqrt{g}\,d^4x$$

$$=\int_B \overset{\text{symmetric}}{T^{ab}}\underbrace{\left(\xi_{b;a}+\xi_{a;b}\right)}_{=2\xi_{b;a}+\underbrace{(\xi_{a;b}-\xi_{b;a})}_{\text{antisymmetric}}}\sqrt{g}\,d^4x$$

$$=\int_B 2\,T^{ab}\,\xi_{b;a}\sqrt{g}\,d^4x$$

$$\int_B \overbrace{T^a{}_{;a}\sqrt{g}\,d^4x}^{=\mathrm{div}\ldots} = \int_{\partial B} i_v\Omega = 0$$

because $\xi=0$ on ∂B by assumption, i.e. also $v^a = T^{ab}\xi_b = 0$ there.

Thus:
$$\int_B T^{ak}{}_{;k}\,\xi_a\sqrt{g}\,d^4x = 0 \text{ for all } \xi$$

$$\Rightarrow \boxed{T^{ak}{}_{;k}=0}$$

Consequence of ✓ diffeomorphism invariance.

But: How to calculate $T^{\mu\nu}(x)=\frac{2}{\sqrt{|g(x)|}}\frac{\delta S^{(\text{matter})}}{\delta g_{\mu\nu}(x)}$?

Recall: $S=\int L(\psi,\nabla\psi)\sqrt{g}\,d^4x$

$$\frac{\partial S}{\partial\lambda}\Big|_{\lambda=0}=\int_B\Bigg(\frac{\partial L}{\partial g_{ab}}\delta g_{ab}+\sum_i\frac{\partial L}{\partial\psi_{(i)}{}^{a\ldots b}{}_{c\ldots d}}\underbrace{\delta\psi_{(i)}{}^{a\ldots b}{}_{c\ldots d}}_{=0\text{ because }\delta\psi=0}+\int_B L\underbrace{\frac{\partial\sqrt{g}}{\partial g_{ab}}}_{=\frac{1}{2}g^{ab}\sqrt{g}}\delta g_{ab}$$

Exercise: prove this.

$$+\sum_i\frac{\partial L}{\partial\psi_{(i)}{}^{a\ldots b}{}_{c\ldots d;e}}\underbrace{\delta\left(\psi_{(i)}{}^{a\ldots b}{}_{c\ldots d;e}\right)}_{(*)}\Bigg)\sqrt{g}\,d^4x$$

(*) Notice: $\delta\left(\psi_{(i)}{}^{a\ldots b}{}_{c\ldots d;e}\right)\neq0$ even though $\delta\psi=0$, because ; contains Γ and if $\delta g\neq0$ then $\delta\Gamma\neq0$:

$$\delta\left(\psi_{(i)}{}^{a\ldots b}{}_{c\ldots d;e}\right)=\sum_{k,m,n}\frac{\partial\psi_{(i)}{}^{a\ldots b}{}_{c\ldots d;e}}{\partial\Gamma^k{}_{mn}}\delta\Gamma^k{}_{mn}$$

Recall: $\delta\Gamma^k{}_{mn}$ is a tensor. It is:

$$\delta\Gamma^a{}_{bc}=\frac{1}{2}g^{ad}\left(\delta g_{db;c}+\delta g_{dc;b}-\delta g_{bc;d}\right)$$

(easiest to prove in geodesic i.e. normal cds)

$$\Rightarrow \delta\left(\psi_{(i)}{}^{a\ldots b}{}_{c\ldots d;e}\right)=\sum_{k,m,n}\frac{\partial\psi}{\partial\Gamma}\,\frac{1}{2}g^{ad}\left(\delta g_{db;c}+\delta g_{dc;b}-\delta g_{bc;d}\right)$$

