Title: Weyl-Ambient Metrics, Obstruction Tensors and Their Roles in Holography

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Abstract: Weyl geometry is a natural extension of conformal geometry with Weyl covariance mediated by a Weyl connection. We generalize the Fefferman-Graham (FG) ambient construction for conformal manifolds to a corresponding construction for Weyl manifolds. We first introduce the Weyl-ambient metric motivated by the Weyl-Fefferman-Graham (WFG) gauge, which is a generalization of the FG gauge for asymptotically locally AdS (AlAdS) spacetimes. Then, the Weyl-ambient space as a pseudo-Riemannian geometry induces a codimension-2 Weyl geometry. Through the Weyl-ambient construction, we investigate Weyl-covariant quantities on the Weyl manifold and define Weyl-obstruction tensors. We show that Weyl-obstruction tensors appear as poles in the Fefferman-Graham expansion of the AlAdS bulk metric for even boundary dimensions. Under holographic renormalization, we demonstrate that Weyl-obstruction tensors can be used as the building blocks for the Weyl anomaly of the dual quantum field theory.

Zoom link https://pitp.zoom.us/j/91781363979?pwd=NlhjTVlHTlhMSTcxYnk5eExkTWFqdz09

Weyl-Ambient Metrics, Obstruction Tensors and Their Roles in Holography

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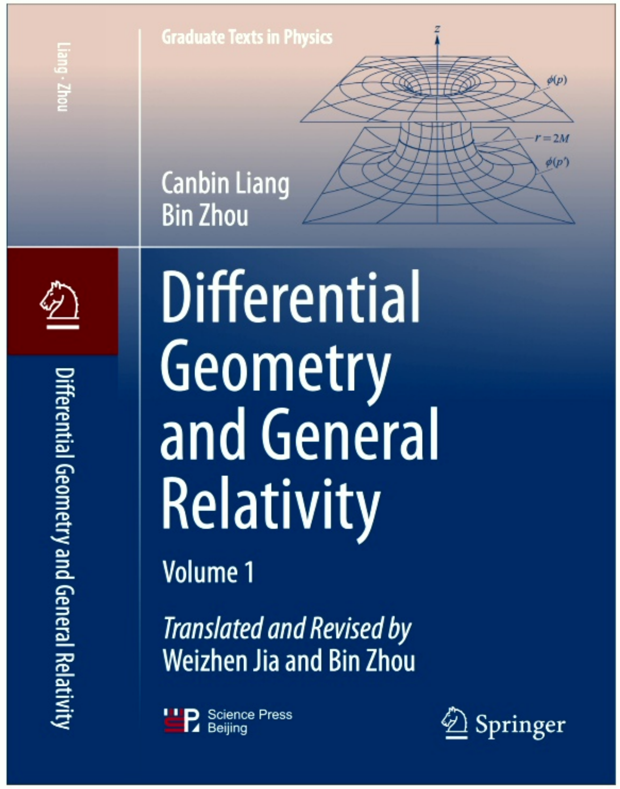
Based on arXiv:2301.06628 [WJ, Manthos Karydas & Rob Leigh]
arXiv:2109.14014 [WJ, Manthos Karydas]

