Title: General Relativity for Cosmology Lecture - 102423

Speakers: Achim Kempf

Collection: General Relativity for Cosmology

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GR for Cosmology, Achim Kempf — Lecture 12

Plan: **I** The dynamics of matter & radiation in curved spacetime

II Energy-momentum tensor

III The dynamics of spacetime itself.

1. **Recall:** On a (pseudo)-Riemannian mfld, equations are well-defined only if defined independently of any chart.

$\Rightarrow$ Any eqn, including the eqns of motions for matter fields must be eqns among tensors and their covariant derivatives.

$\Rightarrow$ Need a tensor field, Ψ, for each species of particle:

e^-, q, gluon, $\pi^\pm$, photon, $W^\pm$, etc..

Notation: $\Psi_{(i)}{}^{a\ldots b}{}_{c\ldots d}$ — $a\ldots b$: contravariant; $c\ldots d$: covariant; (i): species label

Note: any spinor equation can also be expressed as a (complicated) tensor equation (see e.g. Hawking & Ellis, p 59)

Question: Could we have also an additional connection field $\tilde{\Gamma}^r{}_{ij}$?

Yes, we could: But, the difference field $Q^r{}_{ij} := \Gamma^r{}_{ij} - \tilde{\Gamma}^r{}_{ij}$ is actually a tensor field!

$$\Gamma^r{}_{ab} \longrightarrow \frac{\partial \bar{x}^r}{\partial x^s}\frac{\partial^2 x^s}{\partial \bar{x}^a \partial \bar{x}^b} + \frac{\partial \bar{x}^r}{\partial x^s}\frac{\partial x^i}{\partial \bar{x}^a}\frac{\partial x^j}{\partial \bar{x}^b}\Gamma^s{}_{ij}$$

$$\tilde{\Gamma}^r{}_{ab} \longrightarrow \frac{\partial \bar{x}^r}{\partial x^s}\frac{\partial^2 x^s}{\partial \bar{x}^a \partial \bar{x}^b} + \frac{\partial \bar{x}^r}{\partial x^s}\frac{\partial x^i}{\partial \bar{x}^a}\frac{\partial x^j}{\partial \bar{x}^b}\tilde{\Gamma}^s{}_{ij}$$

$$\Rightarrow \left(\Gamma^r{}_{ab} - \tilde{\Gamma}^r{}_{ab}\right) \longrightarrow \frac{\partial \bar{x}^r}{\partial x^s}\frac{\partial x^i}{\partial \bar{x}^a}\frac{\partial x^j}{\partial \bar{x}^b}\left(\Gamma^s{}_{ij} - \tilde{\Gamma}^s{}_{ij}\right)$$

$$\Rightarrow \boxed{Q^r{}_{ab} \longrightarrow \frac{\partial \bar{x}^r}{\partial x^s}\frac{\partial x^i}{\partial \bar{x}^a}\frac{\partial x^j}{\partial \bar{x}^b}\, Q^s{}_{ij}}$$

i.e. $Q^r{}_{ab}$ is a tensor due to having the correct transformation

Eqns of motion of matter fields?

Action principle: (As in special relativity)

Any theory of matter fields can be defined by specifying the so-called Lagrangian function, L, namely a scalar function

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III The dynamics of spacetime itself.

1. **Recall:** On a (pseudo)-Riemannian mfld, equations are well-defined only if defined independently of any chart.

$\Rightarrow$ Any eqn, including the eqns of motions for matter fields must be eqns among tensors and their covariant derivatives.

Yes, we could: But, the difference field $Q^{r}{}_{ij} := \Gamma^{r}{}_{ij} - \tilde{\Gamma}^{r}{}_{ij}$ is actually a tensor field!

$$\Gamma^{r}{}_{ab} \longrightarrow \frac{\partial \tilde{x}^{r}}{\partial x^{s}} \frac{\partial^2 x^{s}}{\partial \tilde{x}^{a} \partial \tilde{x}^{b}} + \frac{\partial \tilde{x}^{r}}{\partial x^{s}} \frac{\partial x^{i}}{\partial \tilde{x}^{a}} \frac{\partial x^{j}}{\partial \tilde{x}^{b}} \Gamma^{s}{}_{ij}$$

