Title: General Relativity for Cosmology Lecture - 101923

Speakers: Achim Kempf

Collection: General Relativity for Cosmology

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GR for Cosmology, Achim Kempf — Lecture 11

Recall: The nontrivial shape of a manifold reveals itself in several ways:

1. Violation of angle sum law, $\alpha + \beta + 90° \neq 180°$.
   ⇝ (can encode shape through deficit angles (used in some quantum gravity approaches))

2. Violation of Pythagoras' law, $a^2 + b^2 \neq c^2$.
   ⇝ Can encode shape through metric distances: $(\mathcal{M}, g)$

3. Nontrivial parallel transport of vectors on loops.
   ⇝ Can encode shape through affine connection: $(\mathcal{M}, \Gamma)$

Observe: Such local descriptions carry redundant information!

This makes it hard to identify the true degrees of freedom, so that they can be quantized.

Why? Two (pseudo-)Riemannian mflds $(\mathcal{M}, g)$, $(\mathcal{M}, g')$ must be considered equivalent, i.e., they are describing the same space(-time), if there exists an isometric, i.e., metric-preserving, isomorphism:

$$\boxed{\varphi: (\mathcal{M}, g) \to (\mathcal{M}, g')}$$

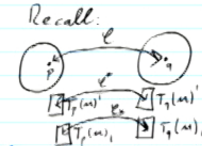
Here: φ is called metric-preserving if, under the pull-back map

$$T\varphi^*: T_p(M)_2 \to T_p(M)_2$$

the metric obeys:

$$T\varphi_*(g) = g'$$

Recall:



⇝ φ can then be considered to be a mere change of chart.

Intuition: $(\mathcal{M}, g)$, $(\mathcal{M}, g')$ that are related by an isometric diffeomorphism are mere cd changes of another, i.e., have the same "shape".

Definition: A (pseudo-) Riemannian structure, say Ξ, is an equivalence class of (pseudo-) Riemannian manifolds which can be mapped into each other via metric-preserving diffeomorphisms, i.e., via changes of coordinates.

⇒ Space(time) will need to be modelled as a (pseudo-)Riemannian

⇒ One would like to be able to reliably identify exactly one representative $(\mathcal{M}, g)$ per class Ξ.

- This would be called a "fixing of gauge".
- Why would this be useful?

A key example of when gauge fixing needed: Quantum gravity

1. Violation of angle sum law, $\alpha + \beta + 90° \neq 180°$.
   ⇝ Can encode shape through deficit angles (used in some quantum gravity approaches)
2. Violation of Pythagoras' law, $a^2 + b^2 \neq c^2$.
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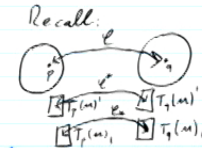

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Definition: A (pseudo-) Riemannian structure, say Ξ, is an equivalence class of (pseudo-) Riemannian manifolds which can be mapped into each other via metric-preserving diffeomorphisms, i.e., via changes of coordinates.

⇒ Space(time) will need to be modelled as a (pseudo-)Riemannian structure, Ξ, i.e., as an equivalence class of pairs $(\mathcal{M}, g)$.

Problem: These equiv. classes are hard to handle because absence or existence of φ is hard to check!

⇒ One would like to be able to reliably identify exactly one representative $(\mathcal{M}, g)$ per class Ξ.

□ This would be called a "fixing of gauge".

□ Why would this be useful?

A key example of when gauge fixing needed: Quantum gravity

We discussed detecting and describing shape through

- deficiency angles
- nontrivial metric distances $(\mathcal{M}, g)$
- nontrivial parallel transport $(\mathcal{M}, \Gamma)$

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diffeomorphisms, i.e., via changes of coordinates.

$\Rightarrow$ $\boxed{\text{Space(-time) will need to be modelled as a (pseudo-)Riemannian structure, } \Xi \text{, i.e., as an equivalence class of pairs } (M,g).}$

Problem: These equiv. classes are hard to handle because absence or existence of ℓ is hard to check!

Recall: Quantum theory can be formulated in path integral form.

Applied to gravity:

Expect to have to handle path integrals of the type:

$$\int_{\text{"all Riemannian structures } \Xi\text{"}} e^{iS'(\Xi)}\, D\Xi$$

But what we initially have is, roughly of the form:

$$\int_{\text{"all } g\text{"}} e^{iS'(g)}\, \delta(?)\, Dg \quad \text{or} \quad \int_{\text{"all } \Gamma\text{"}} e^{iS'(\Xi)}\, \delta(?)\, D\Gamma$$

Here, $\delta(?)$ should be such that from each equivalence class of the g's or the Γ's only exactly one contributes to the path integral.

