Title: General Relativity for Cosmology Lecture - 101723

Speakers: Achim Kempf

Collection: General Relativity for Cosmology

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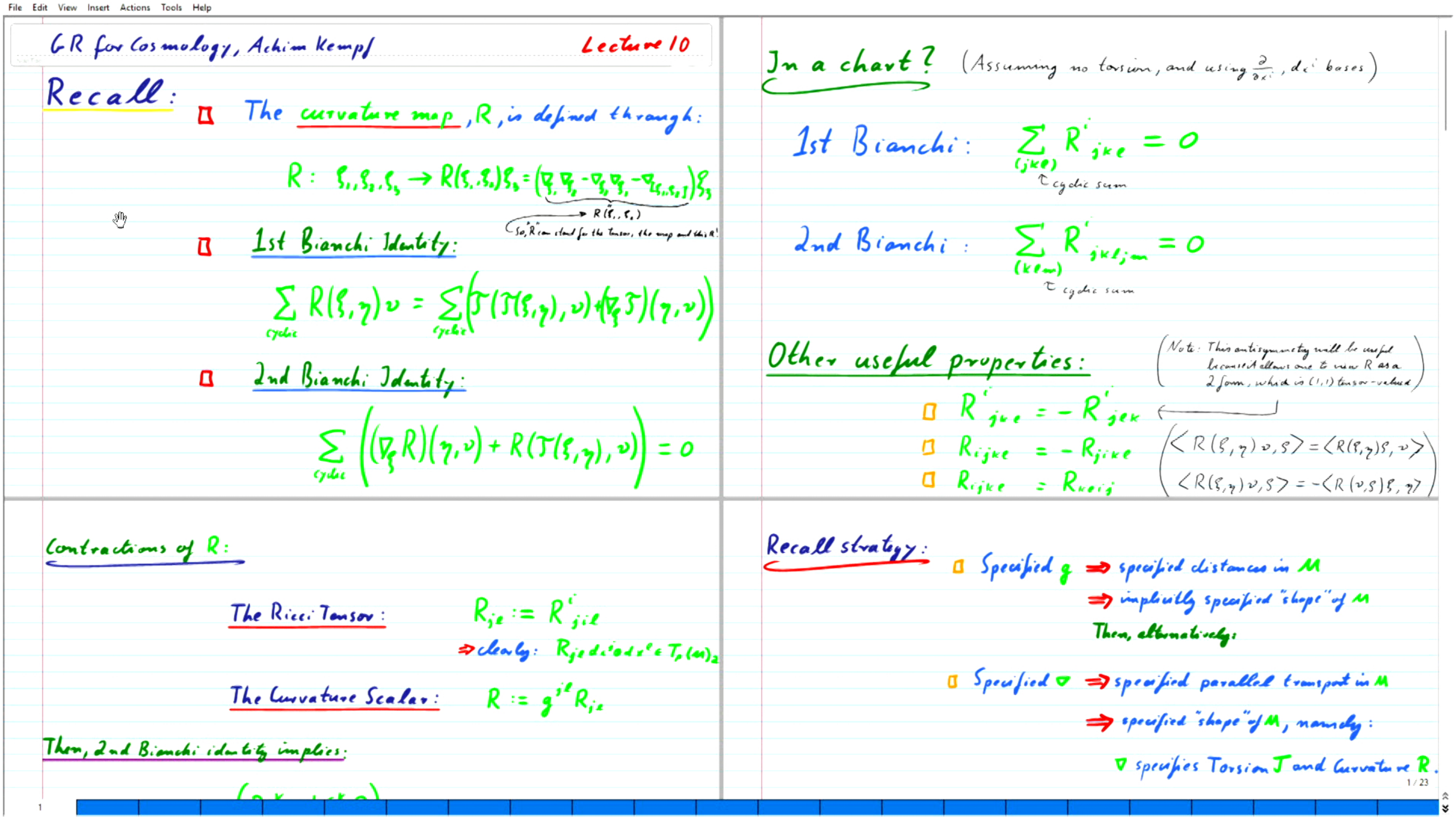
Abstract: Zoom: https://pitp.zoom.us/j/91640855624?pwd=dWVWV2doSnBhUS9JUkhjQVBwY0h0dz09



GR for Cosmology, Achim Kempf — Lecture 10

Recall:

- The curvature map, R, is defined through:

$$R:\ \xi_1,\xi_2,\xi_3 \to R(\xi_1,\xi_2)\xi_3 = \underbrace{\left(\nabla_{\xi_1}\nabla_{\xi_2} - \nabla_{\xi_2}\nabla_{\xi_1} - \nabla_{[\xi_1,\xi_2]}\right)}_{\text{"}R\text{"}(\xi_1,\xi_2)}\xi_3$$

So, "R" can stand for the tensor, the map and this R!

- 1st Bianchi Identity:

$$\sum_{\text{cyclic}} R(\xi,\eta)\nu = \sum_{\text{cyclic}}\Big(T(T(\xi,\eta),\nu) + (\nabla_\xi T)(\eta,\nu)\Big)$$

- 2nd Bianchi Identity:

$$\sum_{\text{cyclic}}\Big((\nabla_\xi R)(\eta,\nu) + R(T(\xi,\eta),\nu)\Big) = 0$$

In a chart?

(Assuming no torsion, and using $\frac{\partial}{\partial x^i}$, dx^i bases)

1st Bianchi: $\sum_{(jkl)} R^i_{\ jkl} = 0$

↑ cyclic sum

2nd Bianchi: $\sum_{(klm)} R^i_{\ jkl;m} = 0$

↑ cyclic sum

Other useful properties:

(Note: This antisymmetry will be useful because it allows one to view R as a 2 form, which is (1,1) tensor-valued)

- $R^i_{\ jkl} = -R^i_{\ jlk}$
- $R_{ijkl} = -R_{jikl}$
- $R_{ijkl} = R_{klij}$

$$\begin{pmatrix} \langle R(\xi,\eta)\nu,\rho\rangle = \langle R(\xi,\eta)\rho,\nu\rangle \\ \langle R(\xi,\eta)\nu,\rho\rangle = -\langle R(\nu,\rho)\xi,\eta\rangle \end{pmatrix}$$

Contractions of R:

The Ricci Tensor: $R_{jl} := R^i_{\ jil}$

⇒ clearly: $R_{jl}\, dx^j \otimes dx^l \in T_p(M)_2$

The Curvature Scalar: $R := g^{jl} R_{jl}$

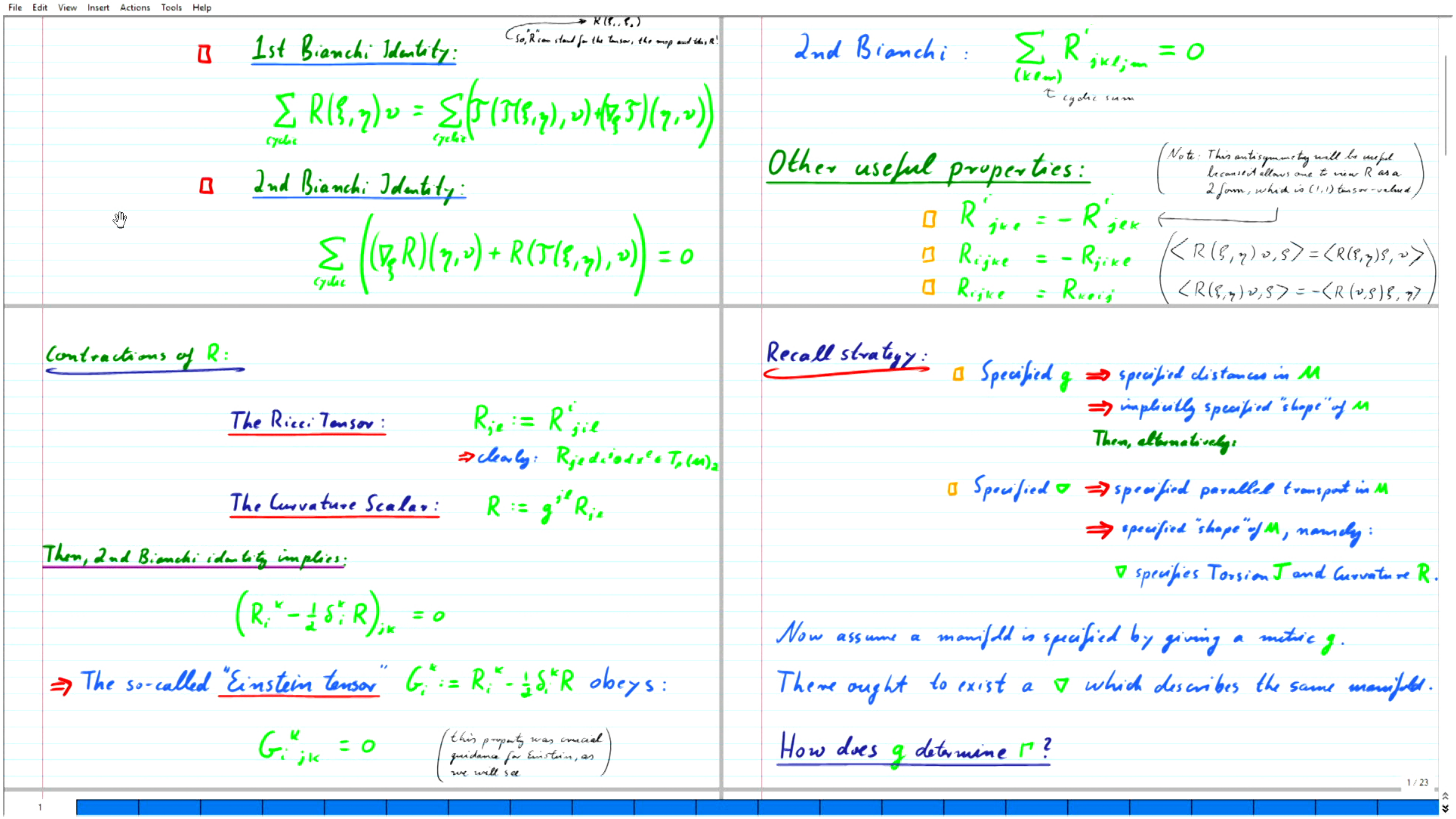
Then, 2nd Bianchi identity implies:

Recall strategy:

- Specified g ⇒ specified distances in M
  ⇒ implicitly specified "shape" of M

  Then, alternatively:
- Specified ∇ ⇒ specified parallel transport in M
  ⇒ specified "shape" of M, namely:

  ∇ specifies Torsion T and Curvature R.