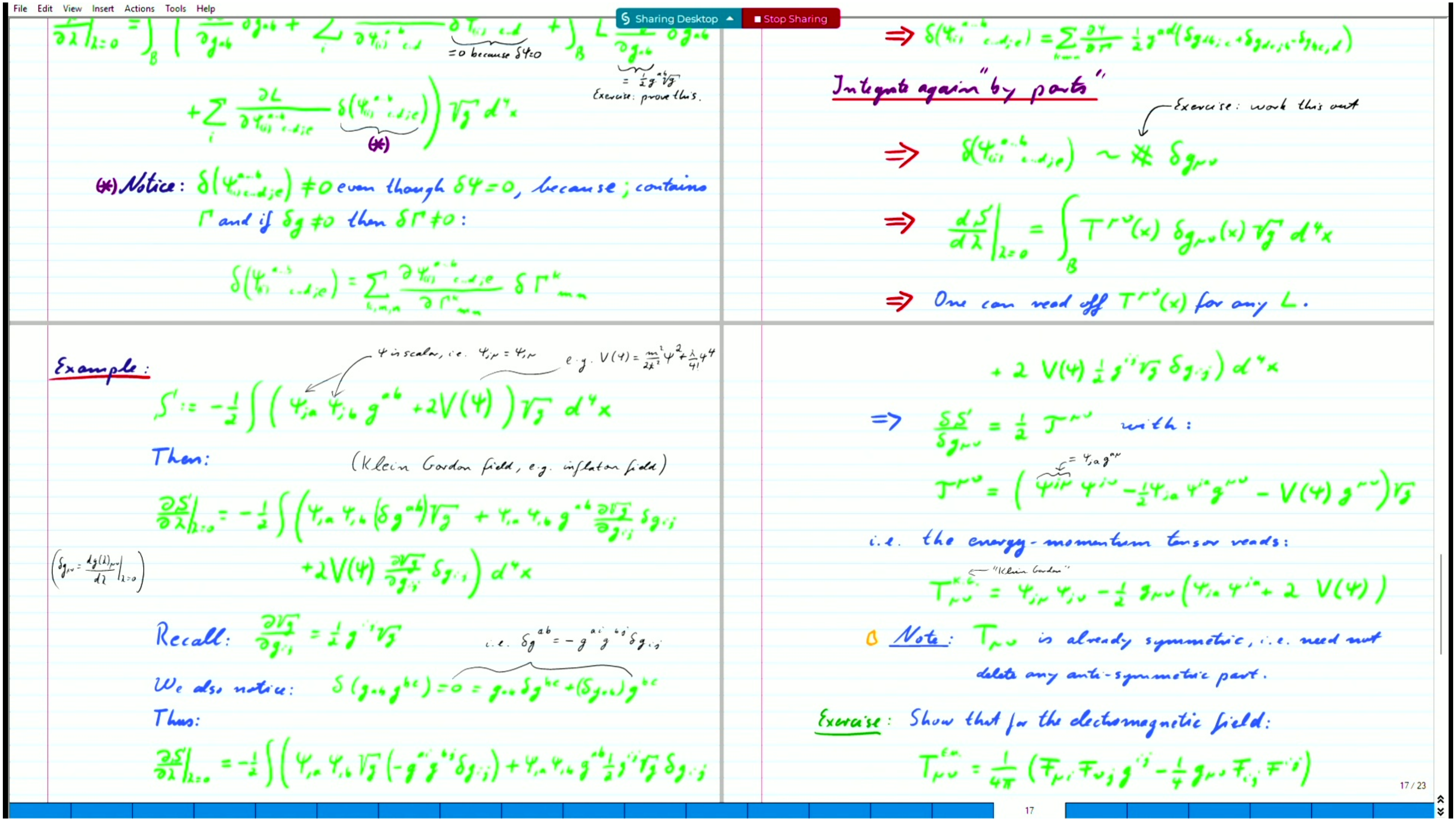
Integrate again "by parts"

Exercise: work this out

$$\Rightarrow \delta\left(\psi_{(i)}{}^{a\ldots b}{}_{c\ldots d;e}\right)\sim \ast\ \delta g_{\mu\nu}$$

$$\Rightarrow \frac{dS}{d\lambda}\Big|_{\lambda=0}=\int_B T^{\mu\nu}(x)\,\delta g_{\mu\nu}(x)\sqrt{g}\,d^4x$$