$$\tilde{\Gamma}^{r}{}_{ab} \longrightarrow \frac{\partial \tilde{x}^{r}}{\partial x^{s}} \frac{\partial^2 x^{s}}{\partial \tilde{x}^{a} \partial \tilde{x}^{b}} + \frac{\partial \tilde{x}^{r}}{\partial x^{s}} \frac{\partial x^{i}}{\partial \tilde{x}^{a}} \frac{\partial x^{j}}{\partial \tilde{x}^{b}} \tilde{\Gamma}^{s}{}_{ij}$$

$$\Rightarrow (\Gamma^{r}{}_{ab} - \tilde{\Gamma}^{r}{}_{ab}) \longrightarrow \frac{\partial \tilde{x}^{r}}{\partial x^{s}} \frac{\partial x^{i}}{\partial \tilde{x}^{a}} \frac{\partial x^{j}}{\partial \tilde{x}^{b}} (\Gamma^{s}{}_{ij} - \tilde{\Gamma}^{s}{}_{ij})$$

$$\Rightarrow \boxed{Q^{r}{}_{ab} \longrightarrow \frac{\partial \tilde{x}^{r}}{\partial x^{s}} \frac{\partial x^{i}}{\partial \tilde{x}^{a}} \frac{\partial x^{j}}{\partial \tilde{x}^{b}} Q^{s}{}_{ij}}$$

i.e. Q^{r}_{ab} is a tensor due to having the correct transformation property according to the physicist's definition of a tensor.

$\Rightarrow$ Introducing an additional connection $\tilde{\Gamma}$ is same as introducing simply a new tensor field Q.

Remark: $\Longrightarrow$ "variations" $\delta\Gamma^{r}{}_{ab}$ will behave tensorially!

$\Psi_{(i)}{}^{\dots}{}_{c\dots d}$ ← covariant

↑ species label

Note: any spinor equation can also be expressed as a (complicated) tensor equation (see e.g. Hawking & Ellis, p 57)

Question:

Could we have also an additional connection field $\tilde{\Gamma}^{r}_{ij}$?

Eqns of motion of matter fields?

Action principle: (As in special relativity)

Any theory of matter fields can be defined by specifying the so-called Lagrangian function, L, namely a scalar function of the matter fields $\Psi_{(i)}{}^{a\dots b}{}_{c\dots d}$ and their first covariant derivatives, and now also of the metric g:

↑ we'll sometimes omit the indices

$$\boxed{L(\Psi) = L^{(\text{matter})}\left(\{\Psi_{(i)}{}^{a\dots b}{}_{c\dots d}\}, \{\Psi_{(i)}{}^{a\dots b}{}_{c\dots d;e}\}, g\right)}$$

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$\Rightarrow \quad (\Gamma^{r}{}_{ab} - \tilde{\Gamma}^{r}{}_{ab}) \longrightarrow \frac{\partial \tilde{x}^{r'}}{\partial x^{r}} \frac{\partial x^{i}}{\partial \tilde{x}^{a}} \frac{\partial x^{j}}{\partial \tilde{x}^{b}} (\Gamma^{r}{}_{ij} - \tilde{\Gamma}^{r}{}_{ij})$

$\Rightarrow \quad \boxed{Q^{r}{}_{ab} \longrightarrow \frac{\partial \tilde{x}^{r'}}{\partial x^{r}} \frac{\partial x^{i}}{\partial \tilde{x}^{a}} \frac{\partial x^{j}}{\partial \tilde{x}^{b}} Q^{r}{}_{ab}}$

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Remark: $\Longrightarrow$ "variations" $\delta\Gamma^{r}{}_{ab}$ will behave tensorially!

- Define the action functional:

$$S[\psi] := \int_B \underbrace{\overbrace{L(\psi)}^{\text{scalar}} \overbrace{\sqrt{g}\, d^4x}^{\Omega = \text{volume form}}}_{n\text{-form}} \quad \in \mathbb{R}$$

B ← some bounded and closed 4-dim region in M.

Thus, each physical field $\psi(x,t)$ (as a function of both space and time) is mapped into a number $S[\psi]$.

- Action principle (or postulate) of classical physics:

In nature, physical fields ψ are such that $S[\psi]$ is extremal in the space of all fields ψ.

by specifying the so-called Lagrangian function, L, namely a scalar function of the matter fields $\psi_{(i)}{}^{a\dots b}{}_{c\dots d}$ and their first covariant derivatives, and now also of the metric g:

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$$\boxed{L(\psi) = L^{(\text{matter})}\left(\{\psi_{(i)}{}^{a\dots b}{}_{c\dots d}\}, \{\psi_{(i)}{}^{a\dots b}{}_{c\dots d;e}\}, g\right)}$$

- Thus: The matter fields ψ obey:

$$\boxed{\frac{\delta S[\psi]}{\delta \psi} = 0} \qquad (*)$$

These will be the eqns of motion for the fields ψ.