☐ **1st Bianchi Identity:**

$R(\xi_\cdot,\xi_\cdot)$ So, "R" can stand for the tensor, the map and this R!

$$\sum_{\text{cyclic}} R(\xi,\eta)v = \sum_{\text{cyclic}}\Big(T(T(\xi,\eta),v)+(\nabla_\xi T)(\eta,v)\Big)$$

☐ **2nd Bianchi Identity:**

$$\sum_{\text{cyclic}}\Big((\nabla_\xi R)(\eta,v)+R(T(\xi,\eta),v)\Big)=0$$

2nd Bianchi: $\sum_{(kl m)} R^i{}_{jkl;m} = 0$

↑ cyclic sum

Other useful properties:

(Note: This antisymmetry will be useful because it allows one to view R as a 2 form, which is (1,1) tensor-valued)

- ☐ $R^i{}_{jkl} = -R^i{}_{jlk}$ ⟵
- ☐ $R_{ijkl} = -R_{jikl}$
- ☐ $R_{ijkl} = R_{klij}$

$$\Big(\langle R(\xi,\eta)v,s\rangle = \langle R(\xi,\eta)s,v\rangle$$
$$\langle R(\xi,\eta)v,s\rangle = -\langle R(v,s)\xi,\eta\rangle\Big)$$

Contractions of R:

The Ricci Tensor: $R_{jl} := R^i{}_{jil}$

⇒ clearly: $R_{jl}\, dx^j \otimes dx^l \in T_p(M)_2$

The Curvature Scalar: $R := g^{jl} R_{jl}$

Then, 2nd Bianchi identity implies:

$$\left(R_i{}^k - \tfrac{1}{2}\delta_i^k R\right)_{;k} = 0$$

⇒ The so-called "Einstein tensor" $G_i{}^k := R_i{}^k - \frac{1}{2}\delta_i{}^k R$ obeys:

$$G_i{}^k{}_{;k} = 0$$

(this property was crucial guidance for Einstein, as we will see)

Recall strategy:

- ☐ Specified g ⇒ specified distances in M
  ⇒ implicitly specified "shape" of M

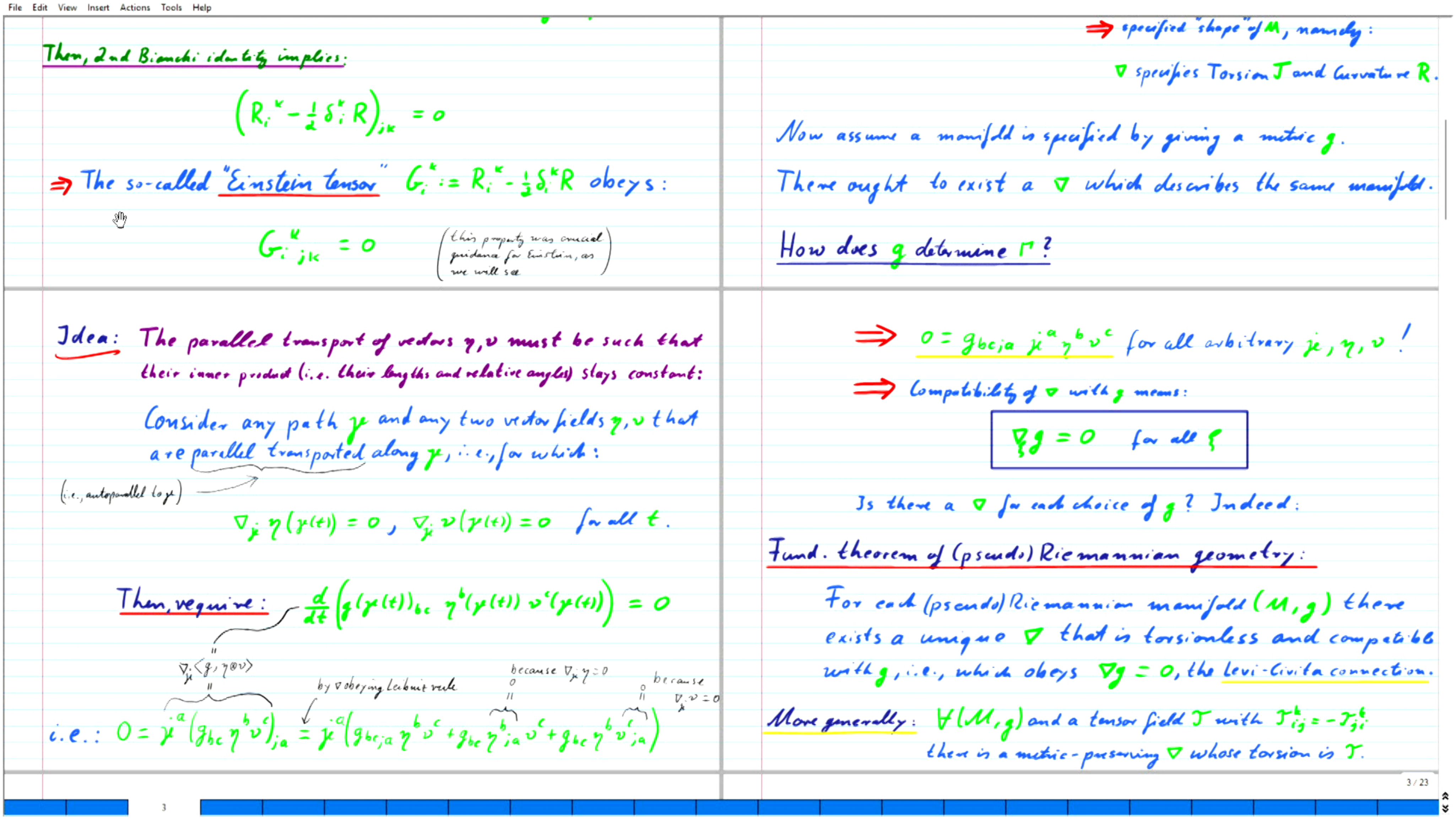
  Then, alternatively:

- ☐ Specified ∇ ⇒ specified parallel transport in M
  ⇒ specified "shape" of M, namely:
  ∇ specifies Torsion T and Curvature R.

Now assume a manifold is specified by giving a metric g.

There ought to exist a ∇ which describes the same manifold.

How does g determine Γ?



Then, 2nd Bianchi identity implies:

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(this property was crucial guidance for Einstein, as we will see)

$\Rightarrow$ specified "shape" of M, namely:

∇ specifies Torsion T and Curvature R.

Now assume a manifold is specified by giving a metric g.

There ought to exist a ∇ which describes the same manifold.

How does g determine Γ?

Idea: The parallel transport of vectors η, v must be such that their inner product (i.e. their lengths and relative angles) stays constant:

Consider any path γ and any two vector fields η, v that are $\underbrace{\text{parallel transported}}$ along γ, i.e., for which:

(i.e., autoparallel to γ) $\rightarrow$

$$\nabla_{\dot\gamma}\,\eta(\gamma(t)) = 0, \quad \nabla_{\dot\gamma}\,v(\gamma(t)) = 0 \quad \text{for all } t.$$

Then, require: $$\underbrace{\frac{d}{dt}\left(g(\gamma(t))_{bc}\,\eta^b(\gamma(t))\,v^c(\gamma(t))\right)}_{=\nabla_{\dot\gamma}\langle g,\,\eta\otimes v\rangle} = 0$$

by ∇ obeying Leibniz rule

because $\nabla_{\dot\gamma}\eta = 0$

because $\nabla_{\dot\gamma}v = 0$

$$\text{i.e.: } 0 = \underbrace{\dot\gamma^a\left(g_{bc}\,\eta^b v^c\right)_{;a}}_{=\nabla_{\dot\gamma}\langle g,\eta\otimes v\rangle} = \dot\gamma^a\left(g_{bc;a}\,\eta^b v^c + g_{bc}\,\underbrace{\eta^b{}_{;a}}_{=0}\,v^c + g_{bc}\,\eta^b\,\underbrace{v^c{}_{;a}}_{=0}\right)$$

$\Rightarrow$ $0 = g_{bc;a}\,\dot\gamma^a\,\eta^b\,v^c$ for all arbitrary $\dot\gamma, \eta, v$!

$\Rightarrow$ Compatibility of ∇ with g means:

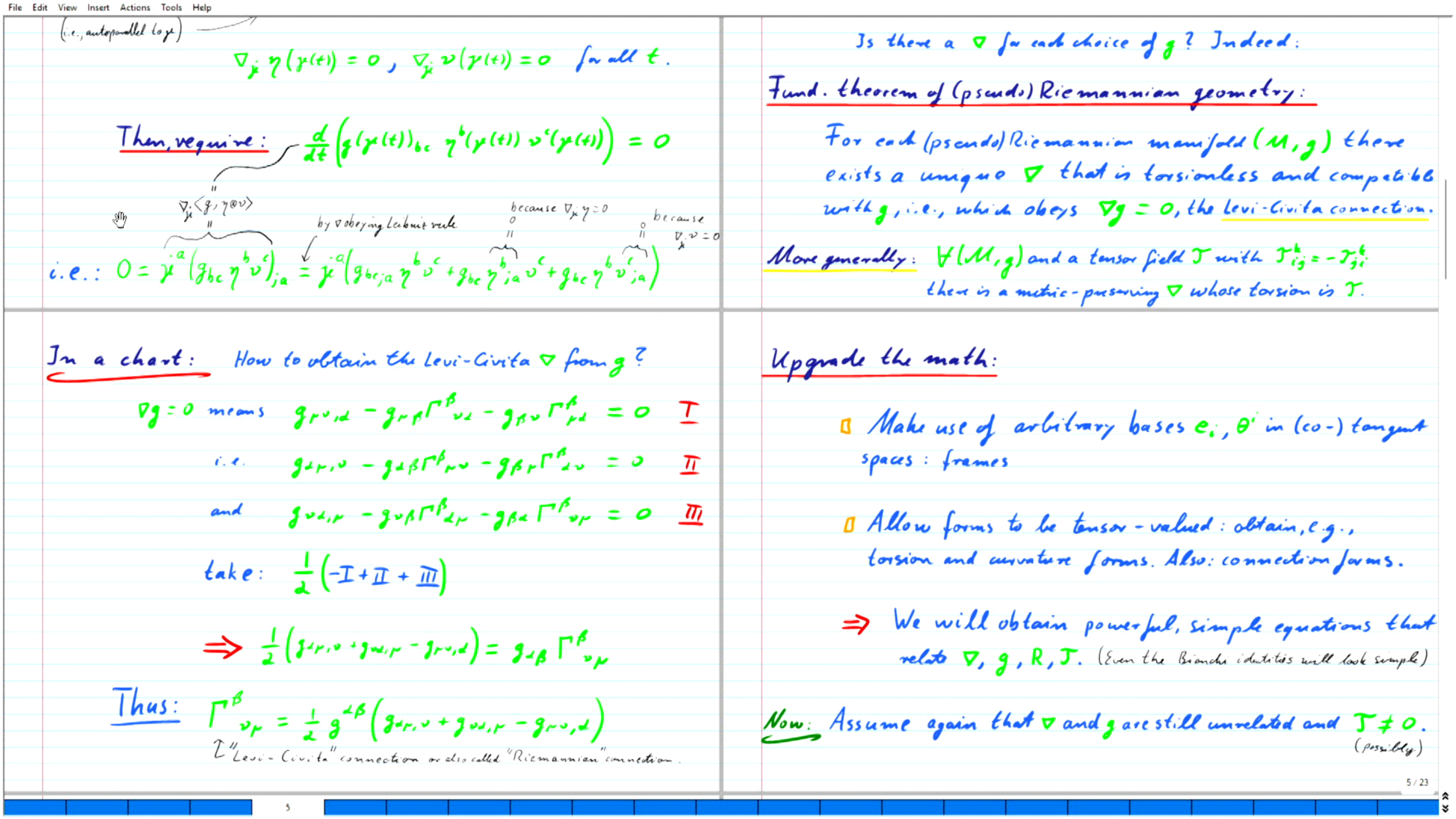
$$\boxed{\nabla_\xi g = 0 \quad \text{for all } \xi}$$

Is there a ∇ for each choice of g? Indeed:

Fund. theorem of (pseudo) Riemannian geometry:

For each (pseudo) Riemannian manifold (M, g) there exists a unique ∇ that is torsionless and compatible with g, i.e., which obeys $\nabla g = 0$, the Levi-Civita connection.

More generally: $\forall (M, g)$ and a tensor field T with $T^k_{ij} = -T^k_{ji}$ there is a metric-preserving ∇ whose torsion is T.