A key example of when gauge fixing needed: Quantum gravity

We discussed detecting and describing shape through

- deficiency angles
- nontrivial metric distances (M,g)
- nontrivial parallel transport (M,Γ)

$\rightsquigarrow$ Much of Quantum Gravity research is concerned with working out suitable $\delta(?)$ for g's or Γ's or other variables formed from them, such as the frame fields (see "Loop quantum gravity").

Q: Can one detect and describe a (pseudo-) Riemannian structure Ξ directly?

A: Possibly yes, using "Spectral Geometry":

Idea: A manifold's vibration spectrum $\underbrace{\{\lambda_n\}}_{\text{Independent of coordinate systems!}}$ depends only on Ξ!

Key question of the field of spectral geometry: (Weyl 1911)

$\boxed{\text{Does the spectrum } \{\lambda_n\} \text{ encode all about the shape, i.e., } \Xi\,?}$

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A: Possibly yes, using "Spectral Geometry":

Independent of coordinate systems!

Idea: A manifold's vibration spectrum $\{\lambda_n\}$ depends only on Ξ!

Key question of the field of spectral geometry: (Weyl 1911)

Does the spectrum $\{\lambda_n\}$ encode **all** about the shape, i.e., Ξ?

Remarks:

- It cannot, if $\mathcal{M}$ has infinite volume, because then the spectrum of Δ will become (almost) completely continuous.
- The spectral geometry of pseudo-Riemannian manifolds is still very little developed.

Theorem:

- Assume $(\mathcal{M}, g)$ is a compact Riemannian manifold without boundary, $\partial\mathcal{M} = \phi$. (implies finite volume)
- Then, each $\operatorname{spec}(\Delta_p)$ is discrete, with finite degeneracies and without accumulation points.

In practice:

We can describe any arbitrarily large part of the universe by a compact Riemannian manifold, $(\mathcal{M}, g)$.

This allows us to describe, e.g., 3-dim. space at any fixed time (or also 4-dim. spacetime after so-called Wick rotation).

Types of waves (incl. sounds) on $\mathcal{M}$:

assumed compact, no boundary

Consider p-form fields $\omega(x)$ on $\mathcal{M}$, with time evolution, e.g.:

1. Schrödinger equation: $i\hbar\partial_t\,\omega(x,t) = -\frac{\hbar^2}{2m}\Delta_p\,\omega(x,t)$
2. Heat equation: $\partial_t\,\omega(x,t) = -\alpha\,\Delta_p\,\omega(x,t)$
3. Klein Gordon (and acoustic) eqn: $-\partial_t^2\,\omega(x,t) = \beta\,\Delta_p\,\omega(x,t)$

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manifolds is still very little developed.

Theorem:

- Assume $(\mathcal{M}, g)$ is a compact (implies finite volume) Riemannian manifold without boundary, $\partial\mathcal{M} = \phi$.
- Then, each $spec(\Delta_p)$ is discrete, with finite degeneracies and without accumulation points.

Types of waves (incl. sounds) on $\mathcal{M}$:

Consider p-form fields $\omega(x)$ on $\mathcal{M}$ (assumed compact, no boundary), with time evolution, e.g.:

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3. Klein Gordon (and acoustic) eqn: $-\partial_t^2\omega(x,t) = \beta\Delta_p\omega(x,t)$

- Each of them can be solved via separation of variables:
- Assume we find an eigenform $\tilde{\omega}(x)$ of Δ on $\mathcal{M}$:

$$\boxed{\Delta_p\tilde{\omega}(x) = \lambda\,\tilde{\omega}(x)}$$

- They exist: Each Δ is self-adjoint, w.r.t. the inner product $(\omega, \nu) = \int_\mathcal{M} \omega\wedge *\nu$.

Then: Schrödinger eqn solved by: $\omega(x,t) := e^{\frac{i\hbar}{2m}\lambda t}\,\tilde{\omega}(x)$

Heat eqn solved by: $\omega(x,t) := e^{-\alpha\lambda t}\,\tilde{\omega}(x)$

Klein Gordon eqn solved by: $\omega_\pm(x,t) := e^{\pm i\sqrt{\beta\lambda}\,t}\,\tilde{\omega}(x)$

$\Rightarrow$ The spectrum $spec(\Delta_p)$ is the overtone spectrum of p-form type waves on the manifold $\mathcal{M}$.