$\Rightarrow$ One can read off $T^{\mu\nu}(x)$ for any L.



$$\frac{}{\partial\lambda}\Big|_{\lambda=0} = \int_B \Big(\frac{}{\partial g_{ab}}\,\delta g_{ab} + \sum_i \frac{}{\partial\psi_{(i)}{}^{a..b}{}_{c..d}}\,\underbrace{\delta\psi_{(i)}{}^{a..b}{}_{c..d}}_{=0 \text{ because } \delta\psi=0} + \int_B L\,\underbrace{\frac{}{\partial g_{ab}}}_{=\frac{1}{2}g^{ab}\sqrt{g}}\,\delta g_{ab}$$

Exercise: prove this.

$$+\sum_i \frac{\partial L}{\partial\psi_{(i)}{}^{a..b}{}_{c..d;e}}\,\underbrace{\delta\left(\psi_{(i)}{}^{a..b}{}_{c..d;e}\right)}_{(*)}\Big)\sqrt{g}\,d^4x$$

(*) **Notice**: $\delta\left(\psi_{(i)}{}^{a..b}{}_{c..d;e}\right) \neq 0$ even though $\delta\psi = 0$, because ; contains Γ and if $\delta g \neq 0$ then $\delta\Gamma \neq 0$:

$$\delta\left(\psi_{(i)}{}^{a..b}{}_{c..d;e}\right) = \sum_{k,m,n} \frac{\partial\psi_{(i)}{}^{a..b}{}_{c..d;e}}{\partial\Gamma^k{}_{mn}}\,\delta\Gamma^k{}_{mn}$$

$$\Rightarrow \delta\left(\psi_{(i)}{}^{a..b}{}_{c..d;e}\right) = \sum_{kmn} \frac{\partial\psi}{\partial\Gamma}\,\frac{1}{2}g^{ad}\left(\delta g_{db;c} + \delta g_{dc;b} - \delta g_{bc;d}\right)$$

Integrate again "by parts"

Exercise: work this out

$$\Rightarrow \quad \delta\left(\psi_{(i)}{}^{a..b}{}_{c..d;e}\right) \sim * \,\delta g_{\mu\nu}$$

$$\Rightarrow \quad \frac{dS}{d\lambda}\Big|_{\lambda=0} = \int_B T^{\mu\nu}(x)\,\delta g_{\mu\nu}(x)\sqrt{g}\,d^4x$$

$\Rightarrow$ One can read off $T^{\mu\nu}(x)$ for any L.

Example:

ψ is scalar, i.e. $\psi_{;\mu} = \psi_{,\mu}$

e.g. $V(\psi) = \frac{m^2}{2\hbar^2}\psi^2 + \frac{\lambda}{4!}\psi^4$

$$S' := -\frac{1}{2}\int \left(\psi_{;a}\psi_{;b}\,g^{ab} + 2V(\psi) \right)\sqrt{g}\,d^4x$$

Then: (Klein Gordon field, e.g. inflaton field)

$$\frac{\partial S'}{\partial\lambda}\Big|_{\lambda=0} = -\frac{1}{2}\int \Big(\psi_{,a}\psi_{,b}\left(\delta g^{ab}\right)\sqrt{g} + \psi_{,a}\psi_{,b}\,g^{ab}\frac{\partial\sqrt{g}}{\partial g_{ij}}\,\delta g_{ij}$$

$$\left(\delta g_{\mu\nu} = \frac{dg(\lambda)_{\mu\nu}}{d\lambda}\Big|_{\lambda=0}\right) \qquad + 2V(\psi)\,\frac{\partial\sqrt{g}}{\partial g_{ij}}\,\delta g_{ij} \Big)\,d^4x$$

Recall: $\frac{\partial\sqrt{g}}{\partial g_{ij}} = \frac{1}{2}g^{ij}\sqrt{g}$

i.e. $\delta g^{ab} = -g^{ai}g^{bj}\,\delta g_{ij}$

We also notice: $\delta\left(g_{ab}g^{bc}\right) = 0 = g_{ab}\,\delta g^{bc} + \left(\delta g_{ab}\right)g^{bc}$