- Definition of $(*)$?

Def: A "variation $\delta\psi$" of the fields $\psi_{(i)}(p)$ ($p \in B \subset M$) in a region B is a one-parameter deformation, $\psi_{(i)}(\lambda, p)$, with $\lambda \in \overbrace{(-\varepsilon, \varepsilon)}^{\text{some finite interval}}$,

← deformation parameter

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and closed 4-dim region in M.

Thus, each physical field $\Psi(x,t)$ (as a function of both space and time) is mapped into a number $S[\Psi]$.

Action principle (or postulate) of classical physics:

In nature, physical fields Ψ are such that $S[\Psi]$ is extremal in the space of all fields Ψ.

so that

i.e. $\lambda = 0$ is non-deformation

1.) $\Psi_{(i)}(0,p) = \Psi_{(i)}(p) \quad \forall p \in M$

2.) $\Psi_{(i)}(\lambda,p) = \Psi_{(i)}(p) \quad \forall \lambda,$ if $p \in M - B$

i.e. no deformation at all outside region B.

Def: Then, we define:

$$\delta\Psi_{(i)}(p) := \left.\frac{\partial \Psi_{(i)}(\lambda,p)}{\partial\lambda}\right|_{\lambda=0}$$

Def: The action principle now reads:

$$\boxed{0 = \left.\frac{\partial S[\Psi]}{\partial\lambda}\right|_{\lambda=0} \quad \text{for all variations } \delta\Psi_{(i)}.}$$

These will be the eqns of motion for the fields Ψ.

Definition of (*)?

Def: A "variation $\delta\Psi$" of the fields $\Psi_{(i)}(p)$ ($p \in B \subset M$) in a region B is a one-parameter deformation, $\Psi_{(i)}(\lambda,p)$, with $\lambda \in \underbrace{(-\varepsilon,\varepsilon)}_{\text{some finite interval}}$,

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Evaluate:

$$0 = \left.\frac{\partial S}{\partial\lambda}\right|_{\lambda=0} = \sum_i \int_B \Bigg[\overbrace{\frac{\partial L}{\partial \Psi_{(i)}{}^{a\dots b}{}_{c\dots d}} \underbrace{\delta\Psi_{(i)}{}^{a\dots b}{}_{c\dots d}}_{\text{recall: } = \left.\frac{d\Psi_{(i)}{}^{a\dots b}{}_{c\dots d}}{d\lambda}\right|_{\lambda=0}}}^{\text{Term I}}$$

$$+ \overbrace{\underbrace{\frac{\partial L}{\partial \Psi_{(i)}{}^{a\dots b}{}_{c\dots d;e}}}_{\text{by assumption, } L \text{ depends also on the 1st cov. derivatives.}} \delta\left(\Psi_{(i)}{}^{a\dots b}{}_{c\dots d;e}\right)}^{\text{Term II}} \Bigg] \sqrt{g}\, d^4x$$

Evaluate terms I, II separately:

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$$+ \overbrace{\underbrace{\frac{\partial L}{\partial \Psi_{(i)}{}^{a\ldots b}{}_{c\ldots d;e}}}_{\substack{\text{by assumption,} \\ L \text{ depends also on} \\ \text{the 1st cov. derivatives.}}} \delta\left(\Psi_{(i)}{}^{a\ldots b}{}_{c\ldots d;e}\right)}^{\text{Term II}} \Bigg] \sqrt{g}\, d^4x$$

Evaluate terms I, II separately:

Term II:

We notice:

Recall: At origin of geodesic coordinate system, $\Gamma^k{}_{ij} = 0$, i.e. $\Psi_{;e} = \Psi_{,e}$. But then $\frac{\partial}{\partial x^i}$ and $\frac{\partial}{\partial \lambda}$ commute. True in any coordinate system.

$$\delta\left(\Psi_{(i)}{}^{a\ldots b}{}_{c\ldots d;e}\right) = \left(\delta\Psi_{(i)}{}^{a\ldots b}{}_{c\ldots d}\right)_{;e}$$

$$\Rightarrow \text{Term II} = \sum_i \int_B \frac{\partial L}{\partial \Psi_{(i)}{}^{a\ldots b}{}_{c\ldots d;e}} \left(\delta\Psi_{(i)}{}^{a\ldots b}{}_{c\ldots d}\right)_{;e} \sqrt{g}\, d^4x$$

$$= \sum_i \int_B \Bigg[\Big(\overbrace{\frac{\partial L}{\partial \Psi_{(i)}{}^{a\ldots b}{}_{c\ldots d;e}} \delta\Psi_{(i)}{}^{a\ldots b}{}_{c\ldots d}}^{=:K^e} \Big)_{;e}$$