Properties of $spec(\Delta_p)$:

- **Expectation:**

  The spectra $spec(\Delta_p)$ for different p carry different information about $\mathcal{M}$:

  E.g., scalar and vector seismic waves travel (and reflect) differently.

- **But recall also:**
  a) $[\Delta, *] = 0$
  b) $[\Delta, d] = 0$
  c) $[\Delta, \delta] = 0$

  This will relate $spec(\Delta_p)$ to $spec(\Delta_{n-p})$, $spec(\Delta_{p+1})$ and $spec(\Delta_{p-1})$:

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- They exist: Each Δ is self-adjoint, w.r.t. the inner product $(\omega,\nu)=\int_M \omega\wedge *\nu$.

Then: Schrödinger eqn solved by: $\omega(x,t) := e^{\frac{i\hbar}{2m}\lambda t}\,\tilde{\omega}(x)$

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$\Rightarrow$ The spectrum spec(Δ_p) is the overtone spectrum of p-form type waves on the manifold M.

E.g., scalar and vector seismic waves travel (and reflect) differently.

- **But recall also:**
  a) $[\Delta, *]=0$
  b) $[\Delta, d]=0$
  c) $[\Delta, \delta]=0$

This will relate spec(Δ_p) to spec(Δ_{n-p}), spec(Δ_{p+1}) and spec(Δ_{p-1}):

Use $[\Delta,*]=0$:

Assume: $\omega\in\Lambda_p$ and $\Delta\omega=\lambda\omega$.

Define: $\nu := *\omega \in \Lambda_{n-p}$

Then:

$$\Delta\nu = \Delta*\omega = *\Delta\omega = *\lambda\omega = \lambda\nu$$

$$\Rightarrow \boxed{\text{spec}(\Delta_p)=\text{spec}(\Delta_{n-p})}$$

Next:

Careful utilization of $[\Delta,d]=0$ and $[\Delta,\delta]=0$ yields much more information about these spectra!

- Notice that: Δ maps exact forms $\omega=d\nu$ into exact forms:

$$\Delta\omega = \Delta d\nu = \overbrace{d\Delta\nu}^{\text{an exact form}}$$

i.e.: $\boxed{\Delta:\ d\Lambda_r \rightarrow d\Lambda_r}$ $d\Lambda_r$ = image of Λ_r under d.

- Analogously: Δ maps co-exact forms $\omega=\delta\beta$ into co-exact forms:

$$\Delta\omega = \Delta\delta\beta = \overbrace{\delta\Delta\beta}^{\text{a co-exact form}}$$

i.e.: $\boxed{\Delta:\ \delta\Lambda_r \rightarrow \delta\Lambda_r}$

- Also: Δ can map forms into 0, namely its eigenspace with eigenvalue 0, denoted Λ_r^0. Λ_r^0 is called the space of "harmonic" p-forms.

$\boxed{\Delta:\ \Lambda_r^0 \rightarrow 0}$

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Then:

$$\Delta v = \Delta * \omega = * \Delta \omega = * \lambda \omega = \lambda v$$

$$\Rightarrow \boxed{\mathrm{spec}(\Delta_p) = \mathrm{spec}(\Delta_{n-p})}$$

Next:

Careful utilization of $[\Delta, d] = 0$ and $[\Delta, \delta] = 0$ yields much more information about these spectra!

$$\Delta \omega = \Delta \delta \beta = \underbrace{\delta \Delta \beta}_{\text{a co-exact form}}$$

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Also: Δ can map forms into 0, namely its eigenspace with eigenvalue 0, denoted Λ_p^0. Λ_p^0 is called the space of "harmonic" p-forms.

$$\boxed{\Delta : \Lambda_p^0 \to 0}$$

Thus: Δ maps $d\Lambda_p$ and $\delta\Lambda_p$ and Λ_p^0 into themselves.

Are there any other forms that Δ could act on? No!

Proposition ("Hodge decomposition"):

$$\boxed{\Lambda_p = d\Lambda_{p-1} \oplus \delta \Lambda_{p+1} \oplus \Lambda_p^0}$$

(Recall that $\oplus$ implies that the three spaces are orthogonal!)

Q: Why useful?

