Title: General Relativity for Cosmology Lecture - 100323

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Collection: General Relativity for Cosmology

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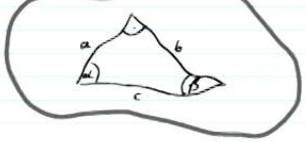
GR for Cosmology, Achim Kempf — **Lecture 8**

How to describe the "shape" of a manifold?

Historically:

E.g., on a potato-shaped surface:



$$a^2+b^2 \neq c^2$$

$$\alpha+\beta+90^\circ \neq 180^\circ$$

Helmholtz & Gauß already considered checking for curvature of space this way.

Recall:

Defined $g_{\mu\nu}(x)$

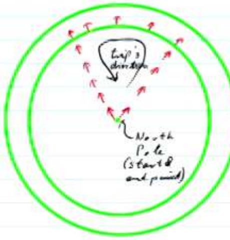
$\Rightarrow$ infinitesimal distances $\Rightarrow$ finite distances $\Rightarrow$ shape

Alternative idea:

A manifold's shape, i.e., its curvature, also reveals itself in the nontriviality of the parallel transport of vectors on the manifold:

Example:

- start with a vector at North Pole.
- parallel transport down to some lower latitude, along that latitude and then back to pole.
- vector will arrive at pole rotated, in spite of having only been parallel transported!



This motivates:

yep: another derivative!

Try to define local shape through "derivative" of vectors with respect to parallel transport!

Recall: Lie derivative insensitive to g. Indeed, for $\xi_1=\frac{\partial}{\partial x^i}$, $\xi_2=\frac{\partial}{\partial x^j}$: we have $[L_{\xi_1},L_{\xi_2}]=L_{[\xi_1,\xi_2]}=0 \Rightarrow$ No shape info from L_ξ!

The Covariant Differentiation, ∇:

Definition: Any linear map of tangent vector fields

$$\nabla: T^1(M)\times T^1(M)\rightarrow T^1(M)$$

$$\nabla: \quad \eta,\xi \rightarrow \nabla_\xi \eta$$

obeying

$$\text{(I)} \quad \nabla_{f\xi}\eta = f\,\nabla_\xi\eta, \ \forall f\in \mathcal{F}(M)$$

Not true for $L_{f\xi}\eta$ because $L_{f\xi}\eta=[f\xi,\eta]\neq f[\xi,\eta]=fL_\xi\eta$

$$\text{(II)} \quad \nabla_\xi(f\eta)=\underbrace{\xi(f)}_{\nabla_\xi f=}\eta+f\,\nabla_\xi\eta \quad \text{(Leibniz rule)}$$

∇ in a chart:

- Choose as bases for $T_x(M)$, e.g.: $\left\{\frac{\partial}{\partial x^i}\right\}$
- Given a covariant derivative ∇, consider its action on basis vectors, such as, e.g.: $\xi=\frac{\partial}{\partial x^i}$, $\eta=\frac{\partial}{\partial x^j}$:

Recall: $L_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j}=0$

$$\boxed{\nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j} := \Gamma^k{}_{ij}(x)\frac{\partial}{\partial x^k}}$$

The $\Gamma^k{}_{ij}$ are called "Christoffel symbol" or "Connection coefficients"

Thus, via the axioms:

function, vector

$$\nabla_\xi\eta=\nabla_{\xi^i\frac{\partial}{\partial x^i}}\left(\eta^j\frac{\partial}{\partial x^j}\right)\overset{(I)}{=}\xi^i\,\nabla_{\frac{\partial}{\partial x^i}}\left(\eta^j\frac{\partial}{\partial x^j}\right)$$

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$a^2 + b^2 \neq c^2$

$\alpha + \beta + 90° \neq 180°$

Helmholtz & Gauß already considered checking for curvature of space this way.

Recall:

Defined $g_{\mu\nu}(x)$

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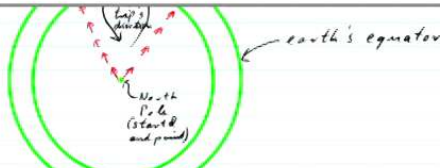

This motivates:

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Recall: Lie derivative insensitive to g. Indeed, for $\xi_1 = \frac{\partial}{\partial x^i}, \xi_2 = \frac{\partial}{\partial x^j}$ we have $[L_{\xi_1}, L_{\xi_2}] = L_{[\xi_1,\xi_2]} = 0 \Rightarrow$ No shape info from L_ξ!