(use Leibniz rule to verify)

$$- \left(\frac{\partial L}{\partial \Psi_{(i)}{}^{a\ldots b}{}_{c\ldots d;e}}\right)_{;e} \delta\Psi_{(i)}{}^{a\ldots b}{}_{c\ldots d} \Bigg] \sqrt{g}\, d^4x$$

One term is a "boundary term":

$$\sum_i \int_B K^e{}_{;e} \sqrt{g}\, d^4x$$

$$= \sum_i \int_B \mathrm{div}_\Omega K$$

$\Leftarrow$

Exercise:

show that for all ξ^μ:

$$\xi^\mu{}_{;\mu}\, \Omega = \mathrm{div}_\Omega \xi$$

if $\Omega = \sqrt{g}\, dx^1 \wedge \ldots \wedge dx^n$

Gauß' theorem $\Rightarrow$

$$= \sum_i \int_{\partial B} i_K \Omega$$

i_K — inner derivation

$$\left(\text{Recall: } \mathrm{div}_\Omega K = L_K \Omega = (i_K \circ d + d \circ i_K)\, \Omega = d \circ i_K \Omega\right)$$

but: $K \propto \delta\Psi$ and $\delta\Psi(p) = 0$ if $p \in \partial B$ by property 2) of variations.

$\Rightarrow \quad = 0$!

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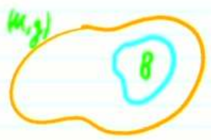

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inner derivation

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$\delta(\Psi_{(i)\ \ldots d;e}) = (\delta\Psi_{(i)\ \ldots d})_{;e}$

$$\Rightarrow \text{Term II} = \sum_i \int_B \frac{\partial L}{\partial \Psi_{(i)\ \ldots d;e}^{a\ldots b}} (\delta\Psi_{(i)\ \ldots d}^{a\ldots b})_{;e} \sqrt{g}\, d^4x$$

$$= \sum_i \int_B \Big[\Big(\underbrace{\frac{\partial L}{\partial \Psi_{(i)\ \ldots d;e}^{a\ldots b}} \delta\Psi_{(i)\ \ldots d}^{a\ldots b}}_{=:K^e} \Big)_{;e} - \Big(\frac{\partial L}{\partial \Psi_{(i)\ \ldots d;e}^{a\ldots b}}\Big)_{;e} \delta\Psi_{(i)\ \ldots d}^{a\ldots b} \Big] \sqrt{g}\, d^4x$$

(use Leibniz rule to verify)

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but: $K \propto \delta\Psi$ and $\delta\Psi(p) = 0$ if $p \in \partial B$ by property 2) of variations.

$\Rightarrow \quad = 0 \quad !$

Thus, term II simplifies and we obtain:

$$0 = \frac{\partial S}{\partial \lambda}\Big|_{\lambda=0} = \sum_i \int_B \Big[\overbrace{\frac{\partial L}{\partial \Psi_{(i)\ \ldots d}^{a\ldots b}} \delta\Psi_{(i)\ \ldots d}^{a\ldots b}}^{\text{Term I}} - \overbrace{\Big(\frac{\partial L}{\partial \Psi_{(i)\ \ldots d;e}^{a\ldots b}}\Big)_{;e} \delta\Psi_{(i)\ \ldots d}^{a\ldots b}}^{\text{Term II}} \Big] \sqrt{g}\, d^4x$$

Since must hold for all variations $\delta\Psi$

$$\Rightarrow \boxed{\frac{\partial L}{\partial \Psi_{(i)\ \ldots d}^{a\ldots b}} - \Big(\frac{\partial L}{\partial \Psi_{(i)\ \ldots d;e}^{a\ldots b}}\Big)_{;e} = 0}$$

"Euler-Lagrange equations"

Given $L(\Psi)$, these eqns yield the eqns. of motion for Ψ.