A: It means that every eigenvector of Δ_p is either in $d\Lambda_{p-1}$ or in $\delta\Lambda_{p+1}$ or in Λ_p^0 but is never a linear combination of vectors in these spaces.

Proof: It is clear that $d\Lambda_{p-1} \subset \Lambda_p$ and $\delta\Lambda_{p+1} \subset \Lambda_p$.

We need to show the orthogonalities and completeness:

- Show that $d\Lambda_{p-1} \perp \delta\Lambda_{p+1}$:

  Indeed, assume $\omega = dv \in \Lambda_p$ and $\alpha = \delta\beta \in \Lambda_p$.

  Then: $(\omega, \alpha) = (dv, \delta\beta) \underset{d^t = \delta}{\overset{\text{use}}{=}} (\overbrace{d d}^{0} v, \beta) = 0$ ✓

Exercise: study the remainder of the proof.

- Show that if $\omega \in \Lambda_p$ and $\omega \perp d\Lambda_{p-1}$ and $\omega \perp \delta\Lambda_{p+1}$ then: $\omega \in \Lambda_p^0$.

  Indeed, assume $\omega \perp d\Lambda_{p-1}$ and $\omega \perp \delta\Lambda_{p+1}$. Then:

  $\forall \alpha: (d\alpha, \omega) = 0$ i.e. $-(\alpha, \delta\omega) = 0 \Rightarrow \delta\omega = 0$

  $\forall \beta: (\delta\beta, \omega) = 0$ i.e. $-(\beta, d\omega) = 0 \Rightarrow d\omega = 0$

  $\Rightarrow \Delta\omega = (d\delta + \delta d)\omega = 0 \Rightarrow \omega \in \Lambda_p^0$ ✓

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$$\boxed{\Lambda_p = d\Lambda_{p-1} \oplus \delta\Lambda_{p+1} \oplus \Lambda_p^0}$$

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Indeed, assume $\omega = d\nu \in \Lambda_p$ and $\alpha = \delta\beta \in \Lambda_p$.

Then: $(\omega, \alpha) = (d\nu, \delta\beta) \underset{d^t = \delta}{\overset{\text{use}}{=}} (\overbrace{dd}^{0}\nu, \beta) = 0$ ✓

Exercise: study the remainder of the proof.

□ Show that if $\omega \in \Lambda_p$ and $\omega \perp d\Lambda_{p-1}$ and $\omega \perp \delta\Lambda_{p+1}$ then: $\omega \in \Lambda_p^0$.

Indeed, assume $\omega \perp d\Lambda_{p-1}$ and $\omega \perp \delta\Lambda_{p+1}$. Then:

$\forall \alpha$: $(d\alpha, \omega) = 0$ i.e. $-(\alpha, \delta\omega) = 0 \Rightarrow \delta\omega = 0$

$\forall \beta$: $(\delta\beta, \omega) = 0$ i.e. $-(\beta, d\omega) = 0 \Rightarrow d\omega = 0$

$\Rightarrow \Delta\omega = (d\delta + \delta d)\omega = 0 \Rightarrow \omega \in \Lambda_p^0$ ✓

□ Show that if $\omega \in \Lambda_p^0$ then $\omega \perp d\Lambda_{p-1}$ and $\omega \perp \delta\Lambda_{p+1}$.

Assume $\omega \in \Lambda_p^0$, i.e., $\Delta\omega = 0$, i.e., $(\delta d + d\delta)\omega = 0$.

$\Rightarrow (\omega, (d\delta + \delta d)\omega) = 0$

$\Rightarrow \overbrace{(\delta\omega, \delta\omega)}^{\geq 0} + \overbrace{(d\omega, d\omega)}^{\geq 0} = 0 \Rightarrow \delta\omega = 0$ and $d\omega = 0$.

(I.e., harmonic forms are closed and co-closed but not exact or co-exact. Thus, $B_p := \dim(\Lambda_p^0)$ measures topological nontriviality. The B_p are called the "Betti numbers".)

$\Rightarrow \forall \alpha \in \Lambda_{p-1}$: $(\alpha, \delta\omega) = 0$, i.e., $(d\alpha, \omega) = 0$.

$\Rightarrow \omega \perp d\Lambda_{p-1}$ ✓

Also: $\forall \beta \in \Lambda_{p+1}$: $(\beta, d\omega) = 0$, i.e., $(\delta\beta, \omega) = 0$.

