Title: General Relativity for Cosmology Lecture - 100523

Speakers: Achim Kempf

Collection: General Relativity for Cosmology

Date: October 05, 2023 - 2:00 PM

URL: https://pirsa.org/23100023

Abstract: Zoom: https://pitp.zoom.us/j/91640855624?pwd=dWVWV2doSnBhUS9JUkhjQVBwY0h0dz09

File Edit View Insert Actions Tools Help

Sharing Desktop | Stop Sharing

(Notice (for essay): See also my paper 1510.02...)

Then, new:

- Specified ∇ $\Rightarrow$ specified parallel transport in M
  $\Rightarrow$ implicitly specified "shape" of M

Question:

How does ∇ determine "shape"? Through:

Torsion & Curvature!

$$\Gamma^k_{(sym)ij} := \frac{1}{2}(\Gamma^k{}_{ij} + \Gamma^k{}_{ji})$$
$$\Gamma^k_{(asym)ij} := \frac{1}{2}(\Gamma^k{}_{ij} - \Gamma^k{}_{ji})$$
$$\Gamma^k_{ij} = \Gamma^k_{(sym)ij} + \Gamma^k_{(asym)ij}$$

$$\Rightarrow \bar{\Gamma}^r_{(asym)ab} = \frac{\partial \bar{x}^r}{\partial x^k}\frac{\partial x^i}{\partial \bar{x}^a}\frac{\partial x^j}{\partial \bar{x}^b}\Gamma^k_{(asym)ij} \quad !$$

Definition: $T^k{}_{ij} := 2\,\Gamma^k_{(asym)ij}$ is the "Torsion tensor"

(Notice: Since Γ is not a tensor, but Γ_{asym} is, Γ_{sym} is not a tensor)

In General Relativity: one assumes torsionless ∇, i.e.: $T = 0$.

Idea: "(Extended) equivalence principle:"

Christoffel Γ will express gravitational and pseudo forces.
Therefore, we require that around each $p \in M$ there exists a chart so that $\Gamma(p) = 0$ (i.e. no such forces in free fall).

This rules out the existence of torsion:

Why? The torsion is a tensor.

$\Rightarrow$ It transforms linearly with invertible Jacobian matrices

$$\bar{T}^i{}_{jk}(\bar{x}) = \frac{\partial \bar{x}^i}{\partial x^a}\frac{\partial x^b}{\partial \bar{x}^j}\frac{\partial x^c}{\partial \bar{x}^k} T^a{}_{bc}(x)$$

$\Rightarrow$ If T_{ij} vanishes in one cds, it vanishes in all cds.

Proposition: Vice versa, if $T^i{}_{jk}(x) = 0 \;\forall x \in M$, then there is for every $p \in M$ a chart with $\Gamma^i{}_{jk}(p) = 0$.

Recall: ξ is autoparallel to a path $\gamma: t \to x(t)$ if
$\xi \in T'(M)$

$$\boxed{\nabla_\gamma \xi = 0}$$

Meaning: ξ is parallel transported along the path γ in M.

Explicitly: $\frac{d\xi^r}{dt} + \Gamma^r{}_{ij}\frac{dx^i}{dt}\xi^j = 0$

Geodesics: A curve $\gamma: t \to x(t)$ is called a geodesic if $\dot{\gamma}$ is autoparallel along γ, i.e. if

$$\boxed{\nabla_{\dot{\gamma}} \dot{\gamma} = 0}$$

Meaning: The path γ in M is such that the path's tangent vectors are parallel translates of each other.

1 1 / 23

Christoffel Γ will express gravitational and pseudo forces. Therefore, we require that around each $p \in M$ there exists a chart so that $\Gamma(p) = 0$ (i.e. no such forces in free fall).

This rules out the existence of torsion:

Why? The torsion is a tensor.

$\Rightarrow$ It transforms linearly with invertible Jacobian matrices

$$\bar{T}^i_{jk}(\bar{x}) = \frac{\partial \bar{x}^i}{\partial x^a} \frac{\partial x^b}{\partial \bar{x}^j} \frac{\partial x^c}{\partial \bar{x}^k} T^a_{bc}(x)$$

$\Rightarrow$ If T_{ij} vanishes in one cds, it vanishes in all cds.

Recall: $\xi \in T'(M)$ is autoparallel to a path $\gamma : t \to x(t)$ if

$$\boxed{\nabla_{\dot\gamma} \xi = 0}$$

Meaning: ξ is parallel transported along the path γ in M.

Explicitly: $\frac{d\xi^r}{dt} + \Gamma^r{}_{ij} \frac{dx^i}{dt} \xi^j = 0$

Geodesics: A curve $\gamma : t \to x(t)$ is called a geodesic if $\dot\gamma$ is autoparallel along γ, i.e. if

$$\boxed{\nabla_{\dot\gamma} \dot\gamma = 0}$$

Meaning: The path γ in M is such that the path's tangent vectors are parallel translates of each other.