Example: A real-valued scalar field Ψ ← real-valued

- Such Ψ describe e.g.:
  - π^0 meson (quark + antiquark)
  - inflaton
- Lagrangian?
  - Choose geodesic cds at arb. point and appeal to equiv. principle.
  - Obtain from spec. relat. Lagrangian:

$$L = -\frac{1}{2}\Big(\Psi_{;a}\Psi_{;b}\, g^{ab} + \frac{m^2}{\hbar^2}\Psi^2\Big)$$

- Euler-Lagrange equation: Klein-Gordon equation

(Exercise: verify) $\boxed{\Psi_{;ab}\, g^{ab} - \frac{m^2}{\hbar^2}\Psi = 0}$

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Since must hold for all variations $\delta\Psi$

$$\Rightarrow \quad \boxed{\frac{\partial L}{\partial \Psi^{a\ldots b}_{;i\ldots d}} - \left(\frac{\partial L}{\partial \Psi^{a\ldots b}_{;i\ldots d;e}}\right)_{;e} = 0}$$

"Euler-Lagrange equations"

Given $L(\Psi)$, these eqns yield the eqns. of motion for Ψ.

- Choose geodesic cds at arb. point and appeal to equiv. principle.
- Obtain from spec. relat. Lagrangian:

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- Euler-Lagrange equation: Klein-Gordon equation

(Exercise: verify) $$\boxed{\Psi_{;ab}\, g^{ab} - \frac{m^2}{\hbar^2}\Psi = 0}$$

Example: The electromagnetic fields

- Assume there are no charges (i.e. there are only EM waves)
- Define the "EM 4-potential" as a real-number-valued one-form A.
- Consider the field strength tensor F:

$$F := dA$$

- Recall that the E and B fields are components of the 2-form F. (up to a factor of 2)

- The Lagrangian (from equiv. principle):

$$L = -\frac{1}{16\pi} F_{ab}F_{cd}\, g^{ac} g^{bd} \qquad \text{(Exercise: write in terms of forms)}$$

- Varying w. resp. to A, the E.L. equations read:

$$\boxed{F_{ab;c}\, g^{bc} = 0}$$ recall: this is $\delta F = 0$

"Maxwell eqns".

- It is also true that

$$F_{ab;c} + F_{ca;b} + F_{bc;a} = 0$$

but this is not an Euler Lagrange eqn. It is simply: $\boxed{dF = 0}$ (which holds because $F = dA$ and $d^2 = 0$)

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$$\boxed{F_{ab;c}\, g^{bc} = 0}$$

recall: this is $\delta F = 0$

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but this is not an Euler Lagrange eqn. It is simply: $\boxed{dF = 0}$ (which holds because $F = dA$ and $d^2 = 0$)

Example: A charged scalar field Ψ, ← complex-valued

(such Ψ describe, e.g., $\pi^{\pm}$ mesons)

together with electromagnetism.

- Equiv. principle yields from spec. relativity:

(Why Ψ complex?
Mixed term is Lorentz force
If Ψ was real, it would be absent:
$-ieA_a\Psi^* \Psi_{;b}\, g^{ab}$
$+ ieA_b\Psi^*_{;a}\Psi\, g^{ab}$
$= ieA_a g^{ab}(\Psi^*_{;a}\Psi - \Psi_{;a}\Psi^*)$
$= 0$ if $\Psi^* = \Psi$)

electric charge constant

$$L = -\frac{1}{2}\left(\Psi^*_{;a} - ieA_a\Psi^*\right)\left(\Psi_{;b} + ieA_b\Psi\right) g^{ab} - \frac{1}{2}\frac{m^2}{\hbar^2}\Psi^*\Psi - \frac{1}{16\pi} F_{ab}F_{cd}\, g^{ac} g^{bd}$$

- Vary w. resp. to Ψ^* ⇒ E.L. eqn:

$$\underbrace{\Psi_{;ab}\, g^{ab} - \frac{m^2}{\hbar^2}\Psi}_{\text{Klein Gordon part}} + \underbrace{ieA_a g^{ab}\left(\Psi_{;b} + ieA_b\Psi\right) + ieA_{a;b}\, g^{ab}\Psi}_{\Psi \text{ is affected by } A} = 0$$

and varying w. resp. to Ψ yields the compl. conj. equation.