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Then, require: $$\frac{d}{dt}\Big(g(\gamma(t))_{bc}\,\eta^b(\gamma(t))\,v^c(\gamma(t))\Big) = 0$$

$$= \nabla_{\dot\gamma}\langle g, \eta\otimes v\rangle =$$

by ∇ obeying Leibniz rule

$$\text{i.e.: } 0 = \overbrace{\dot\gamma^a\left(g_{bc}\eta^b v^c\right)_{;a}} = \dot\gamma^a\Big(g_{bc;a}\eta^b v^c + g_{bc}\overbrace{\eta^b{}_{;a}}^{\substack{\text{because } \nabla_{\dot\gamma}\eta = 0\\ 0\\ \|}} v^c + g_{bc}\eta^b\overbrace{v^c{}_{;a}}^{\substack{0\\ \|}}\Big) \quad \substack{\text{because}\\ \nabla_{\dot\gamma}v = 0}$$

In a chart: How to obtain the Levi-Civita ∇ from g?

$$\nabla g = 0 \text{ means } \quad g_{\mu\nu,\alpha} - g_{\mu\beta}\Gamma^\beta{}_{\nu\alpha} - g_{\beta\nu}\Gamma^\beta{}_{\mu\alpha} = 0 \quad \underline{\text{I}}$$

$$\text{i.e.} \quad g_{\alpha\mu,\nu} - g_{\alpha\beta}\Gamma^\beta{}_{\mu\nu} - g_{\beta\mu}\Gamma^\beta{}_{\alpha\nu} = 0 \quad \underline{\text{II}}$$

$$\text{and} \quad g_{\nu\alpha,\mu} - g_{\nu\beta}\Gamma^\beta{}_{\alpha\mu} - g_{\beta\alpha}\Gamma^\beta{}_{\nu\mu} = 0 \quad \underline{\text{III}}$$

take: $\frac{1}{2}\left(-\text{I} + \text{II} + \text{III}\right)$

$$\Rightarrow \frac{1}{2}\left(g_{\alpha\mu,\nu} + g_{\nu\alpha,\mu} - g_{\mu\nu,\alpha}\right) = g_{\alpha\beta}\Gamma^\beta{}_{\nu\mu}$$

Thus: $$\Gamma^\beta{}_{\nu\mu} = \frac{1}{2}g^{\alpha\beta}\left(g_{\alpha\mu,\nu} + g_{\nu\alpha,\mu} - g_{\mu\nu,\alpha}\right)$$

↳ "Levi-Civita" connection or also called "Riemannian" connection.

Is there a ∇ for each choice of g? Indeed:

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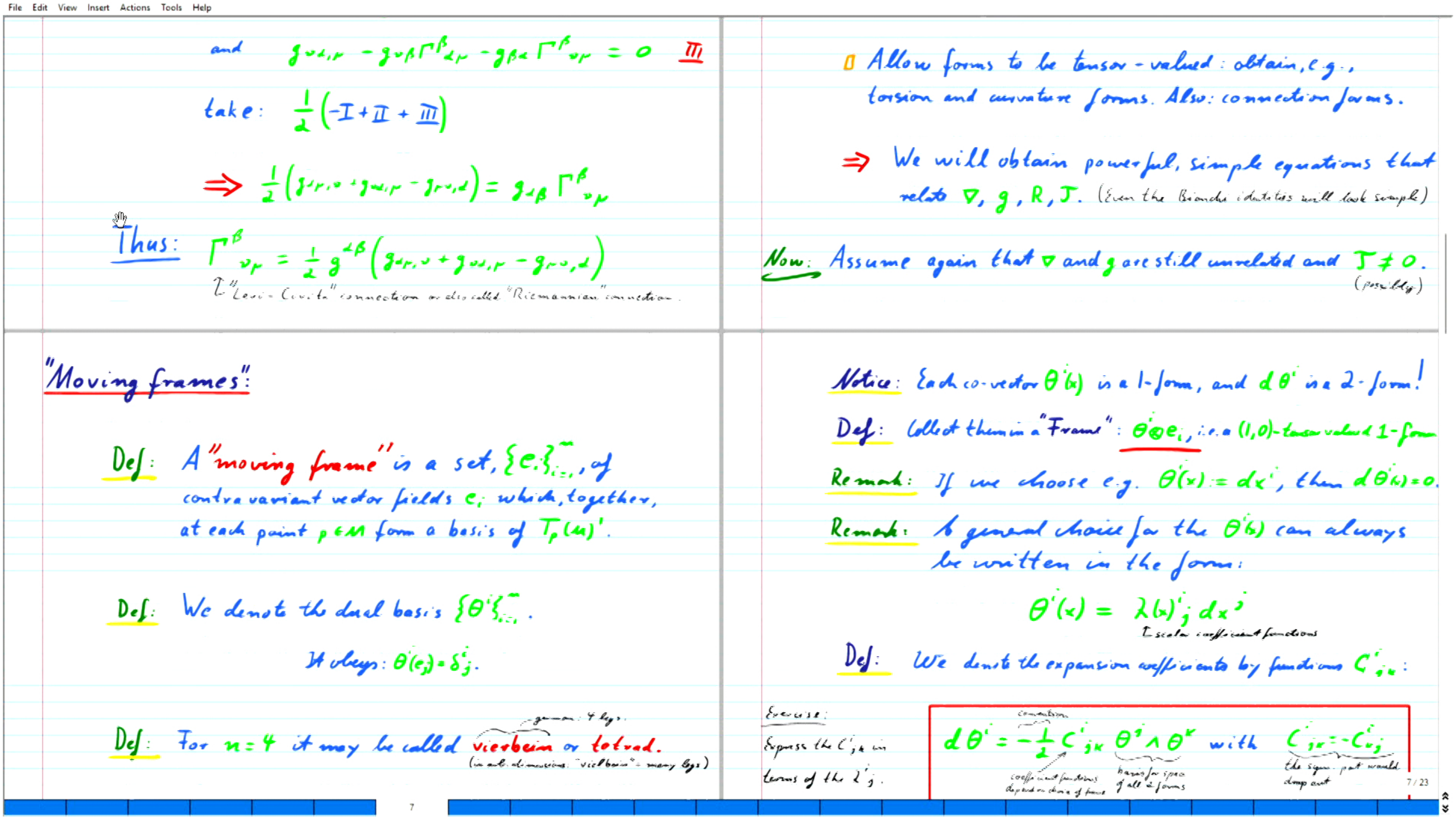
More generally: $\forall (M, g)$ and a tensor field T with $T^k_{ij} = -T^k_{ji}$ there is a metric-preserving ∇ whose torsion is T.

Upgrade the math:

- Make use of arbitrary bases e_i, θ^i in (co-) tangent spaces: frames
- Allow forms to be tensor-valued: obtain, e.g., torsion and curvature forms. Also: connection forms.

$\Rightarrow$ We will obtain powerful, simple equations that relate ∇, g, R, T. (Even the Bianchi identities will look simple)

Now: Assume again that ∇ and g are still unrelated and $T \neq 0$.
(possibly)



and $$g_{\nu\alpha,\mu} - g_{\nu\beta}\Gamma^{\beta}{}_{\alpha\mu} - g_{\beta\alpha}\Gamma^{\beta}{}_{\nu\mu} = 0 \quad \underline{III}$$

take: $\frac{1}{2}\left(-I + \underline{II} + \underline{III}\right)$

$$\Rightarrow \frac{1}{2}\left(g_{\alpha\mu,\nu} + g_{\nu\alpha,\mu} - g_{\mu\nu,\alpha}\right) = g_{\alpha\beta}\Gamma^{\beta}{}_{\nu\mu}$$

Thus: $$\Gamma^{\beta}{}_{\nu\mu} = \frac{1}{2} g^{\alpha\beta}\left(g_{\alpha\mu,\nu} + g_{\nu\alpha,\mu} - g_{\mu\nu,\alpha}\right)$$

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▯ Allow forms to be tensor-valued: obtain, e.g., torsion and curvature forms. Also: connection forms.

⇒ We will obtain powerful, simple equations that relate ∇, g, R, J. (Even the Bianchi identities will look simple)

Now: Assume again that ∇ and g are still unrelated and $J \neq 0$. (possibly)

"Moving frames":

Def: A "moving frame" is a set, $\{e_i\}_{i=1}^{n}$, of contravariant vector fields e_i which, together, at each point $p \in M$ form a basis of $T_p(M)$.

Def: We denote the dual basis $\{\theta^i\}_{i=1}^{n}$.

It obeys: $\theta^i(e_j) = \delta^i_j$.

Def: For $n=4$ it may be called $\overset{\text{german: 4 legs.}}{\text{vierbein}}$ or tetrad.
(in arb. dimensions: "vielbein" = many legs)

Notice: Each co-vector $\theta^i(x)$ is a 1-form, and $d\theta^i$ is a 2-form!

Def: Collect them in a "Frame": $\underline{\theta^i \otimes e_i}$, i.e. a (1,0)-tensor valued 1-form

Remark: If we choose e.g. $\theta^i(x) := dx^i$, then $d\theta^i(x) = 0$.

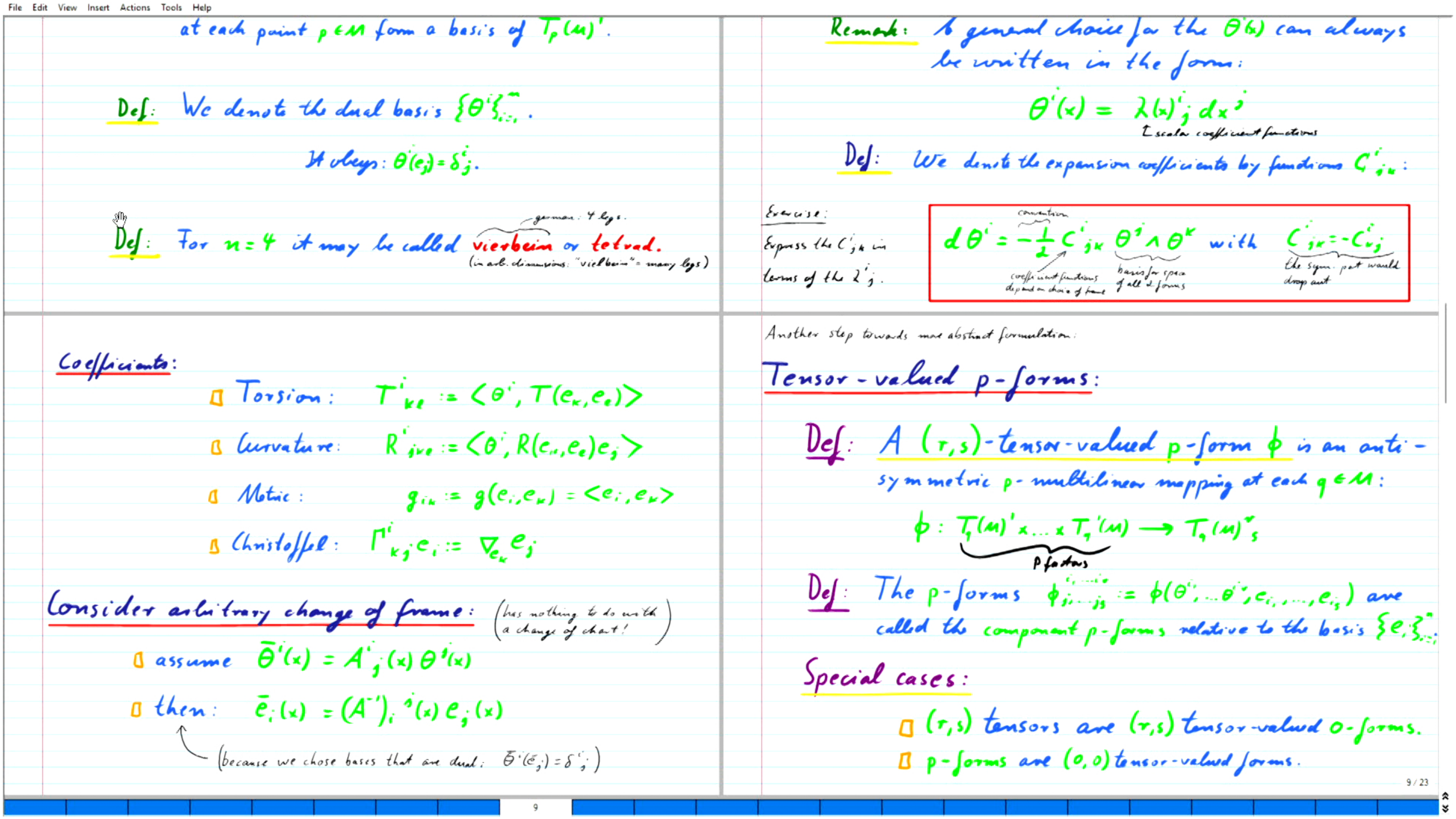
Remark: A general choice for the $\theta^i(x)$ can always be written in the form:

$$\theta^i(x) = \underset{\uparrow\text{ scalar coefficient functions}}{\lambda(x)^i{}_j}\, dx^j$$

Def: We denote the expansion coefficients by functions $C^i{}_{jk}$:

Exercise:
Express the $C^i{}_{jk}$ in terms of the $\lambda^i{}_j$.

$$\boxed{d\theta^i = -\frac{1}{2}\overset{\text{connection}}{\underset{\nwarrow\text{ coefficient functions depend on choice of frame}}{C^i{}_{jk}}}\underbrace{\theta^j \wedge \theta^k}_{\text{basis for space of all 2-forms}} \text{ with } \underbrace{C^i{}_{jk} = -C^i{}_{kj}}_{\text{the sym. part would drop out}}}$$



at each point $p\in M$ form a basis of $T_p(M)'$.