- $\Rightarrow$ In charts, geodesics $x^r(t)$ obey:

$$\frac{d^2 x^r}{dt^2} + \Gamma(x)^r{}_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt} = 0 \qquad (*)$$

- Theory of ordinary differential equations:

$\Rightarrow$ Given $p = \gamma(0)$, each initial condition $\xi = \dot\gamma(0)$ belongs to a unique geodesic γ_ξ of nonzero length.

Subscript indicates initial condition vector

- **Notice:** If $\gamma_\xi(t)$ solves $(*)$ then $\gamma_\xi(\lambda t)$ also solves $(*)$ and for $\lambda \in \mathbb{R}$:

$$\gamma_{\lambda\xi}(t) = \gamma_\xi(\lambda t) \qquad (G)$$

"Exponential map":

- Consider a fixed point $p \in M$.

The exponential map is defined through:

$$\exp_p : T_p'(M) \to M$$

(really from a neighborhood of 0 in $T_p'(M)$ to a neighborhood of p in M.)

$$\exp_p : \quad \xi \to \exp_p(\xi) := \gamma_\xi(1)$$

where γ is the geodesic with $\gamma_\xi(0) = p$, $\dot\gamma_\xi(0) = \xi$.

- **Observe:**

From (G) we obtain:

$$\gamma_\xi(\lambda) = \gamma_{\lambda\xi}(1) = \exp_p(\lambda\xi) \qquad (E)$$

3/23

File Edit View Insert Actions Tools Help

Sharing Desktop | Stop Sharing

$\Rightarrow$ Given $p = \gamma(0)$, each initial condition $\xi = \dot{\gamma}(0)$ belongs to a unique geodesic γ_ξ of nonzero length.

subscript indicates initial condition vector

▫ **Notice:** If $\gamma_\xi(t)$ solves $(*)$ then $\gamma_\xi(\lambda t)$ also solves $(*)$ and for $\lambda \in \mathbb{R}$:

$$\gamma_{\lambda\xi}(t) = \gamma_\xi(\lambda t) \qquad (G)$$

(Exercise: verify)

$$\exp_p : T_p(M) \to M$$

$$\exp_p : \quad \xi \to \exp_p(\xi) := \gamma_\xi(1)$$

where γ is the geodesic with $\gamma_\xi(0) = p$, $\dot{\gamma}_\xi(0) = \xi$.

▫ **Observe:**

From (G) we obtain:

$$\gamma_\xi(\lambda) = \gamma_{\lambda\xi}(1) = \exp_p(\lambda\xi) \qquad (E)$$

"Geodesic" or "Riemann normal" coordinates:

▫ $\exp_p$ is a diffeomorphism from a neighborhood of $0 \in T_p(M) \underset{\text{isomorphic}}{\cong} \mathbb{R}^n$ into a neighborhood of the point $p \in M$.

$\Rightarrow$ $\exp_p$ provides a chart around p:

▫ Choose a basis, say $e_1, e_2, \ldots, e_n$ of $T_p(M)$, then: $\xi = \xi^i e_i$

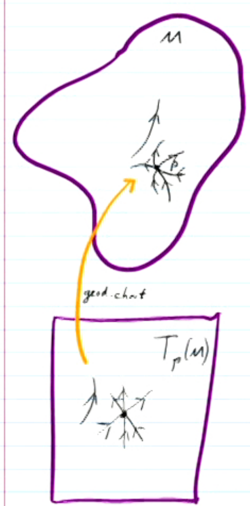
▫ Through $\exp_p$, the ξ^i become the coordinates of points in a neighborhood of $p \in M$:

$$(\xi^1, \ldots, \xi^n) \to \exp_p(\xi^i e_i) \in M$$

▫ These $\{\xi^i\}$ are called "normal" or "geodesic coordinates."

$\Rightarrow$ Geodesics, γ, through p are straight lines in a normal cds about p!

▫ Recall (E):



$$\underbrace{\gamma_\xi(\lambda)}_{\substack{\text{for varying}\\ \lambda \text{ one moves}\\ \text{along the geodesic}\\ \text{in } M.}} = \exp_p(\underbrace{\lambda\xi}_{\substack{\text{for varying } \lambda\\ \text{one moves on}\\ \text{a straight line}\\ \text{in the coordinate}\\ \text{system of the } \xi^i!}})$$

▫ **Thus:** In geodesic cds, geodesics through p are straight lines of constant velocity ξ.

▫ Does this mean $\Gamma^k{}_{ij}(p) = 0$? No!