Perimeter Institute Quantum Gravity Seminar
09.28.2023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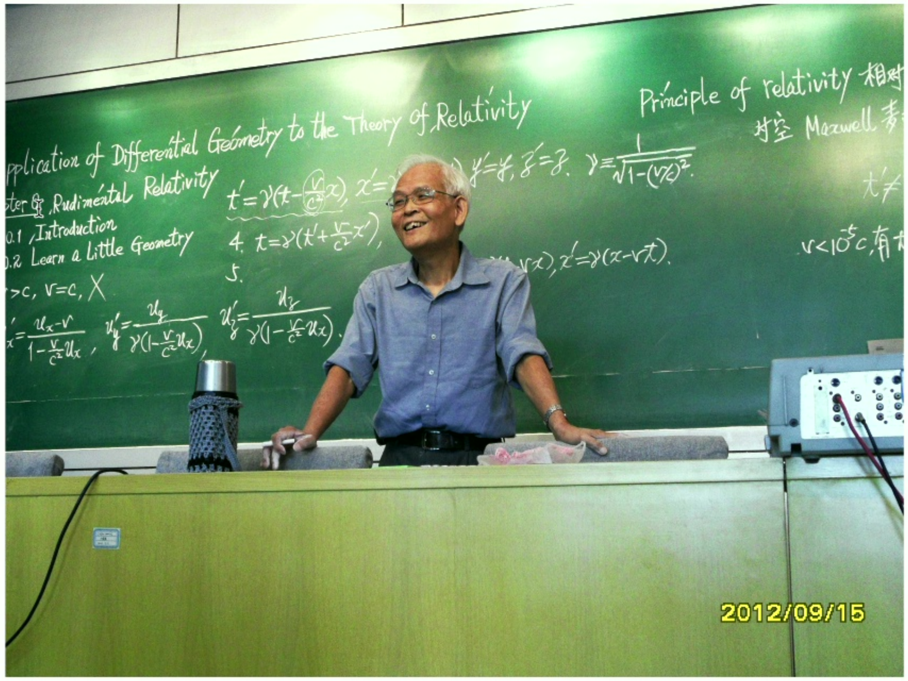

Professor Canbin Liang (1938-2022)

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Outline

Outline

1. Backgrounds
2. Weyl Geometry
3. Ambient Metrics
4. Weyl-Ambient Metric
5. Weyl-Obstruction Tensors
6. Holographic Weyl Anomaly

Backgrounds

Geometries	Riemannian geometry	Conformal geometry	Weyl geometry
Manifold structures	(M, g)	$(M, [g])$	$(M, [g, a])$
Symmetries	$\mathrm{Diff}(M)$	$\mathrm{Diff}(M) \ltimes \mathrm{Weyl}$	$\mathrm{Diff}(M) \ltimes \mathrm{Weyl}$
Covariant quantities	$R, \nabla R, \nabla\nabla R, \cdots$	$W, C, \Omega^{(1)}, \Omega^{(2)}, \cdots$	$\hat{R}, \hat{\nabla}\hat{R}$ $\hat{\nabla}\hat{\nabla}\hat{R}, \cdots$

- $\mathrm{Diff}(M) \ltimes \mathrm{Weyl}$ is represented more naturally on $(M, [g, a])$.
- Weyl geometry can be realized at the codim-2 of Weyl-ambient space.
- QFTs at codim-2 are organized in a Weyl-covariant fashion.

Outline

Weyl Geometry

Under a Weyl transformation

$$g \to \mathcal{B}^{-2}(x)g\,, \qquad \nabla g \to \nabla g - 2\mathrm{d}\ln\mathcal{B}(x)g\,. \tag{1}$$

Introduce the Weyl connection A

$$A \to A - \mathrm{d}\ln\mathcal{B}\,. \tag{2}$$

$$(\nabla g - 2Ag) \to \mathcal{B}^{-2}(\nabla g - 2Ag)\,. \tag{3}$$

Define ∇ with Weyl metricity

$$\nabla g = 2Ag\,. \tag{4}$$

Further define the Weyl-LC connection

$$\hat{\nabla} g \equiv (\nabla - 2A)g = 0\,. \tag{5}$$

For any tensor T with Weyl weight w_T, $T \to \mathcal{B}^{w_T} T$,

$$\hat{\nabla} T \equiv \nabla T + w_T A T\,, \qquad \hat{\nabla} T \to \mathcal{B}^{w_T}\hat{\nabla} T\,. \tag{6}$$