Def: We denote the dual basis $\{\theta^i\}_{i=1}^{m}$.

It obeys: $\theta^i(e_j)=\delta^i_j$.

Def: For $n=4$ it may be called $\overbrace{\text{vierbein}}^{\text{german: 4 legs.}}$ or **tetrad.**
(in arb. dimensions: "vielbein" = many legs)

Coefficients:

- Torsion: $T^i_{\ kl} := \langle \theta^i, T(e_k,e_l)\rangle$
- Curvature: $R^i_{\ jkl} := \langle \theta^i, R(e_k,e_l)e_j\rangle$
- Metric: $g_{ik} := g(e_i,e_k) = \langle e_i, e_k\rangle$
- Christoffel: $\Gamma^i_{\ kj}\, e_i := \nabla_{e_k} e_j$

Consider arbitrary change of frame: (has nothing to do with a change of chart!)

- assume $\bar\theta^i(x) = A^i_{\ j}(x)\,\theta^j(x)$
- then: $\bar e_i(x) = (A^{-1})_i^{\ j}(x)\, e_j(x)$

(because we chose bases that are dual: $\bar\theta^i(\bar e_j)=\delta^i_j$)

Remark: A general choice for the $\theta^i(x)$ can always be written in the form:

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Express the $C^i_{\ jk}$ in terms of the $\lambda^i_{\ j}$.

$$\boxed{d\theta^i = \overbrace{-\frac{1}{2}}^{\text{convention}} C^i_{\ jk}\,\underbrace{\theta^j\wedge\theta^k}_{\text{basis for space of all 2-forms}} \quad \text{with} \quad \underbrace{C^i_{\ jk}=-C^i_{\ kj}}_{\text{the sym. part would drop out}}}$$

coefficient functions depend on choice of frame

Another step towards more abstract formulation:

Tensor-valued p-forms:

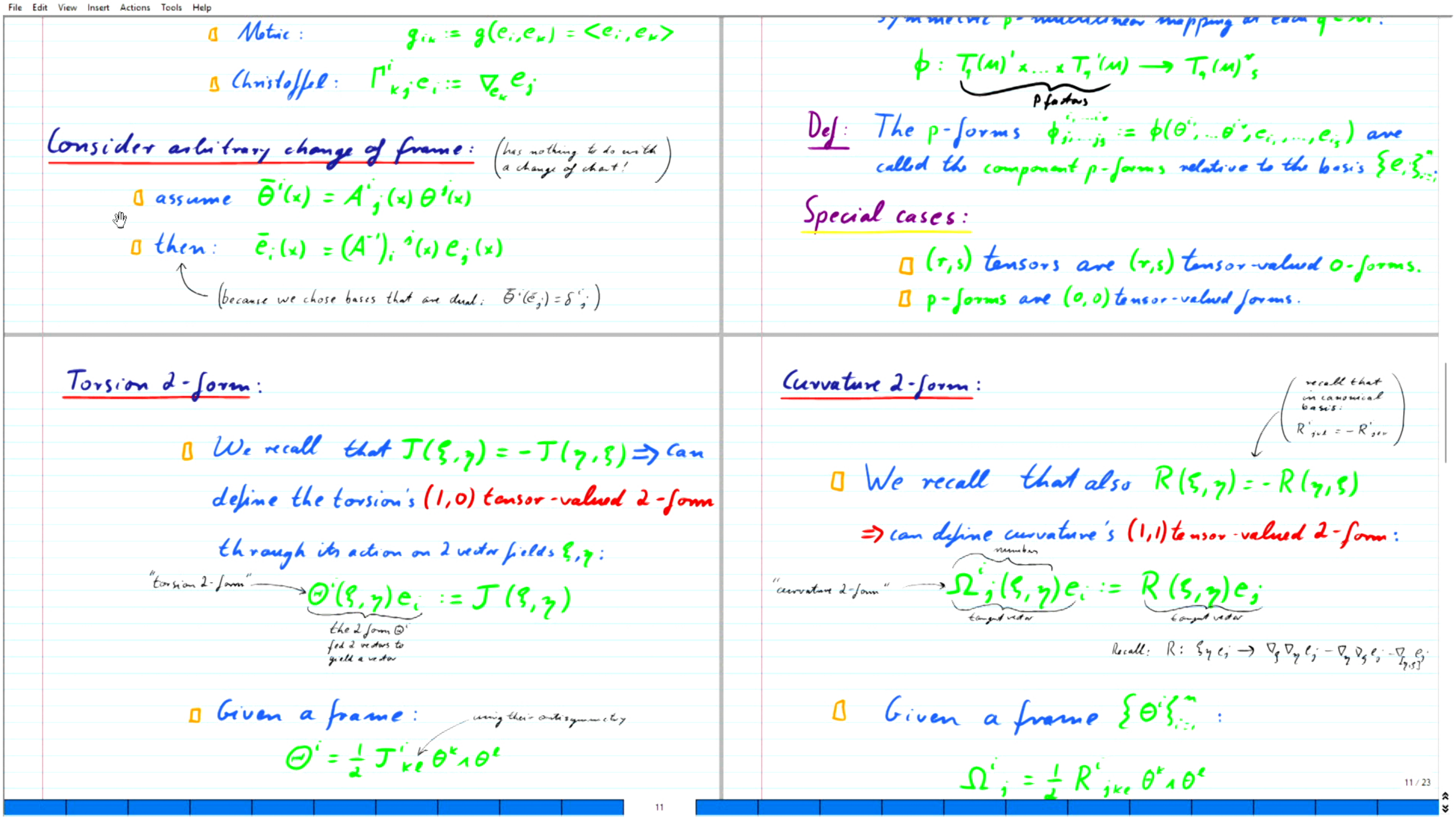
Def: A (r,s)-tensor-valued p-form ϕ is an anti-symmetric p-multilinear mapping at each $q\in M$:

$$\phi : \underbrace{T_q(M)'\times\dots\times T_q'(M)}_{p\text{ factors}} \longrightarrow T_q(M)^r_{\ s}$$

Def: The p-forms $\phi^{i_1\dots i_r}_{j_1\dots j_s} := \phi(\theta^{i_1},\dots\theta^{i_r}; e_{j_1},\dots,e_{j_s})$ are called the component p-forms relative to the basis $\{e_i\}_{i=1}^{n}$.

Special cases:

- (r,s) tensors are (r,s) tensor-valued 0-forms.
- p-forms are $(0,0)$ tensor-valued forms.



- Metric : $g_{ik} := g(e_i, e_k) = \langle e_i, e_k \rangle$
- Christoffel : $\Gamma^i_{kj} e_i := \nabla_{e_k} e_j$

Consider arbitrary change of frame: (has nothing to do with a change of chart!)

- assume $\bar{\theta}^i(x) = A^i{}_j(x)\, \theta^j(x)$
- then: $\bar{e}_i(x) = (A^{-1})_i{}^j(x)\, e_j(x)$

(because we chose bases that are dual: $\bar{\theta}^i(\bar{e}_j) = \delta^i{}_j$)

Torsion 2-form:

- We recall that $T(\xi,\eta) = -T(\eta,\xi) \Rightarrow$ can define the torsion's (1,0) tensor-valued 2-form through its action on 2 vector fields ξ, η:

"torsion 2-form" $\longrightarrow$ $\underbrace{\Theta^i(\xi,\eta)\, e_i}_{\text{the 2 form } \Theta^i \text{ fed 2 vectors to yield a vector}} := T(\xi,\eta)$

- Given a frame:

$\Theta^i = \frac{1}{2} T^i{}_{kl}\, \theta^k \wedge \theta^l$ using their antisymmetry

symmetric p-multilinear mapping at each $q \in M$:

$$\phi : \underbrace{T_q(M)^1 \times \ldots \times T_q{}^1(M)}_{p \text{ factors}} \longrightarrow T_q(M)^r{}_s$$

Def: The p-forms $\phi^{i_1 \ldots i_r}_{j_1 \ldots j_s} := \phi(\theta^{i_1}, \ldots \theta^{i_r}; e_{j_1}, \ldots, e_{j_s})$ are called the component p-forms relative to the basis $\{e_i\}^n_{i=1}$

Special cases:

- (r,s) tensors are (r,s) tensor-valued 0-forms.
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Curvature 2-form:

(recall that in canonical basis: $R^i{}_{jkl} = -R^i{}_{jlk}$)

- We recall that also $R(\xi,\eta) = -R(\eta,\xi)$

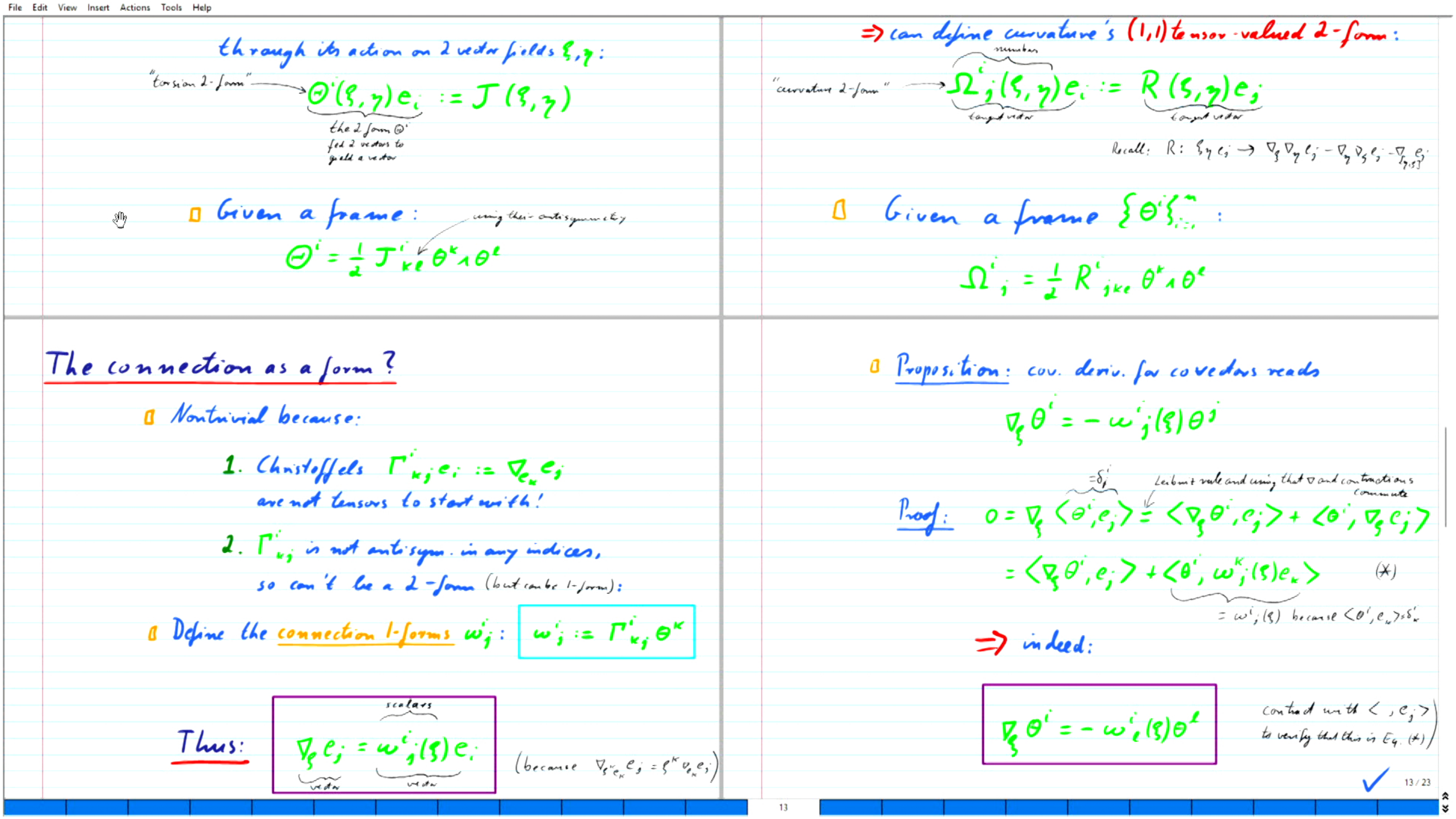
$\Rightarrow$ can define curvature's (1,1) tensor-valued 2-form:

"curvature 2-form" $\longrightarrow$ $\underbrace{\overbrace{\Omega^i{}_j(\xi,\eta)}^{\text{number}}\, e_i}_{\text{tangent vector}} := \underbrace{R(\xi,\eta)\, e_j}_{\text{tangent vector}}$

Recall: $R: \xi\, \eta\, e_j \rightarrow \nabla_\xi \nabla_\eta e_j - \nabla_\eta \nabla_\xi e_j - \nabla_{[\eta,\xi]} e_j$

- Given a frame $\{\theta^i\}^n_{i=1}$:

$\Omega^i{}_j = \frac{1}{2} R^i{}_{jkl}\, \theta^k \wedge \theta^l$



through its action on 2 vector fields ξ, η:

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using their antisymmetry

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Recall: $R: \xi\,\eta\, e_j \rightarrow \nabla_\xi \nabla_\eta e_j - \nabla_\eta \nabla_\xi e_j - \nabla_{[\eta,\xi]} e_j$

- Given a frame $\{\theta^i\}_{i=1}^{n}$:

$$\Omega^i{}_j = \frac{1}{2} R^i{}_{jkl}\, \theta^k \wedge \theta^l$$

The connection as a form?

- Nontrivial because:

  1. Christoffels $\Gamma^i{}_{kj} e_i := \nabla_{e_k} e_j$ are not tensors to start with!
  2. $\Gamma^i{}_{kj}$ is not antisym. in any indices, so can't be a 2-form (but can be 1-form):

- Define the connection 1-forms $\omega^i{}_j$: $\boxed{\omega^i{}_j := \Gamma^i{}_{kj}\, \theta^k}$

Thus: $\boxed{\underbrace{\nabla_\xi e_j}_{\text{vector}} = \underbrace{\overbrace{\omega^i{}_j(\xi)}^{\text{scalars}} e_i}_{\text{vector}}}$ (because $\nabla_{\xi^k e_k} e_j = \xi^k \nabla_{e_k} e_j$)

- Proposition: cov. deriv. for covectors reads

$$\nabla_\xi \theta^i = -\omega^i{}_j(\xi)\theta^j$$

Proof: $0 = \nabla_\xi \overbrace{\langle \theta^i, e_j \rangle}^{=\delta^i_j} = \langle \nabla_\xi \theta^i, e_j \rangle + \langle \theta^i, \nabla_\xi e_j \rangle$

Leibniz rule and using that ∇ and contractions commute

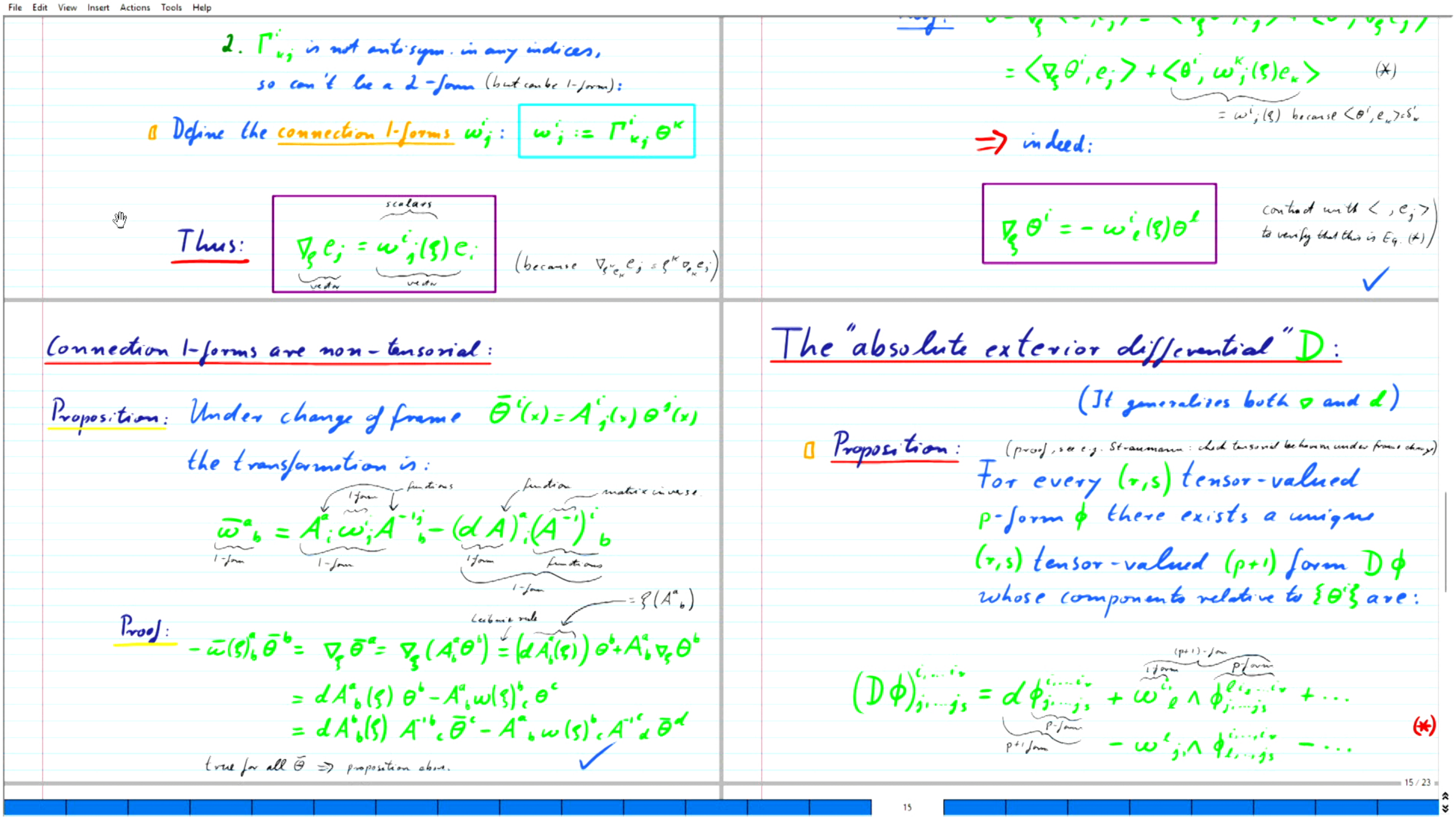
$$= \langle \nabla_\xi \theta^i, e_j \rangle + \underbrace{\langle \theta^i, \omega^k{}_j(\xi) e_k \rangle}_{= \omega^i{}_j(\xi) \text{ because } \langle \theta^i, e_k \rangle = \delta^i_k} \qquad (*)$$

$\Rightarrow$ indeed:

$$\boxed{\nabla_\xi \theta^i = -\omega^i{}_l(\xi)\theta^l}$$

(contract with $\langle \; , e_j \rangle$ to verify that this is Eq. $(*)$)

✓



2. $\Gamma^i{}_{kj}$ is not anti-symm. in any indices,
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Thus:

$$\boxed{\underbrace{\nabla_\xi e_j}_{\text{vector}} = \underbrace{\overbrace{\omega^i{}_j(\xi)}^{\text{scalars}}\, e_i}_{\text{vector}}}$$

(because $\nabla_{\xi^k e_k} e_j = \xi^k \nabla_{e_k} e_j$)

$$= \langle \nabla_\xi \theta^i, e_j \rangle + \underbrace{\langle \theta^i, \omega^k{}_j(\xi) e_k \rangle}_{= \omega^i{}_j(\xi) \text{ because } \langle \theta^i, e_k \rangle = \delta^i_k} \qquad (*)$$

$\Rightarrow$ indeed:

$$\boxed{\nabla_\xi \theta^i = -\omega^i{}_l(\xi)\,\theta^l}$$

(contract with $\langle\ , e_j \rangle$ to verify that this is Eq. $(*)$)

✓

Connection 1-forms are non-tensorial:

Proposition: Under change of frame $\bar{\theta}^i(x) = A^i{}_j(x)\,\theta^j(x)$ the transformation is:

$$\underbrace{\bar{\omega}^a{}_b}_{\text{1-form}} = \underbrace{\overset{\text{functions}}{A^a{}_i}\, \overset{\text{1form}}{\omega^i{}_j}\, \overset{\text{functions}}{A^{-1j}{}_b}}_{\text{1-form}} - \underbrace{\underbrace{(d\overset{\text{function}}{A})^a{}_i}_{\text{1form}}\, \underbrace{(\overset{\text{matrix inverse}}{A^{-1}})^i{}_b}_{\text{functions}}}_{\text{1-form}}$$