5 / 23

5

File Edit View Insert Actions Tools Help

Sharing Desktop | Stop Sharing

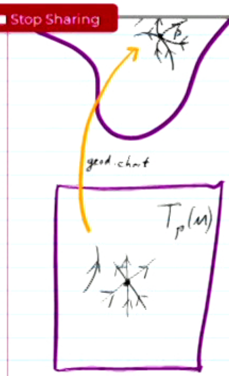
- Choose a basis, say $e_1, e_2, \dots, e_n$ of $T_p^1(M)$, then:

$$\xi = \xi^i e_i$$

- Through $\exp_p$, the ξ^i become the coordinates of points in a neighborhood of $p \in M$:

$$(\xi^1, \dots, \xi^n) \to \exp_p(\xi^i e_i) \in M$$

- These $\{\xi^i\}$ are called "normal" or "geodesic coordinates."



along the geodesic in M

a straight line in the coordinate system of the ξ^i!

- **Thus:** In geodesic cds, geodesics through p are straight lines of constant velocity ξ.

- Does this mean $\Gamma^k{}_{;ij}(p) = 0$? **No!**

Geodesic eqn. at p:

$$\cancel{\frac{d^2x^k}{dt^2}} + \Gamma(p)^k{}_{;ij} \frac{dx^i}{dt}\frac{dx^j}{dt} = 0$$

Thus:

$$\left(\Gamma_{sym}{}^k{}_{;ij}(p) + \Gamma_{asym}{}^k{}_{;ij}(p)\right)\frac{dx^i}{dt}\frac{dx^j}{dt} = 0$$

$= 0$ because of type $(\text{antisymmetric})_{ij}\,(\text{symmetric})^{ij}$

$$\Rightarrow \Gamma_{sym}{}^k{}_{;ij}(p) = 0 \text{ in geodesic cds.}$$

$\Rightarrow$ **Indeed:** If the torsion vanishes, $\Gamma_{asym}{}^k{}_{;ij}(p) = \frac{1}{2} J^k{}_{;ij}(p) = 0$ then for each $p \in M$ there exists a chart in which the entire gravity and pseudo force field vanishes at p:

$$\Gamma^k{}_{;ij}(p) = 0$$

Note:
Quantum fluctuations may induce torsion!
So, let's nevertheless ask:

What would torsion mean, geometrically?

Abstract definition of Torsion:

- Assume ξ_1 and ξ_2 are tangent vectors at $p \in M$:

Then, the Torsion map is defined as:

$$\mathcal{T} : T_p^1(M) \times T_p^1(M) \to T_p^1(M)$$

This will be the amount by which an infinitesimal parallelogram spanned by ξ_1 and ξ_2 does not close.

$$\mathcal{T} : \xi_1, \xi_2 \to \mathcal{T}(\xi_1, \xi_2) := \nabla_{\xi_1}\xi_2 - \nabla_{\xi_2}\xi_1 - [\xi_1, \xi_2]$$

- It is used to define the Torsion tensor, J,

for proof it's a tensor, see Stewart

$$J \in T_{p\,2}^{\,1}(M)$$

through:

we could also write: $= \omega(\mathcal{T}(\xi_1, \xi_2))$

contraction yields a number

feeding 1 covector & 2 vectors to a (1,2) tensor yields a number $\longrightarrow$

$$J(\omega, \xi_1, \xi_2) := \langle \underbrace{\omega}_{\in T_{p,1}(M)}, \underbrace{\mathcal{T}(\xi_1, \xi_2)}_{\in T_p^1(M)} \rangle \in \mathbb{R}$$

File Edit View Insert Actions Tools Help

Sharing Desktop | Stop Sharing

$$\Rightarrow \Gamma^{\kappa}_{(\text{sym}) \, ij}(p) = 0 \quad \text{in geodesic cds.}$$

$\Rightarrow$ **Indeed:** If the torsion vanishes, $\Gamma^{\kappa}_{(\ldots)\, ij}(p) = \frac{1}{2} T^{\kappa}_{\ ij}(p) = 0$ then for each $p \in M$ there exists a chart in which the entire gravity and pseudo force field vanishes at p:

$$\Gamma^{\kappa}_{\ ij}(p) = 0$$

Note: Quantum fluctuations may induce torsion! So, let's nevertheless ask:

What would torsion mean, geometrically?

$$T : T_p(M) \times T_p(M) \to T_p^1(M)$$

parallelogram spanned by ξ_1 and ξ_2 does not close.

$$T : \xi_1, \xi_2 \to T(\xi_1, \xi_2) := \nabla_{\xi_1}\xi_2 - \nabla_{\xi_2}\xi_1 - [\xi_1, \xi_2]$$

- It is used to define the Torsion tensor, T, (for proof it's a tensor, see Straumann)

$$T \in T_{p\,2}^{\,1}(M)$$

through:

feeding 1 covector & 2 vectors to a (1,2) tensor yields a number $\longrightarrow$

$$T(\omega, \xi_1, \xi_2) := \langle \underbrace{\omega}_{\in T_{p,1}(M)}, \underbrace{T(\xi_1, \xi_2)}_{\in T_p^{1}(M)} \rangle \in \mathbb{R}$$