- Vary w. resp. to A_a ⇒ E.L. eqn:

$$\underbrace{\frac{1}{4\pi} F_{ab;c}\, g^{bc}}_{\text{plain Maxwell part}} \underbrace{- ie\Psi\left(\Psi^*_{;a} - ieA_a\Psi^*\right) + ie\Psi^*\left(\Psi_{;a} + ieA_a\Psi\right)}_{A \text{ is affected by } \Psi, \Psi^*} = 0$$

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Equiv. principle yields from spec. relativity:

Why Ψ complex?
Mixed term is Lorentz force
If Ψ was real, it would be absent:

$$-ieA_a\Psi^*\Psi_{;b}\,g^{ab} + ieA_b\Psi^*_{,a}\Psi\,g^{ab} = ieA_a g^{ab}(\Psi^*_{,a}\Psi - \Psi_{,a}\Psi^*) = 0 \text{ if } \Psi^* = \Psi$$

electric charge constant

$$L = -\frac{1}{2}\left(\Psi^*_{;a} - ieA_a\Psi^*\right)\left(\Psi_{;b} + ieA_b\Psi\right)g^{ab} - \frac{1}{2}\frac{m^2}{\hbar^2}\Psi^*\Psi - \frac{1}{16\pi}F_{ab}F_{cd}\,g^{ac}g^{bd}$$

and varying w. resp. to Ψ yields the compl. conj. equation.

Vary w. resp. to $A_a \Rightarrow$ E.L. eqn:

$$\underbrace{\frac{1}{4\pi}F_{ab;c}\,g^{bc}}_{\text{plain Maxwell part}} \underbrace{- ie\Psi\left(\Psi^*_{;a} - ieA_a\Psi^*\right) + ie\Psi^*\left(\Psi_{;a} + ieA_a\Psi\right)}_{A \text{ is affected by } \Psi, \Psi^*.} = 0$$

Dirac equation: (Brief treatment of basics only of Dirac spinors)

In special relativity: (with units such that $\hbar = 1$)

$$\boxed{\left(i\,\gamma^\mu\frac{\partial}{\partial x^\mu} - m\right)\Psi(x) = 0} \quad \text{"Dirac equation"} \quad \text{(D)}$$

where $\Psi = \begin{pmatrix}\Psi_1\\\Psi_2\\\Psi_3\\\Psi_4\end{pmatrix}$ is a "Spinor"

describes spin ½ particles such as electrons and quarks

and the four 4×4 matrices γ^μ obey:

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu} \quad (*)$$

$$\eta^{\mu\nu} = \begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}^{\mu\nu}$$

Why (*)? Equation (*) is specifically chosen so that each component of Ψ obeys the Klein Gordon equation. Indeed:

$$\text{(D)} \Rightarrow \left(-i\gamma^\mu\partial_\mu - m\right)\left(i\gamma^\nu\partial_\nu - m\right)\Psi = 0$$

$$\Rightarrow \left(+\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu + i\gamma^\mu\partial_\mu m - im\gamma^\nu\partial_\nu + m^2\right)\Psi = 0$$

$$\Rightarrow \left(\underbrace{\gamma^\mu\gamma^\nu}_{\text{antisymmetric part not needed, it would drop out.}}\overbrace{\partial_\mu\partial_\nu}^{\text{symmetric under } \mu\leftrightarrow\nu} + m^2\right)\Psi = 0$$

$$\Rightarrow \left(\frac{1}{2}\left(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu\right)\partial_\mu\partial_\nu + m^2\right)\Psi = 0$$

$$\overset{(*)}{\Rightarrow} \mathbb{1}\left(\eta^{\mu\nu}\partial_\mu\partial_\nu + m^2\right)\Psi = 0$$

which is the Klein Gordon equation in flat space.

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$$\boxed{\left(i\,\gamma^\mu \frac{\partial}{\partial x^\mu} - m\right)\Psi(x) = 0}$$ "Dirac equation" (D)

where $\Psi = \begin{pmatrix}\Psi_1\\ \Psi_2\\ \Psi_3\\ \Psi_4\end{pmatrix}$ is a "Spinor"

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and the four 4×4 matrices γ^μ obey:

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\,\eta^{\mu\nu} \qquad (*)$$

$\eta^{\mu\nu} = \begin{pmatrix}-1 & & 0\\ & 1 & \\ 0 & & 1\end{pmatrix}^{\mu\nu}$

(D) $\Rightarrow \left(-i\gamma^\mu\partial_\mu - m\right)\left(i\gamma^\nu\partial_\nu - m\right)\Psi = 0$

$$\Rightarrow \left(+\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu + i\gamma^\mu\partial_\mu m - im\gamma^\nu\partial_\nu + m^2\right)\Psi = 0$$

symmetric under $\mu \leftrightarrow \nu$

$$\Rightarrow \quad \left(\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu + m^2\right)\Psi = 0$$

anti-symmetric part not needed, it would drop out.

$$\Rightarrow \quad \left(\tfrac{1}{2}\left(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu\right)\partial_\mu\partial_\nu + m^2\right)\Psi = 0$$

$$\overset{(*)}{\Rightarrow} \quad 1\left(\eta^{\mu\nu}\partial_\mu\partial_\nu + m^2\right)\Psi = 0$$

which is the Klein Gordon equation in flat space.

