Title: General Relativity for Cosmology Lecture - 100323

Speakers: Achim Kempf

Collection: General Relativity for Cosmology

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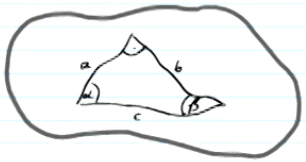
GR for Cosmology, Achim Kempf — Lecture 8

How to describe the "shape" of a manifold?

Historically:

E.g., on a potato-shaped surface:



$a^2+b^2 \neq c^2$

$\alpha+\beta+90° \neq 180°$

Helmholtz & Gauß already considered checking for curvature of space this way.

Recall:

Defined $g_{\mu\nu}(x)$

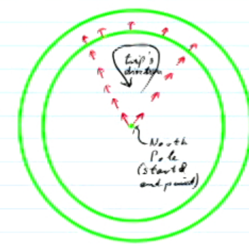
$\Rightarrow$ infinitesimal distances $\Rightarrow$ finite distances $\Rightarrow$ shape

Alternative idea:

A manifold's shape, i.e., its curvature, also reveals itself in the nontriviality of the parallel transport of vectors on the manifold:

Example:

- start with a vector at North Pole.
- parallel transport down to some lower latitude, along that latitude and then back to pole.
- vector will arrive at pole rotated, in spite of having only been parallel transported!



This motivates:

yep: another derivation!

Try to define local shape through "derivative" of vectors with respect to parallel transport!

Recall: Lie derivative insensitive to g. Indeed, for $\xi_1=\frac{\partial}{\partial x^i}, \xi_2=\frac{\partial}{\partial x^j}$: we have $[L_{\xi_1}, L_{\xi_2}] = L_{[\xi_1,\xi_2]} = 0 \Rightarrow$ No shape info from L_ξ!

The Covariant Differentiation, ∇:

Definition: Any linear map of tangent vector fields

$$\nabla : T^1(M) \times T^1(M) \to T^1(M)$$

$$\nabla : \quad \eta, \xi \to \nabla_\xi \eta$$

obeying

(I) $\nabla_{f\xi}\eta = f\nabla_\xi \eta, \ \forall f \in \mathcal{F}(M)$

Not true for $L_{f\xi}\eta$

because $L_{f\xi}\eta = [f\xi,\eta] \neq f[\xi,\eta] = fL_\xi\eta$

(II) $\nabla_\xi(f\eta) = \underbrace{\xi(f)}^{\nabla_\xi f =}\eta + f\nabla_\xi\eta$ (Leibniz rule)

∇ in a chart:

- Choose as bases for $T_x(M)$, e.g.: $\left\{\frac{\partial}{\partial x^i}\right\}$
- Given a covariant derivative ∇, consider its action on basis vectors, such as, e.g.,: $\xi = \frac{\partial}{\partial x^i}, \eta = \frac{\partial}{\partial x^j}$:

Recall: $L_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = 0$

$$\boxed{\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} := \Gamma^k{}_{ij}(x)\frac{\partial}{\partial x^k}}$$

The Γ^k_{ij} are called "Christoffel symbol" or "Connection coefficients"

Thus, via the axioms:

function, vector

$$\nabla_\xi \eta = \nabla_{\xi^i\frac{\partial}{\partial x^i}}\left(\eta^j \frac{\partial}{\partial x^j}\right) \overset{(I)}{=} \xi^i \nabla_{\frac{\partial}{\partial x^i}}\left(\eta^j\frac{\partial}{\partial x^j}\right)$$

$a + b \neq c$

$\alpha + \beta + 90° \neq 180°$

Helmholtz & Gauß already considered checking for curvature of space this way.