(contraction yields a number; we could also write: $= \omega(T(\xi_1, \xi_2))$)

Compare with prior definition:

- Choose canonical bases $\omega := dx^k$, $\xi_1 := \frac{\partial}{\partial x^i}$, $\xi_2 := \frac{\partial}{\partial x^j}$:

- $$T^{\kappa}_{\ ij} := dx^{\kappa}\left(T\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)\right)$$

$$= \left\langle dx^{\kappa}, T\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)\right\rangle \quad \text{(more convenient notation)}$$

$$= \left\langle dx^{\kappa}, \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} - \nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i} - \left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right]\right\rangle$$

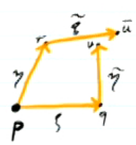
Recall: $\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \Gamma^{\kappa}_{\ ij} \frac{\partial}{\partial x^{\kappa}}$ $\qquad \left(\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j} - \frac{\partial}{\partial x^j}\frac{\partial}{\partial x^i}\right) f = 0 \quad \forall f$

$$= \left\langle dx^{\kappa}, \Gamma^{r}_{\ ij} \frac{\partial}{\partial x^r} - \Gamma^{r}_{\ ji} \frac{\partial}{\partial x^r} \right\rangle = \Gamma^{r}_{\ ij}\delta^{\kappa}_{r} - \Gamma^{r}_{\ ji}\delta^{\kappa}_{r}$$

- $$\Rightarrow \boxed{T^{\kappa}_{\ ij} = \Gamma^{\kappa}_{\ ij} - \Gamma^{\kappa}_{\ ji}}$$

Geometric meaning of torsion? Parallelograms would not close!

Travel from p infinitesimally in ξ and then η direction, and compare with the reverse. (In flat space: $x^{r} + \eta^{r} + \xi^{r} = x^{r} + \xi^{r} + \eta^{r}$.)



$\xi, \eta \in T_p^1$

$\tilde{\xi} \in T_r^1$

$\tilde{\eta} \in T_q^1$

Recall parallel transport: $\nabla_{\dot{\gamma}} v = 0$

$$\frac{dv^k}{dt} + \Gamma^{k}_{\ ij}\frac{dx^i}{dt}v^j = 0$$

$\tilde{\xi}(r) = ?$

$$\tilde{\xi}^k(x^i + \eta^i) \simeq \xi^k(x^i) + \frac{d\xi^k}{dt}(x^i) \qquad \text{Now use } v := \xi, \ \frac{dx^i}{dt} = \eta^i:$$

$$= \xi^k(x^i) - \Gamma(x)^k_{\ ij}\eta^i\xi^j$$

$$\Rightarrow \text{Cds. of } \bar{u}: \quad x^a + \eta^a + \xi^a - \Gamma(x)^a_{\ ij}\eta^i\xi^j$$

9 / 23

9

File Edit View Insert Actions Tools Help

Sharing Desktop ■ Stop Sharing

□ $T^k{}_{ij} := dx^k\left(T\left(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}\right)\right)$

$= \left\langle dx^k, T\left(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}\right)\right\rangle$ (more convenient notation)

$= \left\langle dx^k, \nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j} - \nabla_{\frac{\partial}{\partial x^j}}\frac{\partial}{\partial x^i} - \underbrace{\left[\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}\right]}\right\rangle$

Recall: $\nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j} = \Gamma^r{}_{ij}\frac{\partial}{\partial x^r}$ $\qquad \left(\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j} - \frac{\partial}{\partial x^j}\frac{\partial}{\partial x^i}\right)f = 0 \quad \forall f$

$= \left\langle dx^k, \Gamma^r{}_{ij}\frac{\partial}{\partial x^r} - \Gamma^r{}_{ji}\frac{\partial}{\partial x^r}\right\rangle = \Gamma^r{}_{ij}\delta^k_r - \Gamma^r{}_{ji}\delta^k_r$

□ $\Rightarrow$ $\boxed{T^k{}_{ij} = \Gamma^k{}_{ij} - \Gamma^k{}_{ji}}$

$\xi, \eta \in T_p$

$\tilde\xi \in T_r$

$\tilde\eta \in T_q$

Recall parallel transport: $\nabla_{\dot\gamma} v = 0$

$$\frac{dv^k}{dt} + \Gamma^k{}_{ij}\frac{dx^i}{dt}v^j = 0$$

$\tilde\xi(r) = ?$

$\tilde\xi^k(x^i+\eta^i) \approx \xi^k(x^i) + \frac{d\xi^k}{dt}(x^i)$ Now use $v := \xi$, $\frac{dx^i}{dt} = \eta^i$:

$= \xi^k(x^i) - \Gamma(x)^k{}_{ij}\eta^i\xi^j$

$\Rightarrow$ Cds. of $\bar u$: $x^a + \eta^a + \xi^a - \Gamma(x)^a{}_{ij}\eta^i\xi^j$

Analogously obtain: Cds. of u: $x^a + \xi^a + \eta^a - \Gamma(x)^a{}_{ij}\xi^i\eta^j$

Torsion!