The Covariant Differentiation, ∇:

Definition: Any linear map of tangent vector fields

$$\nabla : T^1(M) \times T^1(M) \rightarrow T^1(M)$$

$$\nabla : \quad \eta, \xi \rightarrow \nabla_\xi \eta$$

obeying

Not true for $L_{f\xi}\eta$

because $L_{f\xi}\eta = [f\xi, \eta] \neq f[\xi,\eta] = f L_\xi \eta$

(I) $\nabla_{f\xi}\eta = f\nabla_\xi \eta, \ \forall f \in \mathcal{F}(M)$

(II) $\nabla_\xi(f\eta) = \underbrace{\xi(f)}^{\nabla_\xi f =}\eta + f\nabla_\xi \eta$ (Leibniz rule)

is called a covariant derivative or affine connection.

Note:

For now, let us assume a metric has not (yet) been specified, so we are free to choose ∇, and this choice defines the shape of M!

∇ in a chart:

- Choose as bases for $T_x(M)^1$, e.g.: $\left\{\frac{\partial}{\partial x^i}\right\}$
- Given a covariant derivative ∇, consider its action on basis vectors, such as, e.g.,: $\xi = \frac{\partial}{\partial x^i}, \eta = \frac{\partial}{\partial x^j}$:

Recall: $L_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = 0$

$$\boxed{\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} := \Gamma^k{}_{ij}(x) \frac{\partial}{\partial x^k}}$$

The Γ^k_{ij} are called "Christoffel symbol" or "Connection coefficients"

Thus, via the axioms:

function

vector

$$\nabla_\xi \eta = \nabla_{\xi^i \frac{\partial}{\partial x^i}}\left(\eta^j \frac{\partial}{\partial x^j}\right) \overset{(I)}{=} \xi^i \nabla_{\frac{\partial}{\partial x^i}}\left(\eta^j \frac{\partial}{\partial x^j}\right)$$

← recall: $\nabla_\alpha f = \alpha(f) \ \forall \alpha \in T^1_p(M), f \in \mathcal{F}_p(M)$

$$\overset{(II)}{=} \xi^i\left(\eta^j{}_{,i}\frac{\partial}{\partial x^j} + \eta^j \Gamma^k{}_{ij}(x)\frac{\partial}{\partial x^k}\right)$$

$$= \left(\xi^i \eta^k{}_{,i} + \xi^i \eta^j \Gamma^k{}_{ij}\right)\frac{\partial}{\partial x^k}$$

Notation: $\eta^k{}_{;i} := \eta^k{}_{,i} + \eta^j \Gamma^k{}_{ij}$

↳ semi-colon for covariant derivative

Thus: $\nabla_\xi \eta = \xi^i \eta^k{}_{;i} \frac{\partial}{\partial x^k}$ (*)

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On the other hand:

$$\nabla_{\partial/\partial\bar{x}^a} \frac{\partial}{\partial \bar{x}^b} = \nabla_{\underbrace{\frac{\partial x^i}{\partial \bar{x}^a}}_{\in F(M)}\frac{\partial}{\partial x^i}} \left(\frac{\partial x^j}{\partial \bar{x}^b}\frac{\partial}{\partial x^j}\right) \quad \text{use axiom } (b) \Rightarrow$$

$$= \frac{\partial x^i}{\partial \bar{x}^a} \nabla_{\frac{\partial}{\partial x^i}} \left(\underbrace{\frac{\partial x^j}{\partial \bar{x}^b}}_{\in F(M)}\frac{\partial}{\partial x^j}\right) \quad \text{use Leibniz rule } (c) \Rightarrow$$

$$= \underbrace{\frac{\partial x^i}{\partial \bar{x}^a}\left[\left(\frac{\partial}{\partial x^i}\right.\right.}\left.\left.\frac{\partial x^j}{\partial \bar{x}^b}\right)\frac{\partial}{\partial x^j} + \frac{\partial x^j}{\partial \bar{x}^b}\Gamma^k{}_{ij}\frac{\partial}{\partial x^k}\right]$$

$$= \left(\frac{\partial}{\partial \bar{x}^a}\frac{\partial x^j}{\partial \bar{x}^b}\right)\frac{\partial}{\partial x^j} + \frac{\partial x^i}{\partial \bar{x}^a}\frac{\partial x^j}{\partial \bar{x}^b}\Gamma^k{}_{ij}\frac{\partial}{\partial x^k} \quad (\text{II})$$