Thus:

$$\frac{\partial S'}{\partial\lambda}\Big|_{\lambda=0} = -\frac{1}{2}\int \Big(\psi_{,a}\psi_{,b}\sqrt{g}\left(-g^{ai}g^{bj}\,\delta g_{ij}\right) + \psi_{,a}\psi_{,b}\,g^{ab}\,\frac{1}{2}g^{ij}\sqrt{g}\,\delta g_{ij}$$

$$+ 2V(\psi)\,\frac{1}{2}g^{ij}\sqrt{g}\,\delta g_{ij} \Big)\,d^4x$$

$$\Rightarrow \quad \frac{\delta S'}{\delta g_{\mu\nu}} = \frac{1}{2}\mathcal{T}^{\mu\nu} \quad \text{with:}$$

$$\mathcal{T}^{\mu\nu} = \Big(\underbrace{\psi^{;\mu}}_{=\psi_{;a}g^{a\mu}}\psi^{;\nu} - \frac{1}{2}\psi_{;a}\psi^{;a}g^{\mu\nu} - V(\psi)\,g^{\mu\nu} \Big)\sqrt{g}$$

i.e. the energy-momentum tensor reads:

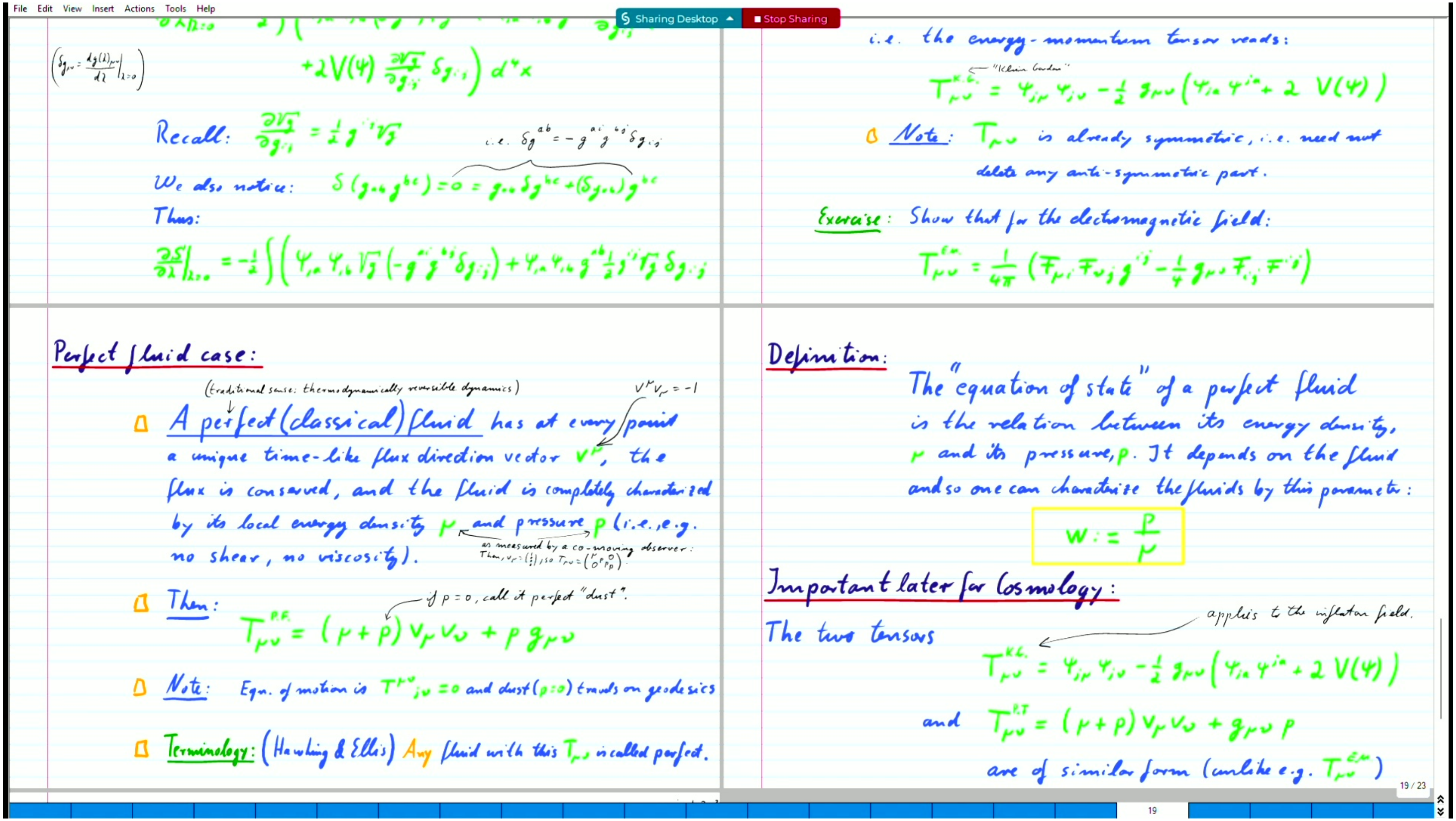
$$T^{K.G.}_{\mu\nu} = \psi_{;\mu}\psi_{;\nu} - \frac{1}{2}g_{\mu\nu}\left(\psi_{;a}\psi^{;a} + 2\,V(\psi)\right)$$