$\Rightarrow \omega \perp \delta\Lambda_{p+1}$ ✓

Conclusion so far:

In the Hodge decomposition, Δ maps every term into itself, i.e., Δ can be diagonalized in each $d\Lambda_r, \delta\Lambda_r, \Lambda_r^0$ separately.

$$\begin{cases} \quad\vdots \\ \Lambda_{p-1} = d\Lambda_{p-2} \oplus \delta\Lambda_p \oplus \Lambda_{p-1}^0 \\ \Lambda_p = d\Lambda_{p-1} \oplus \delta\Lambda_{p+1} \oplus \Lambda_p^0 \\ \Lambda_{p+1} = d\Lambda_p \oplus \delta\Lambda_{p+2} \oplus \Lambda_{p+1}^0 \\ \quad\vdots \end{cases}$$

$\Rightarrow$ Δ has eigenvectors and -values on each of these subspaces, for all r:

$\mathrm{spec}(\Delta|_{d\Lambda_r})$, $\mathrm{spec}(\Delta|_{\delta\Lambda_r})$, $\mathrm{spec}(\Delta|_{\Lambda_r^0}) = \{0\}$ ...

These spectra are related!

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$\Rightarrow \underbrace{(\delta\omega,\delta\omega)}_{\geq 0} + \underbrace{(d\omega,d\omega)}_{\geq 0} = 0 \Rightarrow \delta\omega = 0$ and $d\omega = 0$.

(I.e., harmonic forms are closed and co-closed but not exact or co-exact. Thus, $B_p := \dim(\Lambda^\circ_p)$ measures topological nontriviality. The B_p are called the "Betti numbers".)

$\Rightarrow \forall \alpha \in \Lambda_{p-1}: \quad (\alpha, \delta\omega) = 0$, i.e., $(d\alpha, \omega) = 0$.

$\Rightarrow \omega \perp d\Lambda_{p-1}$ ✓

Also: $\forall \beta \in \Lambda_{p+1}: (\beta, d\omega) = 0$, i.e., $(\delta\beta, \omega) = 0$.

$\Rightarrow \omega \perp \delta\Lambda_{p+1}$ ✓

itself, i.e., Δ can be diagonalized in each $d\Lambda_r, \delta\Lambda_r, \Lambda^\circ_r$ separately.

$$\Lambda_{p+1} = d\Lambda_p \oplus \delta\Lambda_{p+2} \oplus \Lambda^\circ_{p+1}$$
$$\vdots$$

$\Rightarrow \Delta$ has eigenvectors and -values on each of these subspaces, for all r:

$$\mathrm{spec}(\Delta|_{d\Lambda_r}), \quad \mathrm{spec}(\Delta|_{\delta\Lambda_r}), \quad \mathrm{spec}(\Delta|_{\Lambda^\circ_r}) = \{0\} \ \ldots$$

These spectra are related!

Proposition: $\mathrm{spec}(\Delta|_{d\Lambda_r}) = \mathrm{spec}(\Delta|_{\delta\Lambda_{r+1}})$

and for each eigenvector in one there is one in the other.

This means:

$$\vdots$$
$$\Lambda_{p-1} = d\Lambda_{p-2} \oplus \delta\Lambda_p \oplus \Lambda^\circ_{p-1}$$
$$\Lambda_p = d\Lambda_{p-1} \oplus \delta\Lambda_{p+1} \oplus \Lambda^\circ_p$$
$$\Lambda_{p+1} = d\Lambda_p \oplus \delta\Lambda_{p+2} \oplus \Lambda^\circ_{p+1}$$
$$\vdots$$

same spectrum

same spectrum

Proof:

Assume: $\lambda \in \mathrm{spec}(\Delta|_{d\Lambda_r})$ with eigenvector $\omega \in d\Lambda_r$.

Define: $v := \delta\omega \in \delta\Lambda_{r+1}$

Then: $\Delta v = \Delta\delta\omega = \delta\Delta\omega = \lambda\delta\omega = \lambda v$

$\Rightarrow \lambda \in \mathrm{spec}(\Delta|_{\delta\Lambda_{r+1}})$ and v is the eigenvector.

Conversely:

Assume: $\lambda \in \mathrm{spec}(\Delta|_{\delta\Lambda_{r+1}})$ with eigenvector $\omega \in \delta\Lambda_{r+1}$.

Define: $v := d\omega \in d\Lambda_r$

Then: $\Delta v = \Delta d\omega = d\Delta\omega = \lambda d\omega = \lambda v$

$\Rightarrow \lambda \in \mathrm{spec}(\Delta|_{d\Lambda_r})$ and v is the eigenvector.