$\Rightarrow$ Cd. distance from u to $\bar u$ is: $\left(\Gamma(x)^a{}_{ji} - \Gamma(x)^a{}_{ij}\right)\eta^i\xi^j = T^a{}_{ji}\eta^j\xi^i$ ✓

Comment: We had:

$$\tilde\xi^k(x^i+\eta^i) \approx \xi^k(x^i) + \frac{d\xi^k}{dt}(x^i) = \xi^k(x^i) - \Gamma(x)^k{}_{ij}\eta^i\xi^j$$

this is also: $= \xi^k(x^i) - \left(\eta^i\xi^k{}_{,i} + \Gamma(x)^k{}_{ij}\eta^i\xi^j\right) + \eta^i\xi^k{}_{,i}$

$= \xi^k(x^i) - \eta^i\xi^k{}_{;i} + \eta^i\xi^k{}_{,i}$

Thus: cd distance from u to $\bar u$ is:

$$\left(\cancel{x^a} + \cancel{\eta^a} + \cancel{\xi^a} - \eta^i\xi^k{}_{;i} + \eta^i\xi^k{}_{,i}\right) - \left(\cancel{x^a} - \cancel{\xi^a} - \cancel{\eta^a} + \xi^i\eta^k{}_{;i} - \eta^i\xi^k{}_{,i}\right) = T^a{}_{ji}\eta^j\xi^i$$

Recall that indeed: $T: \eta, \xi \to T(\eta,\xi) = \nabla_\eta\xi - \nabla_\xi\eta - [\eta,\xi]$

Curvature:

Assume ξ_1, ξ_2 and ξ_3 are tangent vectors at $p \in M$.

□ The curvature map, R, is defined through:

$$R: \xi_1,\xi_2,\xi_3 \to \underbrace{R(\xi_1,\xi_2)}_{\text{an operator, or map, acting on } \xi_3}\xi_3 = \overbrace{\left(\nabla_{\xi_1}\nabla_{\xi_2} - \nabla_{\xi_2}\nabla_{\xi_1} - \nabla_{[\xi_1,\xi_2]}\right)\xi_3}^{\in T_p(M)}$$

□ It defines the curvature tensor, R,

can be fed one covector and 3 vectors to yield a number

$$R \in T^1_3(M)$$

through:

$$R(\omega,\xi_1,\xi_2,\xi_3) := \langle\omega, \overbrace{R(\xi_1,\xi_2)\xi_3}^{\omega(R(\xi_1,\xi_2)\xi_3)}\rangle \in \mathbb{R}$$

11 11/23

File Edit View Insert Actions Tools Help

Sharing Desktop | Stop Sharing

$$\tilde{\xi}^k(x^i+\eta^i) \approx \xi^k(x^i) + \frac{d\xi^k}{dt}(x^i) = \xi^k(x^i) - \Gamma(x)^k{}_{ij}\,\eta^i\xi^j$$

this is also:

$$= \xi^k(x^i) - \left(\eta^i\xi^k{}_{;i} + \Gamma(x)^k{}_{ij}\,\eta^i\xi^j\right) + \eta^i\xi^k{}_{,i}$$

$$= \xi^k(x^i) - \eta^i\xi^k{}_{;i} + \eta^i\xi^k{}_{,i}$$

Thus: cd distance from u to $\tilde{u}$ is:

$$\left(\cancel{x^k} + \cancel{\eta^k} + \cancel{\xi^k} - \eta^i\xi^k{}_{;i} + \eta^i\xi^k{}_{,i}\right) - \left(\cancel{x^k} - \cancel{\xi^k} - \cancel{\eta^k} + \xi^i\eta^k{}_{;i} - \eta^i\xi^k{}_{,i}\right) = T^k_{j;i}\,\eta^i\xi^j$$

Recall that indeed: $T: \eta,\xi \to T(\eta,\xi) = \nabla_\eta\xi - \nabla_\xi\eta - [\eta,\xi]$