Recall:

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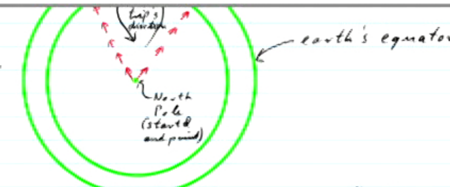

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(I) $\nabla_{f\xi}\eta = f \nabla_\xi \eta, \quad \forall f \in \mathcal{F}(M)$

(II) $\nabla_\xi(f\eta) = \overbrace{\xi(f)}^{\nabla_\xi f =}\eta + f\nabla_\xi \eta$ (Leibniz rule)

is called a covariant derivative or affine connection.

Note:

For now, let us assume a metric has not (yet) been specified, so we are free to choose ∇, and this choice defines the shape of M!

∇ in a chart: □ Choose as bases for $T_x(M)^1$, e.g.: $\left\{\frac{\partial}{\partial x^i}\right\}$

□ Given a covariant derivative ∇, consider its action on basis vectors, such as, e.g.,: $\xi = \frac{\partial}{\partial x^i}, \eta = \frac{\partial}{\partial x^j}$:

Recall: $L_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = 0$

$$\boxed{\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} := \Gamma^k{}_{ij}(x) \frac{\partial}{\partial x^k}}$$

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function

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$$\nabla_\xi \eta = \nabla_{\xi^i \frac{\partial}{\partial x^i}}\left(\eta^j \frac{\partial}{\partial x^j}\right) \overset{(I)}{=} \xi^i \nabla_{\frac{\partial}{\partial x^i}}\left(\eta^j \frac{\partial}{\partial x^j}\right)$$

$\leftarrow$ recall: $\nabla_\alpha f = \alpha(f) \quad \forall \alpha \in T_p^1(M), f \in \mathcal{F}_p(M)$

$$\overset{(II)}{=} \xi^i \left(\eta^j{}_{,i} \frac{\partial}{\partial x^j} + \eta^j \Gamma^k{}_{ij}(x) \frac{\partial}{\partial x^k}\right)$$

$$= \left(\xi^i \eta^k{}_{,i} + \xi^i \eta^j \Gamma^k{}_{ij}\right) \frac{\partial}{\partial x^k}$$

Notation: $\eta^k{}_{;i} := \eta^k{}_{,i} + \eta^j \Gamma^k{}_{ij}$

$\llcorner$ semi-colon for covariant derivatives

Thus: $\nabla_\xi \eta = \xi^i \eta^k{}_{;i} \frac{\partial}{\partial x^k}$ $(*)$

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On the other hand:

$$\nabla_{\frac{\partial}{\partial \bar{x}^a}} \frac{\partial}{\partial \bar{x}^b} = \nabla_{\underbrace{\frac{\partial x^i}{\partial \bar{x}^a}}_{\in F(M)} \frac{\partial}{\partial x^i}} \left(\frac{\partial x^j}{\partial \bar{x}^b} \frac{\partial}{\partial x^j} \right)$$ use axiom (b) $\Rightarrow$

$$= \frac{\partial x^i}{\partial \bar{x}^a} \nabla_{\frac{\partial}{\partial x^i}} \left(\underbrace{\frac{\partial x^j}{\partial \bar{x}^b}}_{\in F(M)} \frac{\partial}{\partial x^j} \right)$$ use Leibniz rule (c) $\Rightarrow$

$$= \underbrace{\frac{\partial x^i}{\partial \bar{x}^a} \left[\left(\frac{\partial}{\partial x^i} \right.\right.}\left.\frac{\partial x^j}{\partial \bar{x}^b} \right) \frac{\partial}{\partial x^j} + \frac{\partial x^j}{\partial \bar{x}^b} \Gamma^k{}_{ij} \frac{\partial}{\partial x^k} \Big]$$

$$= \left(\frac{\partial}{\partial \bar{x}^a} \frac{\partial x^j}{\partial \bar{x}^b} \right) \frac{\partial}{\partial x^j} + \frac{\partial x^i}{\partial \bar{x}^a} \frac{\partial x^j}{\partial \bar{x}^b} \Gamma^k{}_{ij} \frac{\partial}{\partial x^k} \qquad (\mathrm{II})$$

This term is indep. of Γ
$\Rightarrow$ Γ can be zero in one coordinate system and nonzero in another!