← "Klein Gordon"

Note: $T_{\mu\nu}$ is already symmetric, i.e. need not delete any anti-symmetric part.

Exercise: Show that for the electromagnetic field:

$$T^{e.m.}_{\mu\nu} = \frac{1}{4\pi}\left(F_{\mu i}F_{\nu j}\,g^{ij} - \frac{1}{4}g_{\mu\nu}F_{ij}F^{ij} \right)$$



$\left(\delta g_{\mu\nu} = \frac{dg(\lambda)_{\mu\nu}}{d\lambda}\Big|_{\lambda=0}\right)$

$$+2V(\varphi)\frac{\partial\sqrt{g}}{\partial g_{ij}}\,\delta g_{ij}\Big)\,d^4x$$

Recall: $\frac{\partial\sqrt{g}}{\partial g_{ij}} = \frac{1}{2}g^{ij}\sqrt{g}$

i.e. $\delta g^{ab} = -g^{ai}g^{bj}\,\delta g_{ij}$

We also notice: $\delta(g_{ab}g^{bc}) = 0 = \overbrace{g_{ab}\,\delta g^{bc} + (\delta g_{ab})\,g^{bc}}$

Thus:

$$\frac{\partial S}{\partial\lambda}\Big|_{\lambda=0} = -\frac{1}{2}\int\Big(\varphi_{,a}\varphi_{,b}\sqrt{g}\left(-g^{ai}g^{bj}\delta g_{ij}\right) + \varphi_{,a}\varphi_{,b}\,g^{ab}\frac{1}{2}g^{ij}\sqrt{g}\,\delta g_{ij}$$

i.e. the energy-momentum tensor reads:

$$T^{K.G.}_{\mu\nu} = \varphi_{;\mu}\varphi_{;\nu} - \frac{1}{2}g_{\mu\nu}\left(\varphi_{;a}\varphi^{;a} + 2\,V(\varphi)\right)$$

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$$T^{E.M.}_{\mu\nu} = \frac{1}{4\pi}\left(F_{\mu i}F_{\nu j}\,g^{ij} - \frac{1}{4}g_{\mu\nu}F_{ij}F^{ij}\right)$$

Perfect fluid case:

(traditional sense: thermodynamically reversible dynamics)

$V^\mu V_\mu = -1$

- A perfect (classical) fluid has at every point a unique time-like flux direction vector V^μ, the flux is conserved, and the fluid is completely characterized by its local energy density μ and pressure p (i.e., e.g. no shear, no viscosity).

  as measured by a co-moving observer: Then, $v_\mu = \left(\begin{smallmatrix}1\\0\\0\\0\end{smallmatrix}\right)$, so $T_{\mu\nu} = \left(\begin{smallmatrix}\mu & & & 0\\ & p & & \\ & & p & \\ 0 & & & p\end{smallmatrix}\right)$.