In general relativity:

- By choosing an orthonormal tetrad, $\{\theta^i\}$, we achieve
$$g^{\mu\nu} = \eta^{\mu\nu} \quad \forall p \in M$$
i.e. one set of matrices γ^μ obeying $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}$ suffices.

- This motivates:
$$\left(i\gamma^\mu\nabla_\mu - m\right)\Psi = 0$$

- But what is the covariant derivative of a spinor?
$$\nabla_{e_\mu}\Psi = ?$$

Recall: The covariant derivative of a vector yields the infinitesimal Lorentz transformation by which the vector rotates under infinitesimal parallel transport.

Idea: The covariant derivative of a spinor should yield the rotation of the spinor by the same infinitesimal Lorentz transformation.

Recall: Infinitesimal parallel transport of a vector e_ξ in direction e_μ:

$$e_\xi \rightarrow e_\xi + \nabla_{e_\mu} e_\xi = e_\xi + \omega^\sigma_\xi(e_\mu)\,e_\sigma$$

Recall: the 1-form takes values that are infinitesimal Lorentz transformations.

(Recall intuition why parallel transport yields Lorentz transformation: Parallel transport preserves the lengths of vectors, i.e. they can at most "rotate" and in 3+1 dim this is Lorentz transformations.)

This is an infinitesimal Lorentz transformation $\Lambda_\xi{}^\sigma$:

$$e_\xi \rightarrow \Lambda_\xi^\sigma e_\sigma \quad \text{with} \quad \Lambda^\sigma_\xi = \delta^\sigma_\xi + \omega^\sigma_\xi(e_\mu)$$

because ω^σ_ξ obeys: $\omega_{\sigma\xi} = -\omega_{\xi\sigma}$. (Which is the defining equation for infinit. Lorentz transformation)

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i.e. one set of matrices γ^μ obeying $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu$ …
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because ω_ρ^σ obeys: $\omega_{\sigma\rho} = -\omega_{\rho\sigma}$. (Which is the defining equation for infinitesimal Lorentz transformations)

Now that we know the inf. Lorentz transf. for any inf. parallel transport:

⤳ Strategy: Apply the same inf. Lorentz transformation on spinors for their parallel transport.

To this end: Recall from Special Relativity how an infinitesimal Lorentz transformation acts on a spinor:

- Assume $\{s_i\}_{i=1}^4$ are ON basis in Spinor space, i.e.

$$\Psi = \Psi^i(x)\, s_i$$

these are Spinor indices: $i = 1,2,3,4$

- How do the s_i transform under Lorentz transformations? I.e., what is $\nabla_{e_\mu} s_i = ?$ (In analogy to $\nabla_{e_a} e_\mu = \omega^\nu{}_\mu(e_a) e_\nu$)

- From special relativity it is known that under infinitesimal Lorentz transformations,

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu$$

vectors transform as

$$e_\mu \to e_\mu + \omega_\mu^\nu e_\nu$$

and the Dirac spinors transform as:

$$s_i \to s_i \underbrace{- \tfrac{1}{4}\omega_\mu{}^\nu [\gamma^\mu, \gamma_\nu]\, s_i}$$

⇒ under infinitesimal Lorentz transf. the spinor "rotates" by this amount.

Where does $[\gamma^\mu, \gamma^\nu]$ come from?

Recall that e.g. translations in space are generated by momentum operators, $e^{-i\vec{a}\hat{p}} f(x) e^{i\vec{a}\hat{p}} = f(x+\vec{a})$, if they obey the commutation relations $[x_i, p_j] = i\delta_{ij}$.

Similarly, Lorentz transformations are generated by operators $M^{\mu\nu}$: antisym. antisym.

$$e^{-i\omega_{\mu\nu} M^{\mu\nu}} f\, e^{i\omega_{\mu\nu} M^{\mu\nu}} = \Lambda(f)$$

if these $M^{\mu\nu}$ obey certain commutation relations. In spinor space, the unique objects that obey these commutation relations are the $M^{\mu\nu} = [\gamma^\mu, \gamma^\nu]$.

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Lorentz transformation acts on a spinor:

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Similarly, Lorentz transformations are generated by operators $M^{\mu\nu}$: antisym. antisym.