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This means:

$$\Lambda_{p-1} = \underbrace{d\Lambda_{p-2}} \oplus \underbrace{\delta\Lambda_p} \oplus \Lambda^{\circ}_{p-1}$$

same spectrum

$$\Lambda_p = \underbrace{d\Lambda_{p-1}} \oplus \underbrace{\delta\Lambda_{p+1}} \oplus \Lambda^{\circ}_p$$

same spectrum

$$\Lambda_{p+1} = \underbrace{d\Lambda_p} \oplus \underbrace{\delta\Lambda_{p+2}} \oplus \Lambda^{\circ}_{p+1}$$

$\Rightarrow \lambda \in spec\left(\Delta|_{\delta\Lambda_{r+1}}\right)$ and v is the eigenvector.

Conversely:

Assume: $\lambda \in spec\left(\Delta|_{\delta\Lambda_{r+1}}\right)$ with eigenvector $w \in \delta\Lambda_{r+1}$.

Define: $v := dw \in d\Lambda_r$

Then: $\Delta v = \Delta d w = d\Delta w = \lambda dw = \lambda v$

$\Rightarrow \lambda \in spec\left(\Delta|_{d\Lambda_r}\right)$ and v is the eigenvector. ✓

Re-use $[\Delta, *]=0$:

Proposition: $* : \boxed{\underbrace{d\Lambda_r}_{\cap\, \Lambda_{r+1}} \to \underbrace{\delta\Lambda_{n-r}}_{\cap\, \Lambda_{n-r-1}}}$

i.e.: $*$: exact $r+1$ forms $\to$ co-exact $n-r-1$ forms

Proof: Assume $w = d\rho \in d\Lambda_r$

Define $v := *w$

$\Rightarrow v = *d\rho = (-1)^{r(n-r)} \underbrace{*d*}_{=\,\delta} *\rho$

$= \delta\alpha \in \delta\Lambda_{n-r}$ for $\alpha = (-1)^{r(n-r)} *\rho$

Proposition: $* : \boxed{\delta\Lambda_r \to d\Lambda_{n-r}}$

Proof: Exercise.

Recall: $*$ preserves the spectrum of Δ as we showed already.

$\Rightarrow$

Summary:

$$\Lambda_{p-1} = \underbrace{d\Lambda_{p-2}} \oplus \underbrace{\delta\Lambda_p} \oplus \Lambda^{\circ}_{p-1}$$

same spectrum

$$\Lambda_p = \underbrace{d\Lambda_{p-1}} \oplus \underbrace{\delta\Lambda_{p+1}} \oplus \Lambda^{\circ}_p$$

same spectrum

$$\Lambda_{p+1} = \underbrace{d\Lambda_p} \oplus \underbrace{\delta\Lambda_{p+2}} \oplus \Lambda^{\circ}_{p+1}$$

Now we also found:

$$\Lambda_p = \underbrace{d\Lambda_{p-1}} \oplus \underbrace{\delta\Lambda_{p+1}} \oplus \Lambda_p^{\circ}$$

same spectrum   same spectrum   same spectrum

$$\Lambda_{n-p} = \overbrace{d\Lambda_{n-p-1}} \oplus \overbrace{\delta\Lambda_{n-p+1}} \oplus \Lambda^{\circ}_{n-p}$$

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Proof: Assume $\omega = d\rho \in d\Lambda_r$

Define $\nu := *\omega$

$$\Rightarrow \nu = *d\rho = (-1)^{r(n-r)} \underbrace{*d*}_{=\,\delta}*\rho$$
$$= \delta\alpha \in \delta\Lambda_{n-r} \text{ for } \alpha = (-1)^{r(n-r)} *\rho$$

□ **Proposition:** $*: \boxed{\delta\Lambda_r \to d\Lambda_{n-r}}$

Proof: Exercise.

Recall: $*$ preserves the spectrum of Δ as we showed already.