- Then:

$$T^{P.F.}_{\mu\nu} = (\mu + p)\,V_\mu V_\nu + p\,g_{\mu\nu}$$

  if $p=0$, call it perfect "dust".

- Note: Eqn. of motion is $T^{\mu\nu}{}_{;\nu} = 0$ and dust ($p=0$) travels on geodesics

- Terminology: (Hawking & Ellis) Any fluid with this $T_{\mu\nu}$ is called perfect.

Definition:

The "equation of state" of a perfect fluid is the relation between its energy density, μ and its pressure, p. It depends on the fluid and so one can characterize the fluids by this parameter:

$$\boxed{w := \frac{p}{\mu}}$$

Important later for Cosmology:

The two tensors

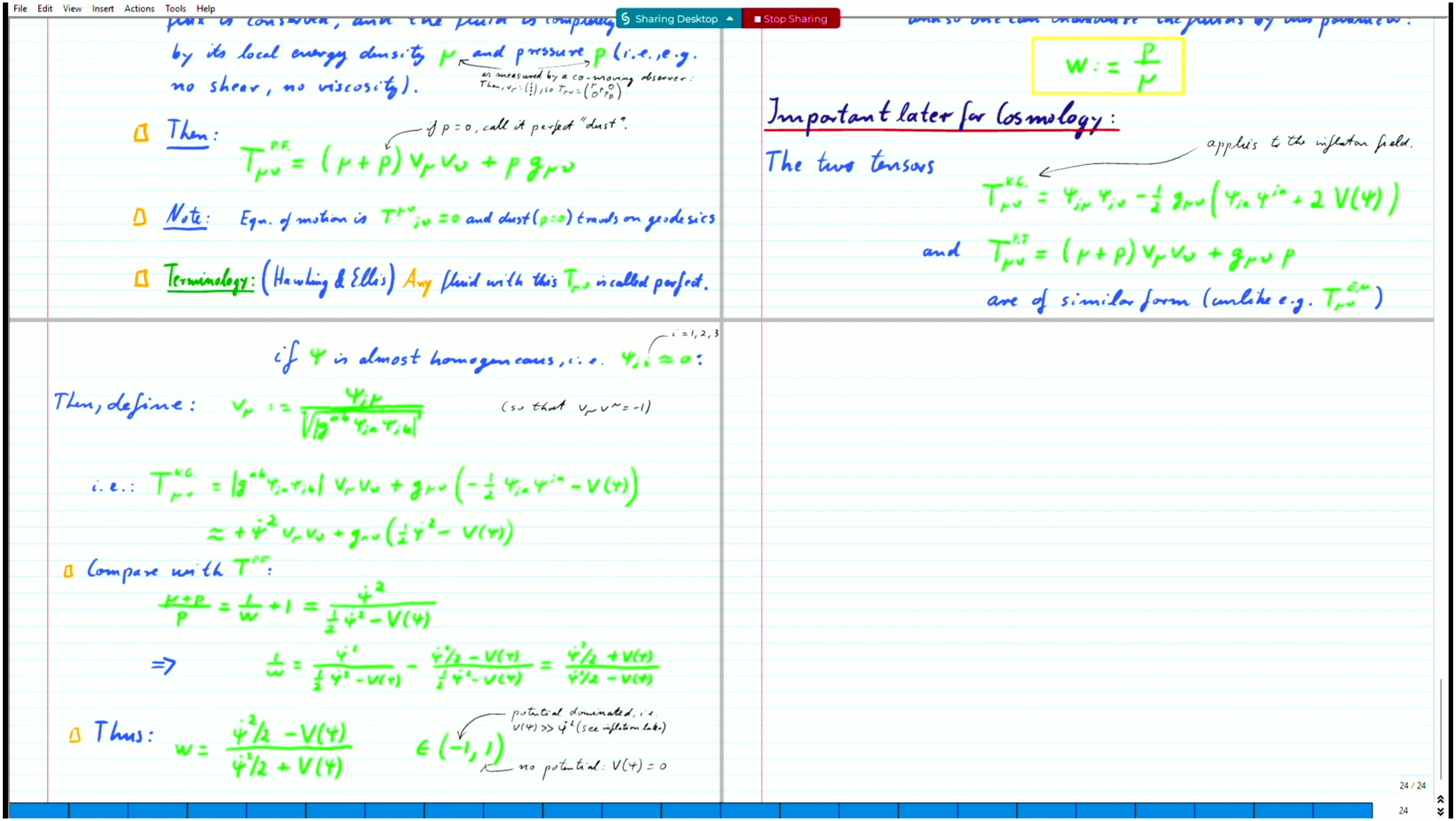
$$T^{K.G.}_{\mu\nu} = \varphi_{;\mu}\varphi_{;\nu} - \frac{1}{2}g_{\mu\nu}\left(\varphi_{;a}\varphi^{;a} + 2\,V(\varphi)\right)$$