Only this term would be there, if the $\Gamma^k{}_{ij}$ were tensor coefficients in the $\frac{\partial}{\partial x^i}$, dx^j bases.

(Can be shown to be equivalent)

Physicists' definition of ∇: Any set of n^3 functions $\Gamma^a_{bc}(x)$ which transform this way are defining a cov. derivative ∇.

The "absolute" covariant derivative ∇:

Consider the covariant derivative but: without choosing a direction vector ξ:

$$\nabla : T_x(M)^1 \to T_x(M)^1_1$$

$$\nabla : \eta = \eta^i(x) \frac{\partial}{\partial x^i} \to \nabla \eta(x) = \eta^k{}_{;i}(x)\, dx^i \otimes \frac{\partial}{\partial x^k}$$

(I.e. feed the open covariant slot of $\nabla \eta$ with contravariant ξ.)

Indeed: The contraction of $\nabla \eta$ with ξ yields:

$$\nabla \eta(\xi) = \eta^k{}_{;i} \, dx^i(\xi) \frac{\partial}{\partial x^k} = \eta^k{}_{;i} \xi^i \frac{\partial}{\partial x^k} = \nabla_\xi \eta$$ ok with (*)

$dx^i(\xi) = \xi(x^i) = \xi^j \frac{\partial}{\partial x^j} x^i = \xi^j \delta^i_j = \xi^i$

We defined ∇ algebraically. Now, extract the

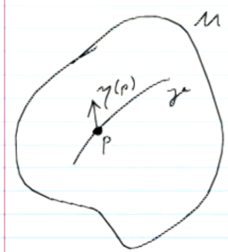
Geometric meaning of ∇: (If ∇ describes infinitesimal parallel transport it should also describe finite parallel transport)

Definition: Assume ∇ is given. Choose a path $\gamma : \mathbb{R} \to M$.

Then, a tangent vector field η is called auto-parallel along γ, if

$$\nabla_{\dot\gamma} \eta = 0$$

i.e. if η doesn't change under parallel transport along the path γ.



Note: We'll see that they always exist, i.e., we can always parallel transport a vector η finite distances.

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$$\nabla : T_x(M)' \to T_x(M)'_1$$

$$\nabla : \eta = \eta^i(x)\frac{\partial}{\partial x^i} \to \nabla\eta(x) = \eta^k{}_{;i}(x)\, dx^i \otimes \frac{\partial}{\partial x^k}$$

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$$dx^i(\xi) = \xi(x^i) = \xi^j\frac{\partial}{\partial x^j}x^i = \xi^j\delta^i_j = \xi^i$$

- In a chart,

$$\eta = \eta^i(x)\frac{\partial}{\partial x^i}$$

and

$$\gamma : [a,b] \to M$$
$$\gamma : t \to x^i(t)$$

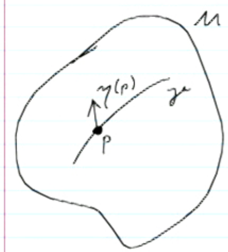
and the tangent vector:

$$\dot\gamma(x(t)) = \frac{dx^k}{dt}\frac{\partial}{\partial x^k}$$

Thus:
$$\nabla_{\dot\gamma}\eta = \nabla_{\frac{dx^k}{dt}\frac{\partial}{\partial x^k}}\left(\eta^i\frac{\partial}{\partial x^i}\right) = \frac{dx^k}{dt}\nabla_{\frac{\partial}{\partial x^k}}\left(\eta^i\frac{\partial}{\partial x^i}\right)$$

$$= \frac{dx^k}{dt}\left(\frac{\partial\eta^i}{\partial x^k}\frac{\partial}{\partial x^i} + \eta^i\Gamma^j{}_{ki}\frac{\partial}{\partial x^j}\right)$$

$$= \left(\frac{d\eta^i}{dt} + \frac{dx^k}{dt}\eta^j\Gamma^i{}_{kj}\right)\frac{\partial}{\partial x^i} \overset{!}{=} 0$$



Then, a tangent vector field η is called auto-parallel along γ, if

$$\nabla_{\dot\gamma}\eta = 0$$

i.e. if η doesn't change under parallel transport along the path γ.