Proof:

$$-\bar{\omega}(\xi)^a{}_b\,\bar{\theta}^b = \nabla_\xi \bar{\theta}^a = \nabla_\xi (A^a{}_b \theta^b) \overset{\text{Leibniz rule}}{=} \big(\overbrace{dA^a{}_b(\xi)}^{=\xi(A^a{}_b)}\big)\,\theta^b + A^a{}_b \nabla_\xi \theta^b$$

$$= dA^a{}_b(\xi)\,\theta^b - A^a{}_b\,\omega(\xi)^b{}_c\,\theta^c$$

$$= dA^b{}_b(\xi)\, A^{-1b}{}_c\,\bar{\theta}^c - A^a{}_b\,\omega(\xi)^b{}_c\, A^{-1c}{}_d\,\bar{\theta}^d$$

true for all $\bar{\theta}$ $\Rightarrow$ proposition above. ✓

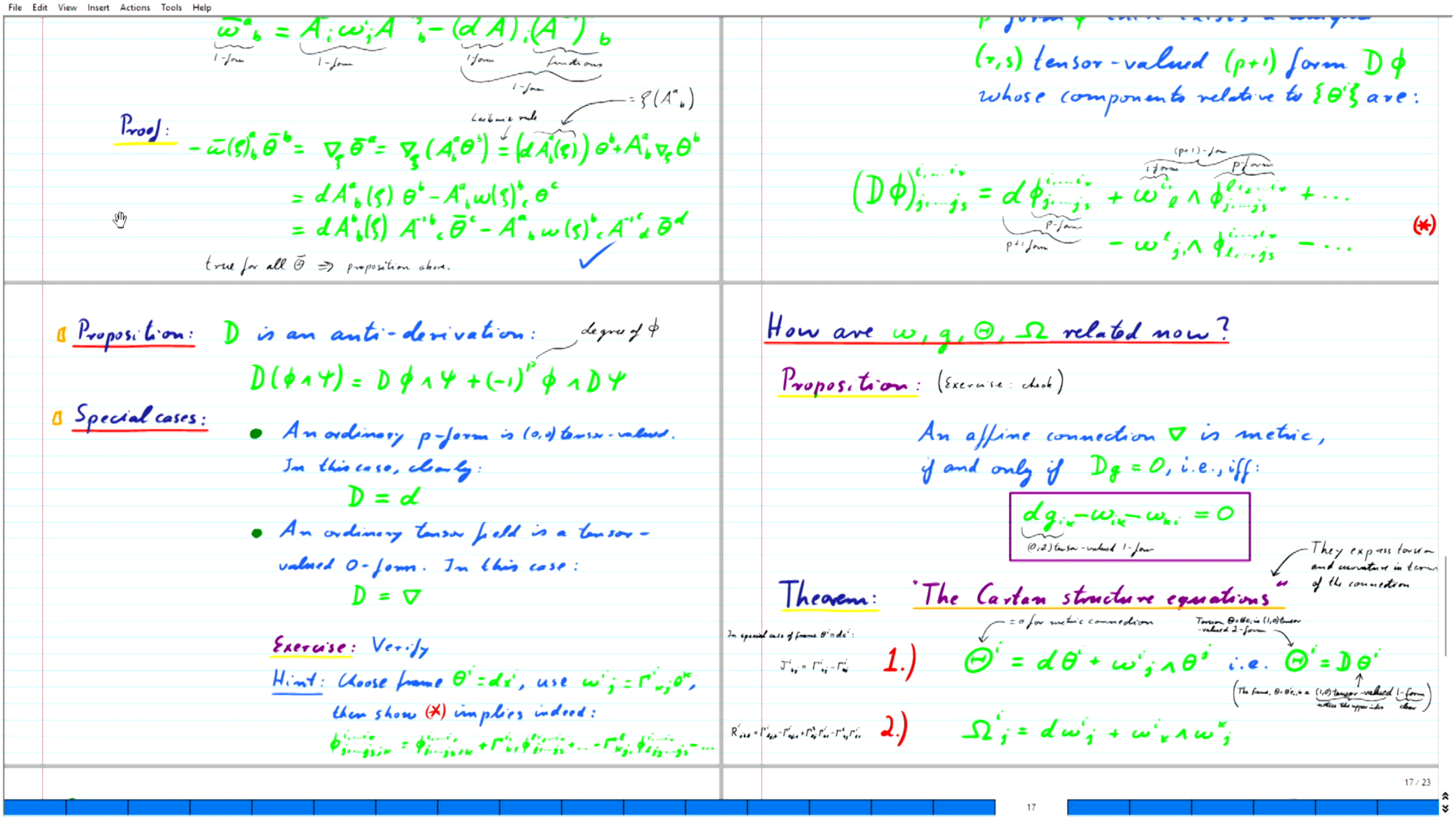
The "absolute exterior differential" D:

(It generalizes both ∇ and d)

Proposition: (proof, see e.g. Straumann: check tensorial behavior under frame change)

For every (r,s) tensor-valued p-form ϕ there exists a unique (r,s) tensor-valued $(p+1)$ form $D\phi$ whose components relative to $\{\theta^i\}$ are:

$$(D\phi)^{i_1\ldots i_r}_{j_1\ldots j_s} = \underbrace{d\,\underbrace{\phi^{i_1\ldots i_r}_{j_1\ldots j_s}}_{p\text{-form}}}_{p+1\text{ form}} + \overbrace{\overbrace{\omega^{i_1}{}_l}^{1\text{ form}} \wedge \overbrace{\phi^{l\,i_2\ldots i_r}_{j_1\ldots j_s}}^{p\text{-form}}}^{(p+1)\text{-form}} + \ldots - \omega^l{}_{j_1} \wedge \phi^{i_1\ldots i_r}_{l\ldots j_s} - \ldots \qquad (*)$$



$$\underbrace{\bar\omega^a{}_b}_{1\text{-form}} = \underbrace{A^a{}_i\,\omega^i{}_j\,A^{-1\,j}{}_b}_{1\text{-form}} - \underbrace{\underbrace{(dA)^a{}_i}_{1\text{-form}}\,\underbrace{(A^{-1})^i{}_b}_{\text{functions}}}_{1\text{-form}}$$

Proof:

$$-\bar\omega(\xi)^a{}_b\,\bar\theta^b = \nabla_\xi\bar\theta^a = \nabla_\xi(A^a{}_b\theta^b) \overset{\text{Leibniz rule}}{=} \big(\overbrace{dA^a{}_b(\xi)}^{=\xi(A^a{}_b)}\big)\,\theta^b + A^a{}_b\nabla_\xi\theta^b$$

$$= dA^a{}_b(\xi)\,\theta^b - A^a{}_b\,\omega(\xi)^b{}_c\,\theta^c$$

$$= dA^a{}_b(\xi)\,A^{-1\,b}{}_c\,\bar\theta^c - A^a{}_b\,\omega(\xi)^b{}_c\,A^{-1\,c}{}_d\,\bar\theta^d \quad ✓$$

true for all $\vec\theta$ ⇒ proposition above.

p-form φ there exists a unique (r,s) tensor-valued (p+1) form $D\phi$ whose components relative to $\{\theta^i\}$ are:

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Proposition: D is an anti-derivation:

$$D(\phi\wedge\psi) = D\phi\wedge\psi + (-1)^{\overset{\text{degree of }\phi}{p}}\,\phi\wedge D\psi$$

Special cases:

- An ordinary p-form is (0,0) tensor-valued. In this case, clearly:
  $$D = d$$
- An ordinary tensor field is a tensor-valued 0-form. In this case:
  $$D = \nabla$$

Exercise: Verify

Hint: Choose frame $\theta^i = dx^i$, use $\omega^i{}_j = \Gamma^i{}_{kj}\theta^k$, then show (*) implies indeed:

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How are ω, g, Θ, Ω related now?

Proposition: (Exercise: check)

An affine connection ∇ is metric, if and only if $Dg = 0$, i.e., iff:

$$\boxed{\underbrace{dg_{ik}}_{(0,2)\text{ tensor-valued 1-form}} - \omega_{ik} - \omega_{ki} = 0}$$

Theorem: "The Cartan structure equations" ← They express torsion and curvature in terms of the connection

In special case of frame $\theta^i = dx^i$:

$T^i{}_{kj} = \Gamma^i{}_{kj} - \Gamma^i{}_{jk}$

1.) $\overset{=0\text{ for metric connection}}{\Theta^i} = d\theta^i + \omega^i{}_j\wedge\theta^j$ i.e. $\Theta^i = D\theta^i$

Torsion $\Theta = \Theta^i e_i$ is (1,0) tensor-valued 2-form

(The frame, $\theta = \theta^i e_i$ is a $\underbrace{(1,0)\text{ tensor-valued}}_{\text{notice the upper index}}$ $\underbrace{1\text{-form}}_{\text{clear}}$)

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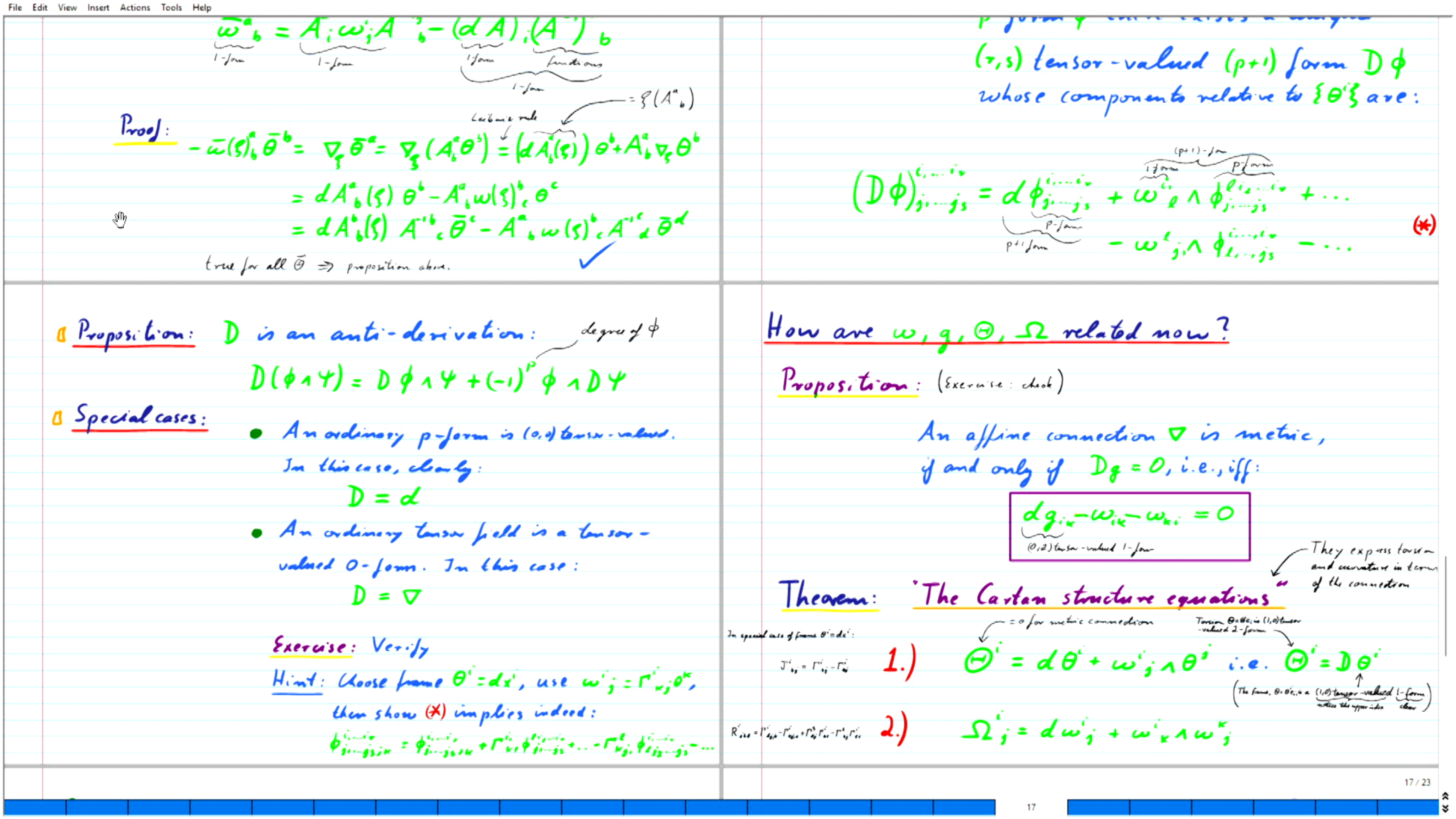
2.) $\Omega^i{}_j = d\omega^i{}_j + \omega^i{}_k\wedge\omega^k{}_j$



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$R^i{}_{jkl} = \Gamma^i{}_{lj,k} - \Gamma^i{}_{kj,l} + \Gamma^n{}_{lj}\Gamma^i{}_{kn} - \Gamma^n{}_{kj}\Gamma^i{}_{ln}$