⋮

Now we also found:

$$\Lambda_p = \underbrace{d\Lambda_{p-1}} \oplus \underbrace{\delta\Lambda_{p+1}} \oplus \Lambda^\circ_p$$

⋮ same spectrum / same spectrum / spectrum

$$\Lambda_{n-p} = d\Lambda_{n-p-1} \oplus \delta\Lambda_{n-p+1} \oplus \Lambda^\circ_{n-p}$$

Example: dim(M)=3 Exercise: do same for dim(M)=4

$$\Lambda_0 = \delta\Lambda_1 \oplus \Lambda^\circ_0$$
$$\Lambda_1 = d\Lambda_0 \oplus \delta\Lambda_2 \oplus \Lambda^\circ_1$$
$$\Lambda_2 = d\Lambda_1 \oplus \delta\Lambda_3 \oplus \Lambda^\circ_2$$
$$\Lambda_3 = d\Lambda_2 \oplus \Lambda^\circ_3$$

Same color means same spectrum of Δ.

Conclusion: There is relatively little independent information in the spectra of p-form waves on M! E.g., when $\dim(M)=3$, then the spectrum of co-vector waves $\mathrm{spec}(\Delta|_{\Lambda_1})$ has already all information of all these spectra.

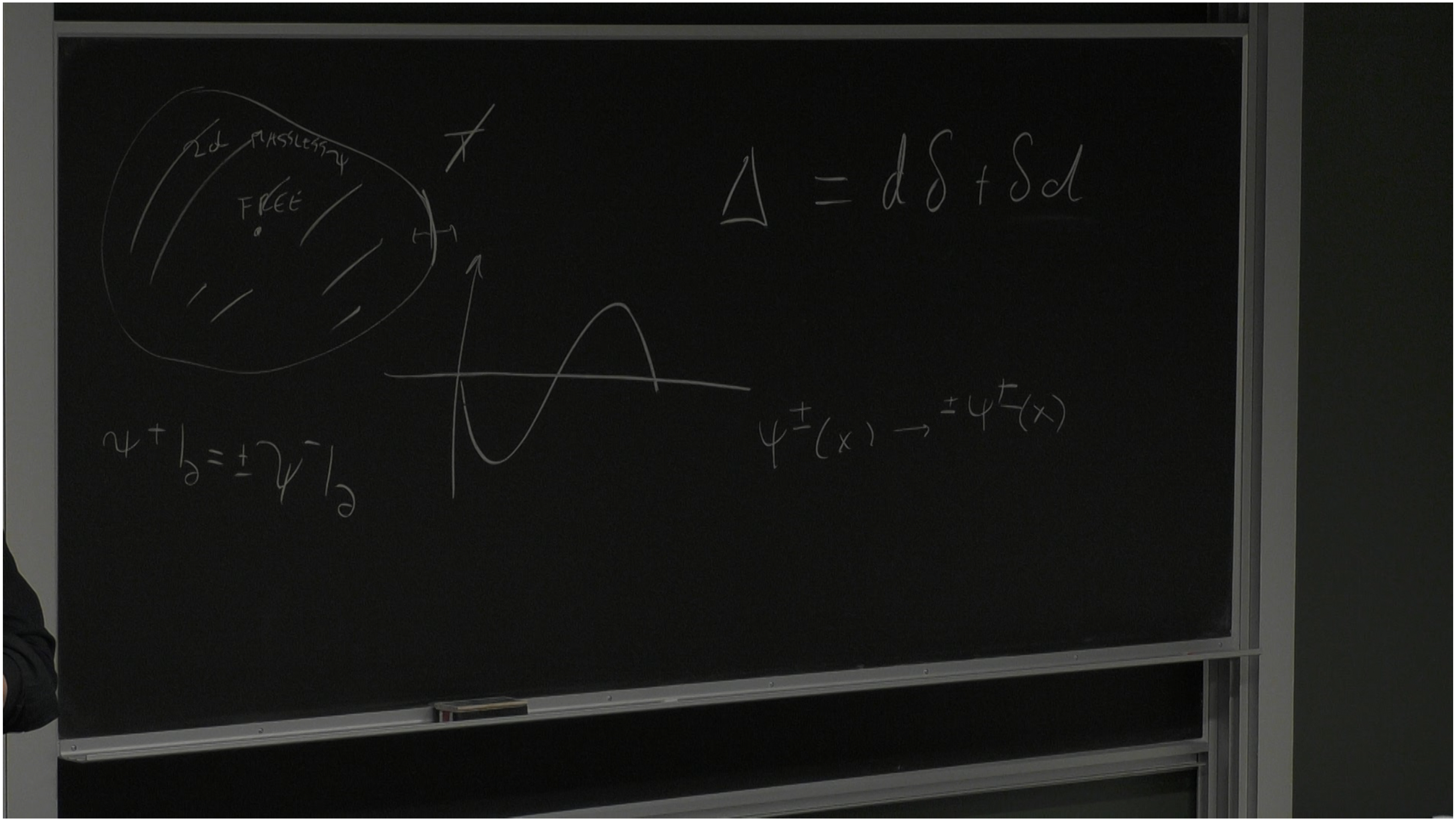
Literature: (neglecting literature on detecting boundary shapes from spectra)

Indeed: The spectra of Δ do not contain sufficient information in general to uniquely identify the Riemannian structure from the spectra alone:

Examples: Cases have been found of pairs (M, g), $(\tilde{M}, \tilde{g})$ that are isospectral for Δ on all Λ_p but that are not diffeomorphically isometric!