Note: We'll see that they always exist, i.e., we can always parallel transport a vector η finite distances.

$\Rightarrow$ η autoparallel to γ means:

$$\frac{d\eta^i}{dt} + \eta^j\frac{dx^k}{dt}\Gamma^i{}_{kj} = 0$$

I.e. this is the condition for the vectors of η being parallel translates of each other, along γ.

- **Conclusion:**

This is 1st order ODEs for η. Thus:

Initial condition $\eta(\gamma(0)) \Rightarrow$ solution $\eta(\gamma(t))$ exists at least locally

$\Rightarrow$ - **Proposition:**

Given a path $\gamma : [t,s] \to M$, the autoparallel transport of a tangent vector η at $\gamma(t)$ to $\gamma(s)$ is unique.

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and the tangent vector:

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$\Rightarrow$ □ **Proposition:**

Given a path $\gamma: [t,s] \to M$, the autoparallel transport of a tangent vector η at $\gamma(t)$ to $\gamma(s)$ is unique.

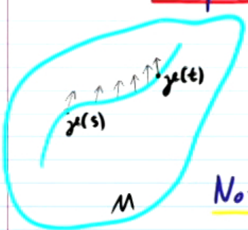
I.e., the path γ defines a parallel transport map τ:

$$\tau(t,s): T^1_{\gamma(t)} \to T^1_{\gamma(s)}$$

$$\tau(t,s): \eta(\gamma(t)) \to \eta(\gamma(s))$$

□ Q: Can one use τ to obtain ∇ as a Newton-Leibniz limit?

□ **Proposition:** (for the proof, see e.g. the text by Straumann)



$$\nabla_{\dot\gamma}\eta(\gamma(t)) = \frac{d}{ds}\bigg|_{s=t}\tau(s,t)\left(\eta(\gamma(s))\right)$$

Note: Since we can choose paths with arbitrary $\dot\gamma$ this equation can be used as a **geometric definition** of ∇.

∇ for arbitrary tensors:

□ The parallel transport map $\tau(s,t)$ transports tangent vectors η from $\gamma(s)$ to $\gamma(t)$.

□ **Definition:** $\tau(s,t)$ also parallel transports the dual vectors w, namely so that contraction is conserved:

$$\boxed{\underbrace{\tau(w)}_{\text{parallel transported } w}\left(\underbrace{\tau(\eta)}_{\text{parallel transported } \eta}\right) = w(\eta)} \qquad (C)$$

□ Extension of τ to tensor products:

$$\tau(S_1 \otimes S_2) := \tau(S_1) \otimes \tau(S_2) \qquad (T)$$

S_1 and S_2 are tensors of arbitrary rank.

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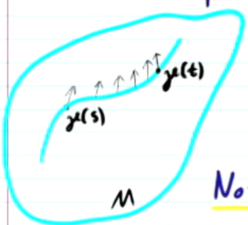

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S_1 and S_2 are tensors of arbitrary rank.

$\rightsquigarrow$

- **Definition:**

Exercise:

Show that when S is a scalar function $S \in \mathcal{F}(M)$, then:
$\nabla_\xi S = \xi(S) = \xi^i \frac{\partial}{\partial x^i} S$

arbitrary tensor, arb. point $\in M$, arb. tangent vector

$$\nabla_\xi S(p) := \nabla_{\dot\gamma} S(\gamma(t))\Big|_{t=0}$$

$$:= \frac{d}{dt}\Big|_{t=0} \tau(t,0)\big(S(\gamma(t))\big)$$

here, γ is any path through p obeying:

$$\dot\gamma(0) = \xi(p), \quad \gamma(0) = p$$

- Absolute covariant derivative:

(for abs. derivative one is not specifying the direction)

$$(\nabla S)(\underbrace{\eta_1,\dots,\eta_p,w_1,\dots,w_r,\xi}_{\text{led to } \nabla S \text{ which is } (r,p) \text{ tensor}}) := \nabla_\xi S(\eta_1,\dots,\eta_p,w_1,\dots,w_r)$$

Properties of ∇:

$*$ ∇ is a derivation: (because ∇ inherits the Leibniz rule from $\frac{d}{ds}$)

$$\nabla_\xi(S_1 \otimes S_2) = \frac{d}{ds}\Big|_{s=t} \tau(S_1 \otimes S_2) = \frac{d}{ds}\Big|_{s=t} \tau(S_1) \otimes \tau(S_2)$$

($\tau(s,t)$, $S_1(\gamma(s))$, $S_2(\gamma(s))$)

$$= \Big[\frac{d}{ds}\Big|_{s=t} \tau(S_1)\Big] \otimes \tau(S_2)\Big|_{s=t} + \tau(S_1)\Big|_{s=t} \otimes \frac{d}{ds}\Big|_{s=t} \tau(S_2)$$

$$= (\nabla_\xi S_1) \otimes S_2 + S_1 \otimes \nabla_\xi S_2 \qquad (A)$$

$*$ Eq. (C') implies that ∇ and contractions do commute.

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Action of ∇ on tensors in a chart?

- Recall: $\boxed{\nabla_\xi \frac{\partial}{\partial x^i} = \xi^l \Gamma^k{}_{li} \frac{\partial}{\partial x^k}}$

- Problem: Find $\nabla_\xi dx^i = ?$

Same strategy will be used below for general tensors.

$\rightsquigarrow$ • Consider $\eta \otimes \omega$. (tangent vector field; cotangent vector field)

• Differentiate:

$$\nabla_\xi(\eta\otimes\omega) \overset{(A)}{=} (\nabla_\xi\eta)\otimes\omega + \eta\otimes\nabla_\xi\omega$$

• Contract: (use that ∇_ξ and contraction commute)

(by exercise above) $\xi(\omega(\eta)) =$

$$\nabla_\xi(\underbrace{\omega(\eta)}_{\text{scalar function}}) = \omega(\nabla_\xi\eta) + (\nabla_\xi\omega)(\eta)$$

$$= \left[\frac{d}{ds}\Big|_{s=t}\tau(S_1)\right]\otimes\tau(S_2)\Big|_{s=t} + \tau(S_1)\Big|_{s=t}\otimes\frac{d}{ds}\Big|_{s=t}\tau(S_2)$$

$$= (\nabla_\xi S_1)\otimes S_2 + S_1\otimes\nabla_\xi S_2 \qquad (A)$$

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(i.e. $\xi(\omega(\eta)) = \omega(\nabla_\xi\eta) + (\nabla_\xi\omega)(\eta)$)

$\Rightarrow$ An expression for $\nabla_\xi(\omega)(\eta)$:

$$(\nabla_\xi\omega)(\eta) = \xi(\omega(\eta)) - \omega(\nabla_\xi\eta) \qquad (*)$$

Now: Choose $\omega := dx^j$ and $\eta := \frac{\partial}{\partial x^i}$

Notation: $\langle\omega,\xi\rangle = \omega(\xi)$ (inner product, contraction)

$$\Rightarrow (\nabla_\xi dx^j)\left(\frac{\partial}{\partial x^i}\right) = \underbrace{\xi\Big(\overbrace{\langle dx^j,\tfrac{\partial}{\partial x^i}\rangle}^{\delta^j_i = \text{const.}}\Big)}_{=0} - \langle dx^j, \nabla_\xi\tfrac{\partial}{\partial x^i}\rangle$$