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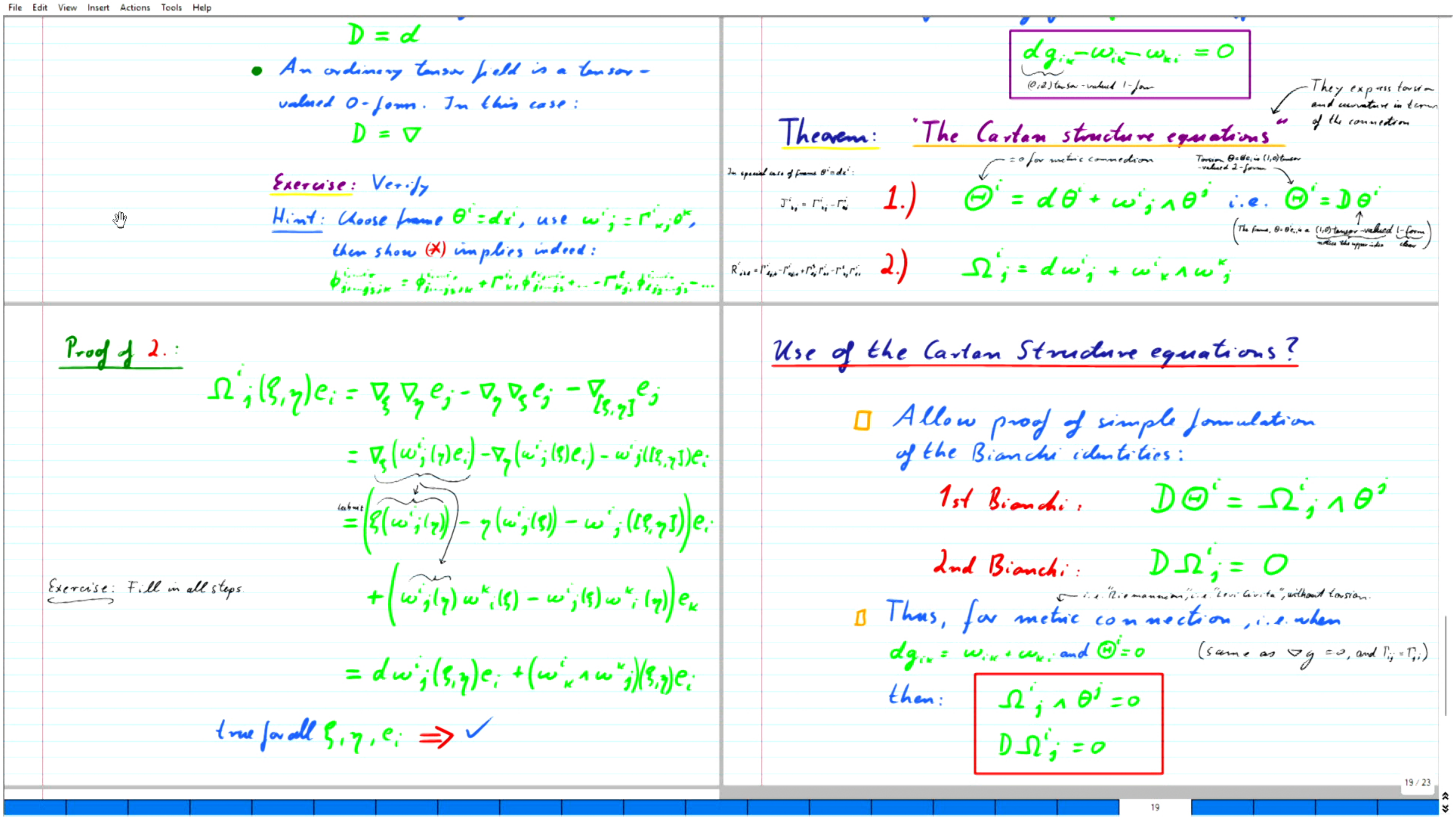

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Hint: Choose frame $\theta^i = dx^i$, use $\omega^i{}_j = \Gamma^i{}_{kj}\theta^k$,
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2.) $\Omega^i{}_j = d\omega^i{}_j + \omega^i{}_k \wedge \omega^k{}_j$

Proof of 2.:

$$\Omega^i{}_j(\xi,\eta)e_i = \nabla_\xi \nabla_\eta e_j - \nabla_\eta \nabla_\xi e_j - \nabla_{[\xi,\eta]} e_j$$

$$= \nabla_\xi\left(\omega^i{}_j(\eta)e_i\right) - \nabla_\eta\left(\omega^i{}_j(\xi)e_i\right) - \omega^i{}_j([\xi,\eta])e_i$$

$$\overset{\text{Leibniz}}{=} \left(\xi(\omega^i{}_j(\eta)) - \eta(\omega^i{}_j(\xi)) - \omega^i{}_j([\xi,\eta])\right)e_i$$

Exercise: Fill in all steps.

$$+ \left(\omega^i{}_j(\eta)\,\omega^k{}_i(\xi) - \omega^i{}_j(\xi)\,\omega^k{}_i(\eta)\right)e_k$$

$$= d\omega^i{}_j(\xi,\eta)e_i + (\omega^i{}_k \wedge \omega^k{}_j)(\xi,\eta)e_i$$

true for all ξ, η, e_i $\Rightarrow$ ✓

Use of the Cartan Structure equations?

- ☐ Allow proof of simple formulation of the Bianchi identities:

  1st Bianchi: $D\Theta^i = \Omega^i{}_j \wedge \theta^j$

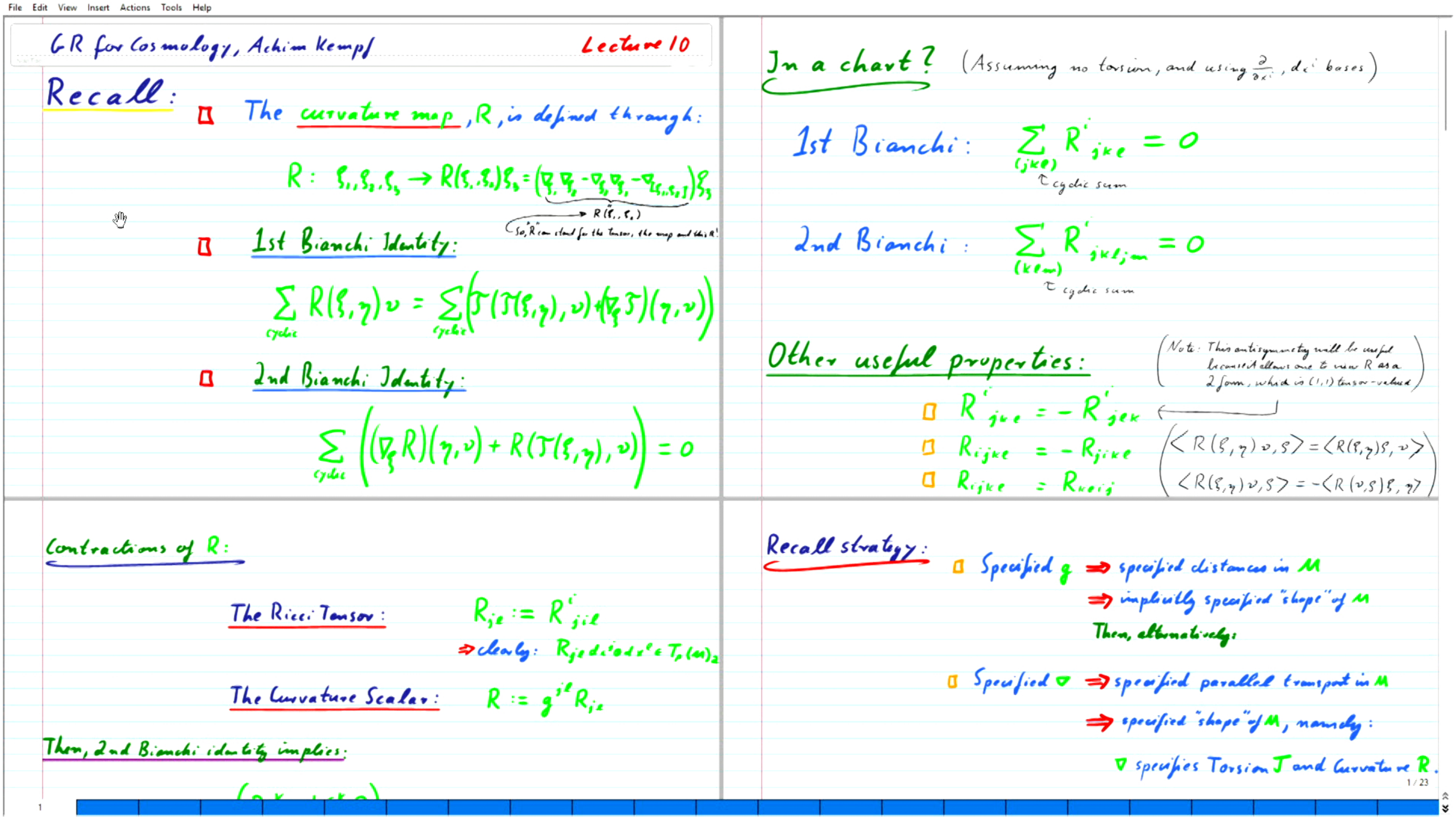
  2nd Bianchi: $D\Omega^i{}_j = 0$

- ☐ Thus, for metric connection, i.e. when $dg_{ik} = \omega_{ik} + \omega_{ki}$ and $\Theta^i = 0$ (same as $\nabla g = 0$, and $\Gamma^i_{ij} = \Gamma^i_{ji}$)

  ← i.e. "Riemannian", i.e. "Levi-Civita", without torsion.

  then:

$$\boxed{\begin{aligned}&\Omega^i{}_j \wedge \theta^j = 0\\ &D\Omega^i{}_j = 0\end{aligned}}$$



GR for Cosmology, Achim Kempf — Lecture 10

Recall:

☐ The curvature map, R, is defined through:

$$R:\ \xi_1,\xi_2,\xi_3 \to R(\xi_1,\xi_2)\xi_3 = \underbrace{\left(\nabla_{\xi_1}\nabla_{\xi_2} - \nabla_{\xi_2}\nabla_{\xi_1} - \nabla_{[\xi_1,\xi_2]}\right)}_{\to\ R(\xi_1,\xi_2)}\xi_3$$

So, "R" can stand for the tensor, the map and this R!