Nevertheless: All examples are of limited significance:

- manifolds that are locally if not globally isometric, or
- manifolds that are isospectral only w.r.t. to some Δ or
- manifolds that are discrete pairs (e.g. mirror images).


2d MASSLESS ψ
FREE
Δ = dδ + δd
ψ±(x) → ±ψ±(x)
ψ+|∂ = ±ψ−|∂

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$$\Lambda_3 = d\Lambda_2 \oplus \Lambda^\circ_3$$

Same color means same spectrum of Δ.

Conclusion: There is relatively little independent information in the spectra of p-form waves on $\mathcal{M}$!
E.g., when $\dim(\mathcal{M})=3$, then the spectrum of co-vector waves $\mathrm{spec}(\Delta|_{\Lambda_1})$ has already all information of all these spectra.

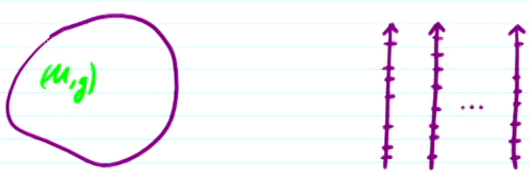
Examples: cases have been found of pairs $(\mathcal{M},g)$, $(\mathcal{M}',g')$ that are isospectral for Δ on all Λ_p but that are not diffeomorphically isometric!

Nevertheless: All examples are of limited significance:
- manifolds that are locally, if not globally isometric, or
- manifolds that are isospectral only w.respect to some Δ or
- manifolds that are discrete pairs (e.g. mirror images).

Fresh approach to spectral geometry (AK)

Strategy: Iterate infinitesimal inverse spectral geometry

Assume both, the mfld and its spectra are given:



A compact Riemannian manifold $(\mathcal{M},g)$ without boundary

The spectra $\{\lambda^{(i)}_n\}$ of Laplacians $\Delta^{(i)}$ on the manifold.

↑ Could be Laplacians not only on forms but also on general tensors.

Perturbation:

Now change the shape of $(\mathcal{M},g)$ slightly, through:

$$g \to g + h$$

This will slightly change the spectra to

$$\{\lambda^{(i)}_n\} \to \{\lambda^{(i)}_n + \mu^{(i)}_n\}$$

Why is this linearization useful?

- One can define a self-adjoint Laplacian $\Delta^{(mn)}$ on $T_2(\mathcal{M})$, with Hilbert basis $\{b_n(x)\}$ and eigenvalues $\{\lambda^{(mn)}_n\}$:

$$\Delta^{(mn)} b_n(x) = \lambda_n b_n(x)$$

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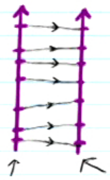
$$\{\lambda_n^{(m)}\} \to \{\lambda_n^{(m)} + \mu_n^{(m)}\}$$

$\Rightarrow$



Notice: Consider only eigenvectors and eigenvalues up to a cutoff scale. Then, there are as many parameters $\{h_n\}_{n=1}^{N}$ as $\{\mu_n\}_{n=1}^{N}$.

$\Rightarrow$ S is a square matrix.

If $\det(S') \neq 0$, then S'^{-1} exists.

$\rightsquigarrow$ should be able to iterate the perturbations?

This is ongoing research.

Remarks:

- Not all h actually change the shape:

  If $h = L_\xi g$ for some vector field ξ, then $g \to g + h$ is merely the infinitesimal change of chart belonging to the flow induced by ξ.

- Symmetric covariant 2-tensors such as h have a canonical decomposition similar to the Hodge decomposition. Thus, Δ has three spectra on $T_2(M)$.

Reference: See also e.g. the video of my talk at PI: http://pirsa.org/15090062

Infinitesimal spectral geometry arose from my paper on how Spacetime could be simultaneously continuous and discrete, in the same way that information can.

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