This term is indep. of Γ $\Rightarrow$ Γ can be zero in one coordinate system and nonzero in another!

Only this term would be there, if the $\Gamma^k{}_{ij}$ were tensor coefficients in the $\frac{\partial}{\partial x^i}$, dx^j bases.

(Can be shown to be equivalent)

Physicists' definition of ∇: Any set of n^3 functions $\Gamma^c_{ab}(x)$ which transform this way are defining a cov. derivative ∇.

The "absolute" covariant derivative ∇:

Consider the covariant derivative but: without choosing a direction vector ξ:

$$\nabla : T_x(M)^1 \to T_x(M)^1_1$$

$$\nabla : \eta = \eta^i(x)\frac{\partial}{\partial x^i} \to \nabla\eta(x) = \eta^k{}_{;i}(x)\, dx^i \otimes \frac{\partial}{\partial x^k}$$

(I.e. feed the open covariant slot of $\nabla\eta$ with contravariant ξ.)

Indeed: The contraction of $\nabla\eta$ with ξ yields:

$$\nabla\eta(\xi) = \eta^k{}_{;i}\, \underbrace{dx^i(\xi)}\, \frac{\partial}{\partial x^k} = \eta^k{}_{;i}\, \xi^i \frac{\partial}{\partial x^k} = \nabla_\xi \eta \quad \text{ok with } (*)$$

$$dx^i(\xi) = \xi(x^i) = \xi^j \frac{\partial}{\partial x^j} x^i = \xi^j \delta^i_j = \xi^i$$

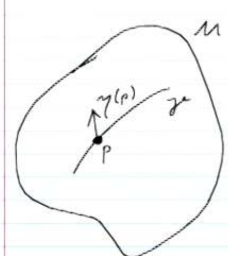
We defined ∇ algebraically. Now, extract the

Geometric meaning of ∇: (If ∇ describes infinitesimal parallel transport it should also describe finite parallel transport)

Definition: Assume ∇ is given. Choose a path $\gamma: \mathbb{R} \to M$. Then, a tangent vector field η is called auto-parallel along γ, if

$$\nabla_{\dot\gamma}\, \eta = 0$$

i.e. if η doesn't change under parallel transport along the path γ.



Note: We'll see that they always exist, i.e., we can always parallel transport a vector η finite distances.

$$\nabla : T_x(M)' \to T_x(M)',$$

$$\nabla : \eta = \eta^i(x)\frac{\partial}{\partial x^i} \to \nabla\eta(x) = \eta^k{}_{;i}(x)\, dx^i \otimes \frac{\partial}{\partial x^k}$$

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$$dx^i(\xi) = \xi(x^i) = \xi^j \frac{\partial}{\partial x^j} x^i = \xi^j \delta^i_j = \xi^i$$

- In a chart,

$$\eta = \eta^i(x)\frac{\partial}{\partial x^i}$$

and

$$\gamma : [a,b] \to M$$
$$\gamma : t \to x^i(t)$$

and the tangent vector:

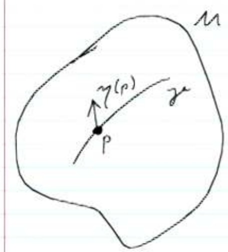
$$\dot\gamma(x(t)) = \frac{dx^k}{dt}\frac{\partial}{\partial x^k}$$