applies to the inflaton field.

and

$$T^{P.F.}_{\mu\nu} = (\mu + p)\,V_\mu V_\nu + g_{\mu\nu}\,p$$

are of similar form (unlike e.g. $T^{E.M.}_{\mu\nu}$)



flux is conserved, and the fluid is completely

by its local energy density μ and pressure p (i.e., e.g. no shear, no viscosity).

as measured by a co-moving observer: Then, $v_\mu = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}$, so $T_{\mu\nu} = \begin{pmatrix}\mu & & & 0\\ & p & & \\ & & p & \\ 0 & & & p\end{pmatrix}$

- **Then:**

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$$\boxed{w := \frac{p}{\mu}}$$

Important later for Cosmology:

The two tensors

applies to the inflaton field.

$$T^{K.G.}_{\mu\nu} = \varphi_{;\mu}\varphi_{;\nu} - \frac{1}{2} g_{\mu\nu}\left(\varphi_{;a}\varphi^{;a} + 2V(\varphi)\right)$$

and

$$T^{P.F.}_{\mu\nu} = (\mu + p)\, v_\mu v_\nu + g_{\mu\nu}\, p$$

are of similar form (unlike e.g. $T^{E.M.}_{\mu\nu}$)

if φ is almost homogeneous, i.e. $\varphi_{;i} \approx 0$: ($i = 1,2,3$)

Then, define: $v_\mu := \dfrac{\varphi_{;\mu}}{\sqrt{|g^{ab}\varphi_{;a}\varphi_{;b}|}}$ (so that $v_\mu v^\mu = -1$)

$$\text{i.e.: } T^{K.G.}_{\mu\nu} = |g^{ab}\varphi_{;a}\varphi_{;b}|\, v_\mu v_\nu + g_{\mu\nu}\left(-\frac{1}{2}\varphi_{;a}\varphi^{;a} - V(\varphi)\right)$$

$$\approx +\dot{\varphi}^2 v_\mu v_\nu + g_{\mu\nu}\left(\frac{1}{2}\dot{\varphi}^2 - V(\varphi)\right)$$

- Compare with $T^{P.F.}$:

$$\frac{\mu + p}{p} = \frac{1}{w} + 1 = \frac{\dot{\varphi}^2}{\frac{1}{2}\dot{\varphi}^2 - V(\varphi)}$$

$$\Rightarrow \quad \frac{1}{w} = \frac{\dot{\varphi}^2}{\frac{1}{2}\dot{\varphi}^2 - V(\varphi)} - \frac{\dot{\varphi}^2/2 - V(\varphi)}{\frac{1}{2}\dot{\varphi}^2 - V(\varphi)} = \frac{\dot{\varphi}^2/2 + V(\varphi)}{\dot{\varphi}^2/2 - V(\varphi)}$$

- **Thus:** $w = \dfrac{\dot{\varphi}^2/2 - V(\varphi)}{\dot{\varphi}^2/2 + V(\varphi)} \in (-1, 1)$

potential dominated, i.e. $V(\varphi) >> \dot{\varphi}^2$ (see inflation later)

no potential: $V(\varphi) = 0$