$$= -\langle dx^j, \xi^l\Gamma^k{}_{li}\tfrac{\partial}{\partial x^k}\rangle$$

$$= -\xi^l\Gamma^j{}_{li}$$

$$\Rightarrow \boxed{\nabla_\xi dx^j = -\xi^l\Gamma^j{}_{li}\,dx^i}$$

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(by exercise above)

$$\xi(\omega(\eta)) = \nabla_\xi(\underbrace{\omega(\eta)}_{\text{scalar function}}) = \omega(\nabla_\xi\eta) + (\nabla_\xi\omega)(\eta)$$

$$= \left[\frac{d}{ds}\Big|_{s=t} \tau(S_1)\right] \otimes \tau(S_2)\Big|_{s=t} + \tau(S_1)\Big|_{s=t} \otimes \frac{d}{ds}\Big|_{s=t} \tau(S_2)$$

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Notation: $\langle\omega,\xi\rangle = \omega(\xi)$ (inner product, contraction)

$$\Rightarrow (\nabla_\xi dx^j)\left(\frac{\partial}{\partial x^i}\right) = \underbrace{\xi\left(\langle dx^j, \tfrac{\partial}{\partial x^i}\rangle\right)}_{=0} - \langle dx^j, \nabla_\xi \tfrac{\partial}{\partial x^i}\rangle$$

($\langle dx^j, \tfrac{\partial}{\partial x^i}\rangle = \delta^j_i = \text{const.}$)

$$= -\langle dx^j, \xi^\ell \Gamma^k{}_{\ell i} \tfrac{\partial}{\partial x^k}\rangle$$

$$= -\xi^\ell \Gamma^j{}_{\ell i}$$

$$\Rightarrow \boxed{\nabla_\xi dx^j = -\xi^\ell \Gamma^j{}_{\ell i}\, dx^i}$$

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$$- S(\omega(\eta_1 \ldots \eta_r, \omega_1, \ldots \omega_s))$$

$$- S'(\nabla_\xi \eta_1, \eta_2, \ldots, \eta_r, \omega_1 \ldots \omega_s) - \ldots$$

$$- S'(\eta_1, \ldots, \nabla_\xi \eta_r, \omega_1 \ldots \omega_s)$$

$$- S'(\eta_1, \ldots, \eta_r, \nabla_\xi \omega_1, \omega_2, \ldots, \omega_s) + \ldots$$

$$- S'(\eta_1 \ldots \eta_r, \omega_1 \ldots \nabla_\xi \omega_s)$$

that $\nabla_\xi S$ reads

$$\nabla_\xi S' = \xi^k S^{i_1 \ldots i_p}_{j_1 \ldots j_q ; k} \frac{\partial}{\partial x^{i_1}} \otimes \ldots \otimes \frac{\partial}{\partial x^{i_p}} \otimes dx^{j_1} \otimes \ldots \otimes dx^{j_q}$$

with:

$$S^{i_1 \ldots i_p}_{j_1 \ldots j_q ; k} := S^{i_1 \ldots i_p}_{j_1 \ldots j_q , k} + \Gamma^{i_1}_{k l} S^{l i_2 \ldots i_p}_{j_1 \ldots j_q}$$
$$+ \ldots + \Gamma^{i_p}_{k l} S^{i_1 \ldots l}_{j_1 \ldots j_q}$$
$$- \Gamma^{l}_{k j_1} S^{i_1 \ldots i_p}_{l \ldots j_q}$$
$$- \ldots - \Gamma^{l}_{k j_q} S^{i_1 \ldots i_p}_{j_1 \ldots l}$$

Special cases:

- Tangent vector fields:

$$\xi^i{}_{;k} = \xi^i{}_{,k} + \xi^j \Gamma^i{}_{kj}$$

- Cotangent vector fields:

$$\omega_{i;k} = \omega_{i,k} - \omega_l \Gamma^l{}_{ki}$$

Recall: Specifying ∇ specifies parallel transport of vectors and this should specify the manifold's shape, but how?

→ Indeed, ∇ specifies Torsion & Curvature

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