Thus:

$$\nabla_{\dot\gamma}\eta = \nabla_{\frac{dx^k}{dt}\frac{\partial}{\partial x^k}}\left(\eta^i\frac{\partial}{\partial x^i}\right) = \frac{dx^k}{dt}\nabla_{\frac{\partial}{\partial x^k}}\left(\eta^i\frac{\partial}{\partial x^i}\right)$$

$$= \underbrace{\frac{dx^k}{dt}\left(\frac{\partial\eta^i}{\partial x^k}\frac{\partial}{\partial x^i} + \eta^j\Gamma^i{}_{kj}\frac{\partial}{\partial x^i}\right)}$$

$$= \left(\frac{d\eta^i}{dt} + \frac{dx^k}{dt}\eta^j\Gamma^i{}_{kj}\right)\frac{\partial}{\partial x^i} \stackrel{!}{=} 0$$



Then, a tangent vector field η is called auto-parallel along γ, if

$$\nabla_{\dot\gamma}\eta = 0$$

i.e. if η doesn't change under parallel transport along the path γ.

Note: We'll see that they always exist, i.e., we can always parallel transport a vector η finite distances.

$\Rightarrow$ η autoparallel to γ means:

$$\frac{d\eta^i}{dt} + \eta^j\frac{dx^k}{dt}\Gamma^i{}_{kj} = 0$$

I.e. this is the condition for the vectors of η being parallel translates of each other, along γ.

- **Conclusion:**

This is 1st order ODEs for η. Thus:

Initial condition $\eta(\gamma(0)) \Rightarrow$ solution $\eta(\gamma(t))$ exists at least locally

$\Rightarrow$ - **Proposition:**

Given a path $\gamma : [t,s] \to M$, the autoparallel transport of a tangent vector η at $\gamma(t)$ to $\gamma(s)$ is unique.

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and the tangent vector:

$$\dot{\gamma}(x(t)) = \frac{dx^k}{dt}\frac{\partial}{\partial x^k}$$

Thus: $\nabla_{\dot{\gamma}}\,\eta = \nabla_{\frac{dx^k}{dt}\frac{\partial}{\partial x^k}}\left(\eta^i\frac{\partial}{\partial x^i}\right) = \frac{dx^k}{dt}\,\nabla_{\frac{\partial}{\partial x^k}}\left(\eta^i\frac{\partial}{\partial x^i}\right)$

$$= \underbrace{\frac{dx^k}{dt}\left(\frac{\partial\eta^i}{\partial x^k}\right.}\left.\frac{\partial}{\partial x^i} + \eta^j\Gamma^i{}_{kj}\frac{\partial}{\partial x^i}\right)$$

$$= \left(\frac{d\eta^i}{dt} + \frac{dx^k}{dt}\eta^j\Gamma^i{}_{kj}\right)\frac{\partial}{\partial x^i} \overset{!}{=} 0$$

This is 1st order ODEs for η. Thus:

Initial condition $\eta(\gamma(0)) \Rightarrow$ solution $\eta(\gamma(t))$ exists at least locally

$\Rightarrow$ □ Proposition:

Given a path $\gamma : [t,s] \to M$, the autoparallel transport of a tangent vector η at $\gamma(t)$ to $\gamma(s)$ is unique.

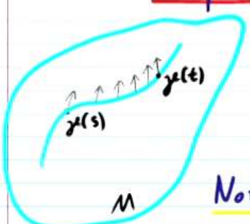
I.e., the path γ defines a parallel transport map τ:

$$\tau(t,s):\ T^1_{\gamma(t)} \to T^1_{\gamma(s)}$$

$$\tau(t,s):\ \eta(\gamma(t)) \to \eta(\gamma(s))$$

□ Q: Can one use τ to obtain ∇ as a Newton-Leibniz limit?

□ Proposition: (for the proof, see e.g. the text by Straumann)



$$\nabla_{\dot{\gamma}}\,\eta(\gamma(t)) = \frac{d}{ds}\bigg|_{s=t} \tau(s,t)\big(\eta(\gamma(s))\big)$$

Note: Since we can choose paths with arbitrary $\dot{\gamma}$ this equation can be used as a geometric definition of ∇.

∇ for arbitrary tensors:

□ The parallel transport map $\tau(s,t)$ transports tangent vectors η from $\gamma(s)$ to $\gamma(t)$.