☐ 1st Bianchi Identity:

$$\sum_{\text{cyclic}} R(\xi,\eta)\nu = \sum_{\text{cyclic}}\Big(T(T(\xi,\eta),\nu) + (\nabla_\xi T)(\eta,\nu)\Big)$$

☐ 2nd Bianchi Identity:

$$\sum_{\text{cyclic}}\Big((\nabla_\xi R)(\eta,\nu) + R(T(\xi,\eta),\nu)\Big) = 0$$

In a chart? (Assuming no torsion, and using $\frac{\partial}{\partial x^i}$, dx^i bases)

1st Bianchi: $\sum_{(jkl)} R^i{}_{jkl} = 0$ ← cyclic sum

2nd Bianchi: $\sum_{(klm)} R^i{}_{jkl;m} = 0$ ← cyclic sum

Other useful properties:

(Note: This antisymmetry will be useful because it allows one to view R as a 2 form, which is (1,1) tensor-valued)

☐ $R^i{}_{jkl} = -R^i{}_{jlk}$

☐ $R_{ijkl} = -R_{jikl}$

☐ $R_{ijkl} = R_{klij}$

$$\left(\begin{array}{l}\langle R(\xi,\eta)\nu,\rho\rangle = \langle R(\xi,\eta)\rho,\nu\rangle \\ \langle R(\xi,\eta)\nu,\rho\rangle = -\langle R(\nu,\rho)\xi,\eta\rangle\end{array}\right)$$

Contractions of R:

The Ricci Tensor: $R_{jl} := R^i{}_{jil}$

⇒ clearly: $R_{jl}\, dx^j \otimes dx^l \in T_p(M)_2$

The Curvature Scalar: $R := g^{jl} R_{jl}$

Then, 2nd Bianchi identity implies:

Recall strategy:

☐ Specified g ⇒ specified distances in M

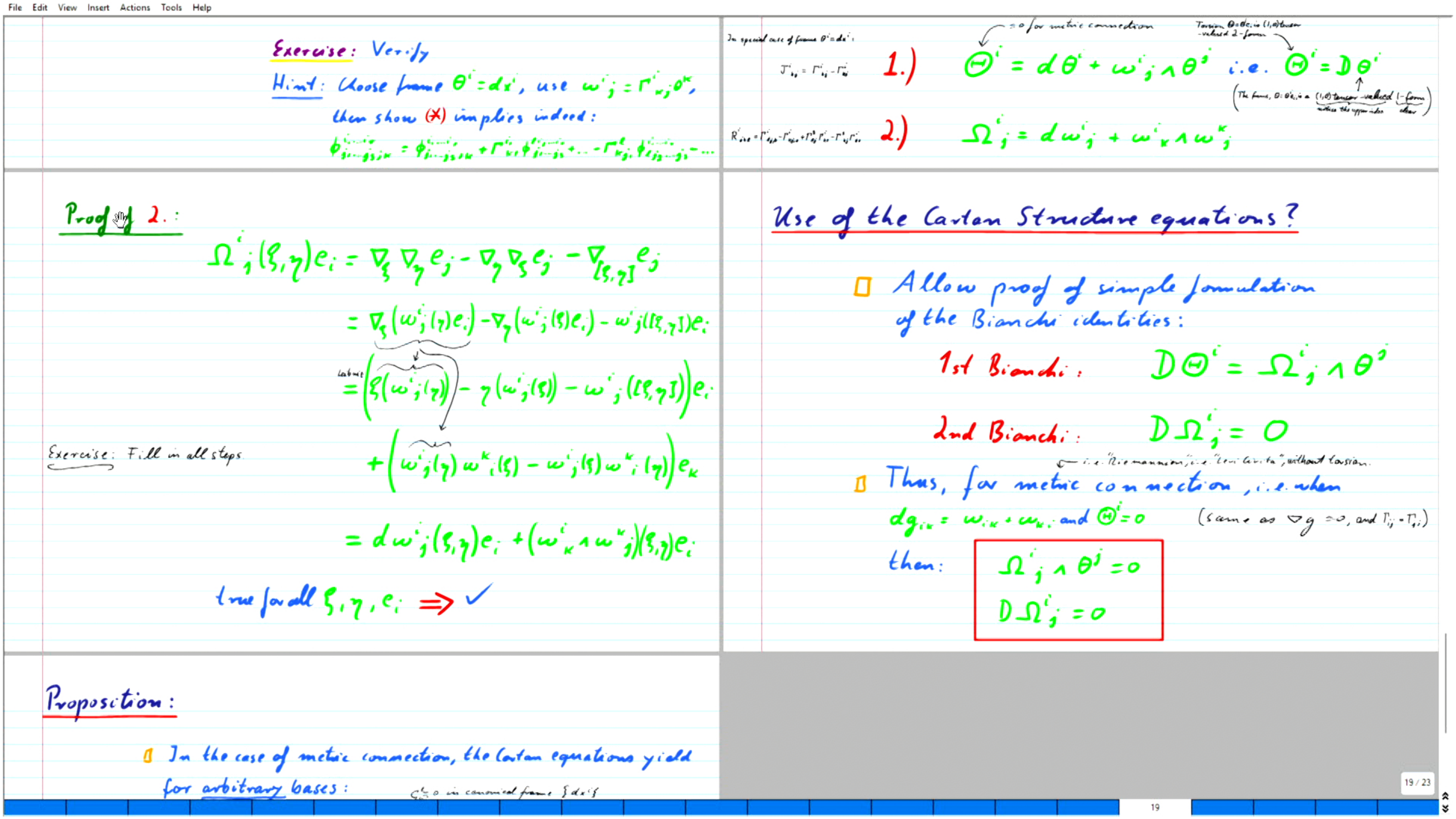
⇒ implicitly specified "shape" of M

Then, alternatively:

☐ Specified ∇ ⇒ specified parallel transport in M

⇒ specified "shape" of M, namely:

∇ specifies Torsion T and Curvature R.



Exercise: Verify

Hint: Choose frame $\theta^i = dx^i$, use $\omega^i{}_j = \Gamma^i{}_{kj}\theta^k$,
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In special case of frame $\theta^i = dx^i$:

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= 0 for metric connection

Torsion $\Theta = \Theta^i e_i$ is (1,0) tensor-valued 2-form

1.) $$\Theta^i = d\theta^i + \omega^i{}_j \wedge \theta^j \quad \text{i.e.} \quad \Theta^i = D\theta^i$$

(The frame, $\theta = \theta^i e_i$ is a $\underbrace{\text{(1,0) tensor-valued}}_{\text{notice the upper index}}$ $\underbrace{\text{1-form}}_{\text{clear}}$)

$R^i{}_{jk\ell} = \Gamma^i{}_{j\ell,k} - \Gamma^i{}_{jk,\ell} + \Gamma^s{}_{j\ell}\Gamma^i{}_{ks} - \Gamma^s{}_{jk}\Gamma^i{}_{\ell s}$

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$$\Omega^i{}_j(\xi,\eta)e_i = \nabla_\xi \nabla_\eta e_j - \nabla_\eta \nabla_\xi e_j - \nabla_{[\xi,\eta]} e_j$$

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Leibniz

$$= \Big(\xi(\omega^i{}_j(\eta)) - \eta(\omega^i{}_j(\xi)) - \omega^i{}_j([\xi,\eta])\Big)e_i$$

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Exercise: Fill in all steps.

$$= d\omega^i{}_j(\xi,\eta)e_i + (\omega^i{}_k \wedge \omega^k{}_j)(\xi,\eta)e_i$$

true for all ξ, η, e_i ⇒ ✓

Use of the Cartan Structure equations?

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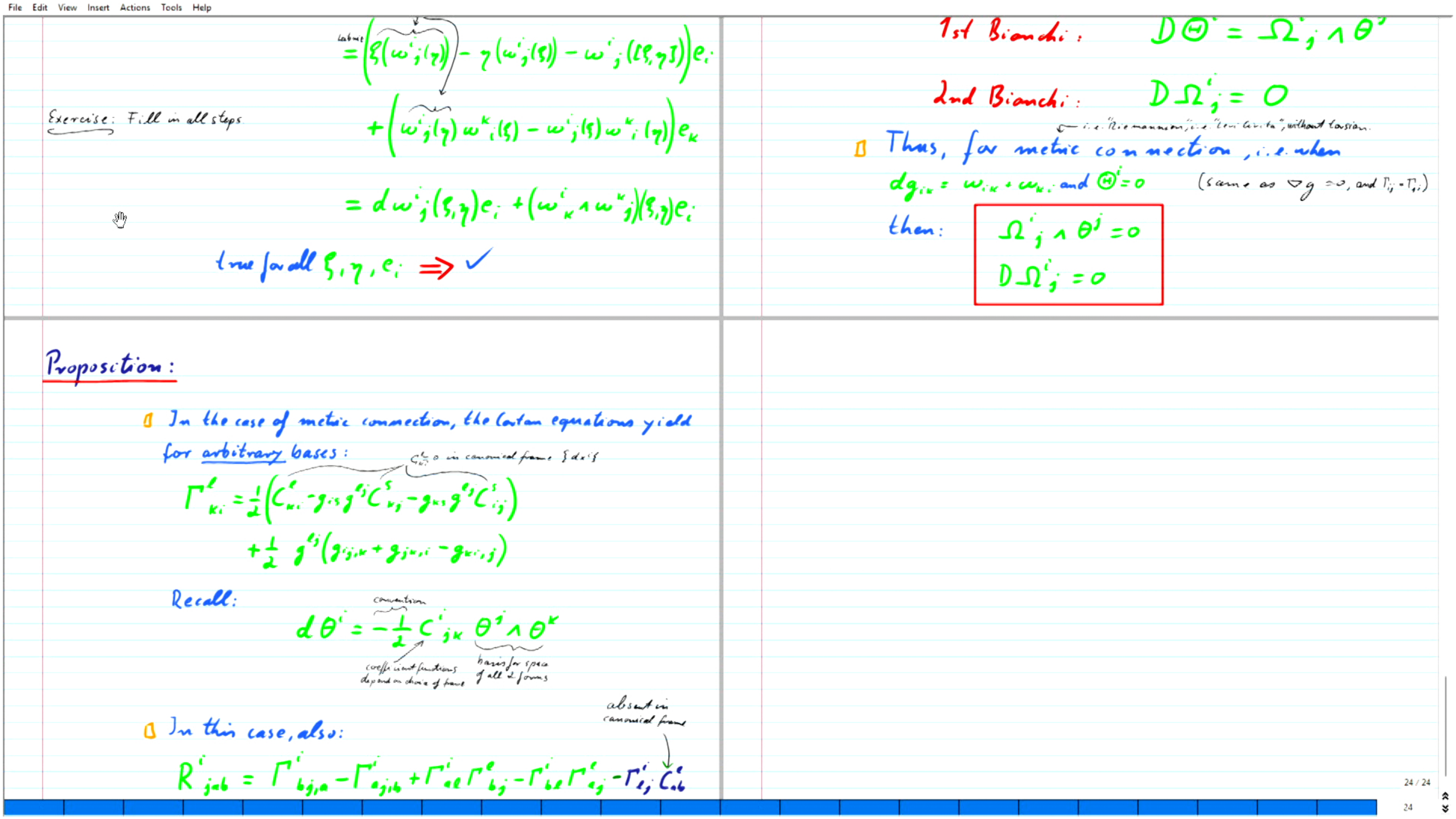
  $dg_{ik} = \omega_{ik} + \omega_{ki}$ and $\Theta^i = 0$ (same as $\nabla g = 0$, and $\Gamma_{ij} = \Gamma_{ji}$)

  then:
  $$\boxed{\begin{aligned} \Omega^i{}_j \wedge \theta^j &= 0 \\ D\Omega^i{}_j &= 0 \end{aligned}}$$

Proposition:

- ☐ In the case of metric connection, the Cartan equations yield for arbitrary bases:

  $\Gamma^k_{ij} = 0$ in canonical frame $\{dx^i\}$



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- Thus, for metric connection, i.e. when $dg_{ik} = \omega_{ik} + \omega_{ki}$ and $\Theta^i = 0$ (same as $\nabla g = 0$, and $\Gamma_{ij} = \Gamma_{ji}$) then:

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Proposition:

- In the case of metric connection, the Cartan equations yield for arbitrary bases:

$C^{\ell}_{ki} = 0$ in canonical frame $\{dx^i\}$

$$\Gamma^{\ell}{}_{ki} = \frac{1}{2}\left(C^{\ell}{}_{ki} - g_{is}g^{\ell j}C^{s}{}_{kj} - g_{ks}g^{\ell j}C^{s}{}_{ij}\right) + \frac{1}{2}g^{\ell j}\left(g_{ij,k} + g_{jk,i} - g_{ki,j}\right)$$

Recall:

$$d\theta^i = \overset{\text{convention}}{-\frac{1}{2}}\, C^i{}_{jk}\, \underbrace{\theta^j \wedge \theta^k}_{\text{basis for space of all 2 forms}}$$

coefficient functions depend on choice of frame

- In this case, also:

absent in canonical frame

$$R^i{}_{jab} = \Gamma^i{}_{bj,a} - \Gamma^i{}_{aj,b} + \Gamma^i{}_{ae}\Gamma^e{}_{bj} - \Gamma^i{}_{be}\Gamma^e{}_{aj} - \Gamma^i{}_{ej}C^e{}_{ab}$$