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$$e_\mu \to e_\mu + \omega_\mu^\nu e_\nu$$

and the Dirac spinors transform as:

$$s_i \to s_i \underbrace{- \tfrac{1}{4}\omega_\mu^\nu [\gamma^\mu, \gamma_\nu]\, s_i}$$

$\Rightarrow$ under infinitesimal Lorentz transf. the spinor "rotates" by this amount.

Apply to GR:

If a vector e_μ is infinitesimally parallel transported in the direction of e_a then it obtains an infinitesimal "rotation", namely the infinitesimal Lorentz transformation

$$\omega^\mu{}_\nu(e_a)$$

which is the value of the connection 1-form, i.e.:

$$e_\mu \to e_\mu + \overbrace{\omega_\mu^\nu(e_a)}^{\text{local value of the connection form}} e_\nu$$

$\longrightarrow$ From this one can immediately read off again the covariant derivative for vectors:

$$\nabla_{e_a} e_\mu = \omega^\nu{}_\mu(e_a) e_\nu$$

- Now, when a spinor s_i is infinitesimally parallel transported in the direction of e_a then it too experiences the infinitesimal rotation, i.e., the infinitesimal Lorentz transformation

$$\omega^\mu{}_\nu(e_a)$$

which is the value of the connection 1 form. Thus:

$$s_i \to s_i - \tfrac{1}{4} \overbrace{\omega(e_a)_\mu^\nu}^{\text{local infinitesimal Lorentz transformation, i.e., local value of the connection 1-form.}} [\gamma^\mu, \gamma_\nu]\, s_i$$

- Since, under infinitesimal parallel transport:

$$s_i \to s_i + \underbrace{\nabla_{e_a} s_i}_{\text{to be determined}}$$

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which is the value of the connection 1-form, i.e.:

$$e_\mu \to e_\mu + \underbrace{\omega_\mu^{\ \nu}(e_a)}_{\text{local value of the connection form}} e_\nu$$

→ From this one can immediately read off again the covariant derivative for vectors:

$$\nabla_{e_a} e_\mu = \omega^{\ \nu}_{\mu}(e_a) e_\nu$$

⇒ The covariant derivative of the basis vectors $\{s_i\}$ of Dirac spinors is:

$$\nabla_{e_a} s_i = -\frac{1}{4}\omega_\mu^{\ \nu}(e_a)[\gamma^\mu, \gamma_\nu] s_i$$

⇒ For general Dirac spinors $\Psi(x) = \Psi^i(x) s_i$ the Leibniz rule for ∇ yields:

scalar coefficient functions

$$\nabla_{e_a}\Psi = \nabla_{e_a}(\Psi^i(x) s_i) = (\nabla_{e_a}\Psi^i(x)) s_i + \Psi^i(x)\nabla_{e_a} s_i$$

i.e.:

$$\boxed{\nabla_{e_a}\Psi = e_a(\Psi) - \frac{1}{4}\omega(e_a)_\mu^{\ \nu}[\gamma^\mu, \gamma_\nu]\Psi}$$

$e_a(\Psi) = s_i\, e_a(\Psi^i)$ — function — vector field

which is the value of the connection 1 form. Thus:

$$s_i \to s_i - \frac{1}{4}\underbrace{\omega(e_a)_\mu^{\ \nu}}_{\text{local infinitesimal Lorentz transformation, i.e., local value of the connection 1-form.}}[\gamma^\mu, \gamma_\nu] s_i$$

□ Since, under infinitesimal parallel transport:

$$s_i \to s_i + \underbrace{\nabla_{e_a} s_i}_{\text{to be determined}}$$

Dirac equation:

The general relativistic Dirac equation

$$\boxed{(i\gamma^\mu \nabla_{e_\mu} - m)\Psi = 0}$$

now takes this explicit form:

$$i\gamma^\mu \underbrace{e_\mu(\Psi)}_{} - i\frac{1}{4}\omega(e_\mu)^\nu_{\ \rho}\gamma^\mu[\gamma^\rho, \gamma_\nu]\Psi - m\Psi = 0$$

in a chart, this becomes a directional derivative of Ψ.

Remark: The relationship between the Dirac operator $D = i\gamma^\mu\nabla_{e_\mu}$ and the Laplace or d'Alembert operator $\Box$ also becomes:

$$D = d + \delta.$$

To this end, one re-interprets the Grassmann algebra of differential forms as a so-called Clifford algebra.