□ Definition: $\tau(s,t)$ also parallel transports the dual vectors w, namely so that contraction is conserved:

$$\boxed{\underbrace{\tau(w)}_{\text{parallel transported } w}\big(\underbrace{\tau(\eta)}_{\text{parallel transported } \eta}\big) = w(\eta)} \qquad (C)$$

□ Extension of τ to tensor products:

$$\tau(S_1 \otimes S_2) := \tau(S_1) \otimes \tau(S_2) \qquad (T)$$

S_1 and S_2 are tensors of arbitrary rank.

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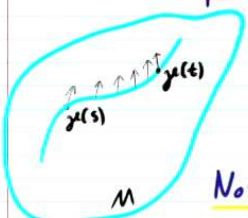

$$\nabla_{\dot\gamma}\,\eta(\gamma(t)) = \frac{d}{ds}\Big|_{s=t} \tau(s,t)\big(\eta(\gamma(s))\big)$$

Note: Since we can choose paths with arbitrary $\dot\gamma$ this equation can be used as a **geometric definition of ∇.**

□ **Definition:** $\tau(s,t)$ also parallel transports the dual vectors ω, namely so that contraction is conserved:

$$\boxed{\underbrace{\tau(\omega)}_{\text{parallel transported }\omega}\big(\underbrace{\tau(\eta)}_{\text{parallel transported }\eta}\big) = \omega(\eta)} \qquad (C')$$

□ Extension of τ to tensor products:

$$\tau(S_1 \otimes S_2) := \tau(S_1) \otimes \tau(S_2) \qquad (T)$$

S_1 and S_2 are tensors of arbitrary rank.

□ **Definition:**

arbitrary tensor, arb. point $\in M$, arb. tangent vector

$$\nabla_\xi S(p) := \nabla_{\dot\gamma} S(\gamma(t))\Big|_{t=0}$$

$$:= \frac{d}{dt}\Big|_{t=0} \tau(t,0)\big(S(\gamma(t))\big)$$

Exercise: Show that when S is a scalar function $S \in \mathcal{F}(M)$, then: $\nabla_\xi S = \xi(S) = \xi^i \frac{\partial}{\partial x^i} S$

here, γ is any path through p obeying:

$$\dot\gamma(0) = \xi(p)\ ,\quad \gamma(0) = p$$

□ Absolute covariant derivative:

(for abs. derivative one is not specifying the direction)

$$\underbrace{(\nabla S)(\eta_1,\dots,\eta_p,\omega_1,\dots,\omega_r,\xi)}_{\text{fed to }\nabla S\text{ which is }(r,p)\text{ tensor}} := \nabla_\xi S(\eta_1,\dots,\eta_p,\omega_1,\dots,\omega_r)$$

Properties of ∇:

$*$ ∇ is a derivation: (because ∇ inherits the Leibniz rule from $\frac{d}{ds}$)

$$\nabla_\xi(S_1\otimes S_2) = \frac{d}{ds}\Big|_{s=t} \tau(S_1 \otimes S_2) = \frac{d}{ds}\Big|_{s=t} \tau(S_1)\otimes\tau(S_2)$$

($\tau(s,t)$; $S_1(\gamma(s))$; $S_2(\gamma(s))$)

$$= \Big[\frac{d}{ds}\Big|_{s=t}\tau(S_1)\Big]\otimes\tau(S_2)\Big|_{s=t} + \tau(S_1)\Big|_{s=t}\otimes\frac{d}{ds}\Big|_{s=t}\tau(S_2)$$

$$= (\nabla_\xi S_1)\otimes S_2 + S_1\otimes\nabla_\xi S_2 \qquad (A)$$

$*$ Eq. (C') implies that ∇ and contractions do commute.

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$\left.\frac{d}{dt}\right|_{t=0}$

here, γ is any path through p obeying:

$$\dot\gamma(0) = \xi(p) \ , \ \gamma(0) = p$$

- Absolute covariant derivative:

(for abs. derivative one is not specifying the direction)

$$(\nabla S')(\underbrace{\eta_1,\dots,\eta_p,\omega_1,\dots,\omega_r,\xi}_{\text{fed to } \nabla S \text{ which is } (r,p) \text{ tensor}}) := \nabla_\xi S'(\eta_1,\dots,\eta_p,\omega_1,\dots,\omega_r)$$

Action of ∇ on tensors in a chart?