In a chart:

$$R^i{}_{jkl} = \left\langle dx^i, R\left(\frac{\partial}{\partial x^k},\frac{\partial}{\partial x^l}\right)\frac{\partial}{\partial x^j}\right\rangle$$

$$= \left\langle dx^i, \left(\nabla_{\frac{\partial}{\partial x^k}}\nabla_{\frac{\partial}{\partial x^l}} - \nabla_{\frac{\partial}{\partial x^l}}\nabla_{\frac{\partial}{\partial x^k}} - \nabla_{\underbrace{[\frac{\partial}{\partial x^k},\frac{\partial}{\partial x^l}]}_{=0}}\right)\frac{\partial}{\partial x^j}\right\rangle$$

$$= \left\langle dx^i, \nabla_{\frac{\partial}{\partial x^k}}\Gamma^s{}_{lj}\frac{\partial}{\partial x^s} - \nabla_{\frac{\partial}{\partial x^l}}\Gamma^s{}_{kj}\frac{\partial}{\partial x^s}\right\rangle$$

$$= \left\langle dx^i, \left(\overbrace{\Gamma^s{}_{lj,k} + \Gamma^r{}_{lj}\Gamma^s{}_{kr}} - \overbrace{\Gamma^s{}_{kj,l} - \Gamma^r{}_{kj}\Gamma^s{}_{lr}}\right)\frac{\partial}{\partial x^s}\right\rangle$$

$$= \Gamma^i{}_{lj,k} - \Gamma^i{}_{kj,l} + \underbrace{\Gamma^s{}_{lj}\Gamma^i{}_{ks} - \Gamma^s{}_{kj}\Gamma^i{}_{ls}}_{\text{(at origin of geodesic cds they vanish.)}}$$

$$R: \xi_1,\xi_2,\xi_3 \to \underbrace{R(\xi_1,\xi_2)}_{\text{an operator, or map, acting on } \xi_3}\xi_3 = \overbrace{\left(\nabla_{\xi_1}\nabla_{\xi_2} - \nabla_{\xi_2}\nabla_{\xi_1} - \nabla_{[\xi_1,\xi_2]}\right)\xi_3}^{\in T_0^1(M)}$$

- It defines the curvature tensor, R,

can be fed one covector and 3 vectors to yield a number

$$R \in T^1_3(M)$$

through:

$$R(\omega,\xi_1,\xi_2,\xi_3) := \overbrace{\langle \omega, R(\xi_1,\xi_2)\xi_3\rangle}^{\equiv\, \omega(R(\xi_1,\xi_2)\xi_3)} \in \mathbb{R}$$

Curvature tensor's meaning?

Intuition:

- Contains derivatives of Γ $\Rightarrow$
- expresses variation in gravitational forces $\Rightarrow$
- expresses the strength and direction of "tidal forces".

Geometry:

- Curvature expresses noncommutativity of two parallel transports, namely:

File Edit View Insert Actions Tools Help

Sharing Desktop | Stop Sharing

$$= \left\langle dx^i, \nabla_{\frac{\partial}{\partial x^k}} \Gamma^s_{\ell j} \frac{\partial}{\partial x^s} - \nabla_{\frac{\partial}{\partial x^\ell}} \Gamma^s_{kj} \frac{\partial}{\partial x^s} \right\rangle$$

$$= \left\langle dx^i, \left(\Gamma^s_{\ell j,k} + \Gamma^r_{\ell j} \Gamma^s_{kr} - \Gamma^s_{kj,\ell} - \Gamma^r_{kj} \Gamma^s_{\ell r} \right) \frac{\partial}{\partial x^s} \right\rangle$$

$$= \Gamma^i_{\ell j,k} - \Gamma^i_{kj,\ell} + \underbrace{\Gamma^s_{\ell j} \Gamma^i_{ks} - \Gamma^s_{kj} \Gamma^i_{\ell s}}_{\text{(at origin of geodesic cds they vanish.)}}$$

$\Rightarrow$

- expresses the strength and direction of "tidal forces."

Geometry:

- Curvature expresses noncommutativity of two parallel transports, namely:

Proposition: (Ricci Identity)

Assume the torsion vanishes and that ξ is a vector field. Then:

$$\boxed{\xi^a{}_{;cd} - \xi^a{}_{;dc} = R^a{}_{cdb}\, \xi^b}$$

(here: $\xi^a{}_{;cd} := \xi^a{}_{;c;d}$ etc.)

Remark: (a bit messy to derive because need Taylor expansion, see, e.g., text by Stewart or Einstein)

It implies that for parallel transport along infinitesimal parallelogram:

$$\boxed{(\hat{\xi} - \xi)^a \approx \eta^c \nu^d R^a{}_{bcd}\, \xi^b}$$

Proof of Ricci identity:

- Assume ξ, η, ν are vector fields.
- Then, $R(\xi,\eta)\nu := \nabla_\xi(\nabla_\eta \nu) - \nabla_\eta(\nabla_\xi \nu) - \nabla_{[\xi,\eta]}\nu$ reads

use: $\nabla_\eta \nu = \nabla_{\eta^i \frac{\partial}{\partial x^i}} \left(\nu^j \frac{\partial}{\partial x^j}\right) = \eta^i \nabla_{\frac{\partial}{\partial x^i}} \left(\nu^j \frac{\partial}{\partial x^j}\right) = \eta^i \nu^j{}_{;i} \frac{\partial}{\partial x^j} \ldots$

in basis: $R^a{}_{bcd}\, \xi^b \eta^c \nu^d = (\nu^a{}_{;d}\, \eta^d)_{;c}\, \xi^c - (\nu^a{}_{;d}\, \xi^d)_{;c}\, \eta^c$

$$- \nu^a{}_{;d} \underbrace{\left(\eta^d{}_{;c}\, \xi^c - \xi^d{}_{;c}\, \eta^c \right)}$$

used Torsion $= T(\xi_1, \xi_2) := \nabla_{\xi_1} \xi_2 - \nabla_{\xi_2} \xi_1 - [\xi_1, \xi_2] = 0$
i.e. $[\xi, \eta] = \nabla_\xi \eta - \nabla_\eta \xi$

Terms cancel:

$$\Rightarrow R^a{}_{bcd}\, \xi^b \eta^c \nu^d = (\nu^a{}_{;d;c} - \nu^a{}_{;c;d})\, \xi^c \eta^d$$

- True $\forall \xi, \eta \Rightarrow R^a{}_{bcd}\, \nu^d = \nu^a{}_{;cb} - \nu^a{}_{;bc}$ ✓

15 / 23

15

$$\boxed{\xi^a{}_{;cd} - \xi^a{}_{;dc} = R^a{}_{cdb}\,\xi^b}$$

(here: $\xi^a{}_{;cd} := \xi^a{}_{;c;d}$ etc.)

Remark: (a bit messy to derive because need Taylor expansion, see, e.g., text by Stewart or Einstein)

It implies that for parallel transport along infinitesimal parallelogram:

$$\boxed{(\hat{\xi}-\xi)^a \approx \eta^b \nu^c R^a{}_{bcd}\,\xi^d}$$

in basis: $R^a{}_{bcd}\,\xi^b \eta^c \nu^d = (\nu^a{}_{;d}\,\eta^d)_{;c}\,\xi^c - (\nu^a{}_{;d}\,\xi^d)_{;c}\,\eta^c$

$$- \nu^a{}_{;d}\underbrace{\left(\eta^d{}_{;c}\,\xi^c - \xi^d{}_{;c}\,\eta^c\right)}_{\substack{\text{used Torsion} = T(\xi_1,\xi_2) := \nabla_{\xi_1}\xi_2 - \nabla_{\xi_2}\xi_1 - [\xi_1,\xi_2] = 0 \\ \text{i.e. } [\xi,\eta] = \nabla_\xi \eta - \nabla_\eta \xi}}$$

Terms cancel:

$$\Longrightarrow R^a{}_{bcd}\,\xi^b \eta^c \nu^d = (\nu^a{}_{;d;c} - \nu^a{}_{;c;d})\,\xi^c \eta^d$$

▫ True $\forall \xi, \eta$ $\Longrightarrow R^a{}_{bcd}\,\nu^d = \nu^a{}_{;cb} - \nu^a{}_{;bc}$ ✓

The "Bianchi Identities":

▫ They are automatic relations among torsion and curvature, by construction.

▫ Preparation: $\boxed{\nabla \text{ for maps!}}$

Consider an arbitrary $\mathcal{F}(M)$-linear map:

$$K: \underbrace{\xi_1 \times \xi_2 \times \dots \times \xi_r}_{\text{tangent vectors}} \longrightarrow \underbrace{K(\xi_1,\dots,\xi_r)}_{\text{tangent vector}}$$

(e.g. Torsion or Curvature map)

i.e. at each $p \in M$:

$$K: T_p(M)^r \longrightarrow T_p(M)'$$

▫ We can view K as a tensor $\tilde{K} \in T_p(M)^1_r$, (as we did for R and T)

namely:

$$\tilde{K}(\omega, \xi_1, \dots, \xi_r) := \langle \omega, K(\xi_1,\dots,\xi_r) \rangle$$

▫ Now let the usual derivative of the tensor $\tilde{K}$ define the derivative of the map K:

$$\langle \omega, \underbrace{(\nabla_\xi K)}_{\substack{\text{new concept:} \\ \text{covariant derivative} \\ \text{of a map } K: T_p(M)^r \to T_p(M)'}}(\xi_1,\dots,\xi_r) \rangle := \underbrace{\nabla_\xi \tilde{K}(\omega, \xi_1 \dots \xi_r)}_{\substack{\text{usual cov. derivative} \\ \text{of a } (1,r) \text{ tensor} \\ \text{when fed one covector \& } r \text{ vectors}}}$$

Using ∇ for map:

17 / 23

17

File Edit View Insert Actions Tools Help

Sharing Desktop | Stop Sharing

Preparation: $\boxed{\nabla \text{ for maps!}}$

Consider an arbitrary $\mathcal{F}(M)$-linear map:

(e.g. Torsion or Curvature map)

$$K: \underbrace{\xi_1 \times \xi_2 \times \dots \times \xi_r}_{\text{tangent vectors}} \longrightarrow \underbrace{K(\xi_1, \dots, \xi_r)}_{\text{tangent vector}}$$

i.e. at each $p \in M$:

$$K: T_p(M)^r \longrightarrow T_p(M)^1$$

Now let the usual derivative of the tensor $\tilde{K}$ define the derivative of the map K:

$$\langle \omega, (\underbrace{\nabla_\xi}_{\substack{\text{new concept:}\\ \text{covariant derivative}\\ \text{of a map } K: T_p(M)^r \to T_p(M)^1}} K)(\xi_1, \dots, \xi_r) \rangle := \underbrace{\nabla_\xi \tilde{K}(\omega, \xi_1 \dots \xi_r)}_{\substack{\text{usual cov. derivative}\\ \text{of a } (1,r) \text{ tensor}\\ \text{when fed one covector \& r vectors}}}$$

Using ∇ for map:

1st Bianchi Identity:

$$\sum_{\text{cyclic}} R(\xi, \eta)\nu = \sum_{\text{cyclic}} \Big(T(T(\xi,\eta),\nu) + (\nabla_\xi T)(\eta,\nu) \Big)$$

2nd Bianchi Identity:

$$\sum_{\text{cyclic}} \Big((\nabla_\xi R)(\eta,\nu) + R(T(\xi,\eta),\nu) \Big) = 0$$

with obvious simplification in case $T = 0$.

Note: They are automatically obeyed equations, just like any set of lin. operators obeys the Jacobi identity with respect to $[\,,\,]$. Indeed that's why:

(or like the homogeneous Maxwell equations)

Proof of 1st Bianchi: (assuming no torsion)

$$\underbrace{\sum_{\text{cyclic}} R(\xi,\eta)\nu}_{\parallel} = 0$$

Indeed:

$$\overbrace{(\nabla_\xi \nabla_\eta - \nabla_\eta \nabla_\xi)\nu - \nabla_{[\xi,\eta]}\nu + \text{cyclic}}$$

skip by 1 cyclically | skip by 1 cyclically

$$= \nabla_\xi(\nabla_\eta \nu - \nabla_\nu \eta) - \nabla_{[\eta,\nu]}\xi + \text{cyclic}$$

Exercise: Prove that without torsion: $\boxed{\nabla_\eta \nu - \nabla_\nu \eta = [\eta,\nu]}$ (easy!)

$$= \underbrace{\nabla_\xi [\eta,\nu] - \nabla_{[\eta,\nu]}\xi}_{\parallel \text{ because again } \nabla_a b - \nabla_b a = [a,b]} + \text{cyclic}$$

$$= [\xi, [\eta,\nu]] + \text{cyclic}$$

19 / 23

19

2nd Bianchi Identity:

$$\sum_{\text{cyclic}} \Big((\nabla_\xi R)(\eta,\nu) + R(T(\xi,\eta),\nu) \Big) = 0$$

with obvious simplification in case $T = 0$.

Note: They are automatically obeyed equations, just like any set of lin. operators obeys the Jacobi identity with respect to $[\,,\,]$. Indeed that's why:

or like the homogeneous Maxwell equations

$$(\nabla_\xi \nabla_\eta - \nabla_\eta \nabla_\xi)\nu - \nabla_{[\xi,\eta]}\nu + \text{cyclic}$$

skip by 1 cyclically, skip by 1 cyclically

$$= \nabla_\xi(\nabla_\eta \nu - \nabla_\nu \eta) - \nabla_{[\eta,\nu]}\xi + \text{cyclic}$$

Exercise: Prove that without torsion: $\boxed{\nabla_\eta \nu - \nabla_\nu \eta = [\eta,\nu]}$ (easy!)

$$= \underbrace{\nabla_\xi[\eta,\nu] - \nabla_{[\eta,\nu]}\xi}_{\text{because again } \nabla_a b - \nabla_b a = [a,b]} + \text{cyclic}$$

$$= [\xi,[\eta,\nu]] + \text{cyclic}$$

$$= 0 \quad \text{by Jacobi identity for all lin. maps.}$$

Recall:

Assume A, B, C are linear maps $V \to V$

Then: $[A,[B,C]] + [C,[A,B]] + [B,[C,A]] = 0$

i.e., the Jacobi identity holds.

Proof: Simply spell out the commutators.

Remark: This means that, e.g., in quantum mechanics, all objects that that need to be representable as operators on the Hilbert space must obey the Jacobi identity, e.g., generators of symmetries.

This is why the Jacobi identity is one of the axioms of Lie Algebras.