- Recall: $\boxed{\nabla_\xi \frac{\partial}{\partial x^i} = \xi^l \Gamma^k{}_{li} \frac{\partial}{\partial x^k}}$

- Problem: Find $\nabla_\xi dx^i = ?$

Same strategy will be used below for general tensors.

$\rightsquigarrow$ • Consider $\underset{\text{tangent vector field}}{\eta} \otimes \underset{\text{cotangent vector field}}{\omega}$.

• Differentiate:

$$\nabla_\xi(\eta\otimes\omega) \overset{(A)}{=} (\nabla_\xi\eta)\otimes\omega + \eta\otimes\nabla_\xi\omega$$

• Contract: (use that ∇_ξ and contraction commute)

(by exercise above) $\xi(\omega(\eta)) = \nabla_\xi(\underbrace{\omega(\eta)}_{\text{scalar function}}) = \omega(\nabla_\xi\eta) + (\nabla_\xi\omega)(\eta)$

$$= \left[\left.\frac{d}{ds}\right|_{s=t} \tau(S_1)\right] \otimes \tau(S_2)\Big|_{s=t} + \tau(S_1)\Big|_{s=t} \otimes \left.\frac{d}{ds}\right|_{s=t} \tau(S_2)$$

$$= (\nabla_\xi S_1)\otimes S_2 + S_1 \otimes \nabla_\xi S_2 \qquad (A)$$

$\ast$ Eq. (G') implies that ∇ and contractions do commute.

(i.e. $\xi(\omega(\eta)) = \omega(\nabla_\xi\eta) + (\nabla_\xi\omega)(\eta)$)

$\Rightarrow$ An expression for $\nabla_\xi(\omega)(\eta)$:

$$(\nabla_\xi\omega)(\eta) = \xi(\omega(\eta)) - \omega(\nabla_\xi\eta) \qquad (\ast)$$

Now: Choose $\omega := dx^j$ and $\eta := \frac{\partial}{\partial x^i}$

(Notation: $\langle\omega,\xi\rangle = \omega(\xi)$ (inner product, contraction))

$$\Rightarrow (\nabla_\xi dx^j)\left(\frac{\partial}{\partial x^i}\right) = \underbrace{\xi\Big(\overbrace{\langle dx^j, \tfrac{\partial}{\partial x^i}\rangle}^{\delta^j_i = \text{const.}}\Big)}_{=0} - \langle dx^j, \nabla_\xi \tfrac{\partial}{\partial x^i}\rangle$$

$$= -\langle dx^j, \xi^l \Gamma^k{}_{li} \tfrac{\partial}{\partial x^k}\rangle$$

$$= -\xi^l \Gamma^j{}_{li}$$

$$\Rightarrow \boxed{\nabla_\xi dx^j = -\xi^l \Gamma^j{}_{li}\, dx^i}$$

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here, γ is any path through p obeying:

$$\dot\gamma(0)=\xi(p)\ ,\quad \gamma(0)=p$$

□ Absolute covariant derivative:

(for abs. derivative one is not specifying the direction)

$$(\nabla S')(\underbrace{\eta_1,\dots,\eta_p,\omega_1,\dots,\omega_r,\xi}_{\text{fed to } \nabla S \text{ which is } (r,p) \text{ tensor}})=\nabla_\xi S'(\eta_1,\dots,\eta_p,\omega_1,\dots,\omega_r)$$

Action of ∇ on tensors in a chart?

□ Recall: $\boxed{\nabla_\xi \frac{\partial}{\partial x^i}=\xi^l\Gamma^k{}_{li}\frac{\partial}{\partial x^k}}$

□ Problem: Find $\nabla_\xi dx^i = ?$

$\rightsquigarrow$ • Consider $\eta\otimes\omega$. (tangent vector field η, cotangent vector field ω)

• Differentiate:

$$\nabla_\xi(\eta\otimes\omega)\overset{(A)}{=}(\nabla_\xi\eta)\otimes\omega+\eta\otimes\nabla_\xi\omega$$

• Contract: (use that ∇_ξ and contraction commute)

(by exercise above)

$$\underbrace{\xi(\omega(\eta))}=\nabla_\xi(\underbrace{\omega(\eta)}_{\text{scalar function}})=\omega(\nabla_\xi\eta)+(\nabla_\xi\omega)(\eta)$$

Same strategy will be used below for general tensors.

$$=\left[\frac{d}{ds}\Big|_{s=t}\tau(S_1)\right]\otimes\tau(S_2)\Big|_{s=t}+\tau(S_1)\Big|_{s=t}\otimes\frac{d}{ds}\Big|_{s=t}\tau(S_2)$$

$$=(\nabla_\xi S_1)\otimes S_2+S_1\otimes\nabla_\xi S_2 \qquad (A)$$

* Eq. (C') implies that ∇ and contractions do commute.

$$\left(\text{i.e. } \xi(\omega(\eta))=\omega(\nabla_\xi\eta)+(\nabla_\xi\omega)(\eta)\right)$$

$\Rightarrow$ An expression for $\nabla_\xi(\omega)(\eta)$:

$$(\nabla_\xi\omega)(\eta)=\xi(\omega(\eta))-\omega(\nabla_\xi\eta) \qquad (*)$$

Now: Choose $\omega := dx^j$ and $\eta := \frac{\partial}{\partial x^i}$

(Notation: $\langle\omega,\xi\rangle=\omega(\xi)$ (inner product, contraction))

$$\Rightarrow\quad (\nabla_\xi dx^j)\left(\frac{\partial}{\partial x^i}\right)=\xi\big(\underbrace{\langle dx^j,\tfrac{\partial}{\partial x^i}\rangle}_{\delta^j_i=\text{const.}}\big)_{=0}-\langle dx^j,\nabla_\xi\tfrac{\partial}{\partial x^i}\rangle$$

$$=-\langle dx^j,\xi^l\Gamma^k{}_{li}\tfrac{\partial}{\partial x^k}\rangle$$

$$=-\xi^l\Gamma^j{}_{li}$$

$$\Rightarrow\quad \boxed{\nabla_\xi dx^j=-\xi^l\Gamma^j{}_{li}\,dx^i}$$

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$$-S(\nabla_\xi \eta_1, \eta_2, \ldots, \eta_r, \omega_1, \ldots \omega_s) - \ldots$$

$$-S(\eta_1, \ldots, \nabla_\xi \eta_r, \omega_1, \ldots \omega_s)$$

$$-S(\eta_1, \ldots, \eta_r, \nabla_\xi \omega_1, \omega_2, \ldots, \omega_s) + \ldots$$

$$-S(\eta_1, \ldots \eta_r, \omega_1, \ldots \nabla_\xi \omega_s)$$

that $\nabla_\xi S$ reads

$$\nabla_\xi S = \xi^k S^{i_1 \ldots i_p}_{j_1 \ldots j_q ; k} \frac{\partial}{\partial x^{i_1}} \otimes \ldots \otimes \frac{\partial}{\partial x^{i_p}} \otimes dx^{j_1} \otimes \ldots \otimes dx^{j_q}$$

with:

$$S^{i_1 \ldots i_p}_{j_1 \ldots j_q ; k} := S^{i_1 \ldots i_p}_{j_1 \ldots j_q , k} + \Gamma^{i_1}_{k\ell} S^{\ell i_2 \ldots i_p}_{j_1 \ldots j_q} + \ldots + \Gamma^{i_p}_{k\ell} S^{i_1 \ldots \ell}_{j_1 \ldots j_q} - \Gamma^{\ell}_{k j_1} S^{i_1 \ldots i_p}_{\ell \ldots j_q} - \ldots - \Gamma^{\ell}_{k j_q} S^{i_1 \ldots i_p}_{j_1 \ldots \ell}$$

Special cases:

- Tangent vector fields:

$$\xi^i_{;k} = \xi^i_{,k} + \xi^j \Gamma^i_{kj}$$

- Cotangent vector fields:

$$\omega_{i;k} = \omega_{i,k} - \omega_\ell \Gamma^\ell_{ki}$$

Recall: Specifying ∇ specifies parallel transport of vectors and this should specify the manifold's shape, but how?

→ Indeed, ∇ specifies Torsion & Curvature