Weyl Geometry

The connection coefficients of ∇ reads

$$\Gamma^{\rho}{}_{\mu\nu} = \frac{1}{2} g^{\rho\sigma}(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\nu\sigma} - \partial_\sigma g_{\mu\nu}) - (A_\mu \delta^\rho{}_\nu + A_\nu \delta^\rho{}_\mu - g^{\rho\sigma} A_\sigma g_{\mu\nu}) .$$

The curvature tensors of $\hat{\nabla}$ and the LC connection $\mathring{\nabla}$ are related in following way:

$$\begin{aligned}
\hat{R}^\mu{}_{\nu\rho\sigma} &= \mathring{R}^\mu{}_{\nu\rho\sigma} + \mathring{\nabla}_\sigma A_\nu \delta^\mu{}_\rho - \mathring{\nabla}_\rho A_\nu \delta^\mu{}_\sigma + \mathring{\nabla}_\rho A^\mu g_{\nu\sigma} - \mathring{\nabla}_\sigma A^\mu g_{\nu\rho} \\
&\quad + A_\nu (A_\sigma \delta^\mu{}_\rho - A_\rho \delta^\mu{}_\sigma) + A^\mu (g_{\nu\sigma} A_\rho - g_{\nu\rho} A_\sigma) \\
&\quad + A^2 (g_{\nu\rho} \delta^\mu{}_\sigma - g_{\nu\sigma} \delta^\mu{}_\rho) , \\
\hat{R}_{\mu\nu} &= \mathring{R}_{\mu\nu} - \frac{(d-2)}{2} F_{\mu\nu} + (d-2)(\mathring{\nabla}_{(\mu} A_{\nu)} + A_\mu A_\nu) \\
&\quad + (\mathring{\nabla} \cdot A - (d-2) A^2) g_{\mu\nu} , \\
\hat{R} &= \mathring{R} + 2(d-1) \mathring{\nabla} \cdot A - (d-1)(d-2) A^2 ,
\end{aligned}$$

Under a Weyl transformation,

$$\hat{R}^\mu{}_{\nu\rho\sigma} \to \hat{R}^\mu{}_{\nu\rho\sigma} , \qquad \hat{R}_{\mu\nu} \to \hat{R}_{\mu\nu} , \qquad \hat{R} \to \mathcal{B}^2 \hat{R} . \tag{7}$$

Outline

Flat Ambient Metrics

Consider the $(d+2)$-dimensional Minkowski spacetime $\tilde{M} = \mathbb{R}^{1,d+1}$

$$\eta = -(\mathrm{d}X^0)^2 + \sum_{i=1}^{d+1} (\mathrm{d}X^i)^2 \,. \tag{8}$$

Euclidean AdS_{d+1} spaces are codimension-1 hyperboloids

$$(X^0)^2 - R^2 = L^2 \,, \qquad R^2 = \sum_{i=1}^{d+1} (X^i)^2 \,. \tag{9}$$

Convert η into the "cone" form

$$\eta = -\mathrm{d}\ell^2 + \frac{\ell^2}{L^2} g^+ \,, \qquad \ell = \sqrt{(X^0)^2 - R^2} \,, \tag{10}$$

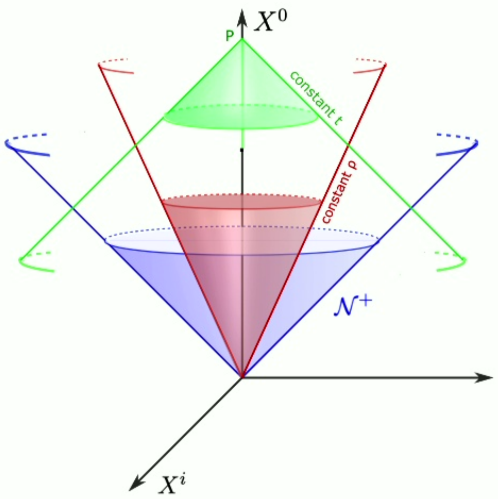
$$g^+ = \frac{L^2}{z^2}\left(\mathrm{d}z^2 + \gamma_{ij}\mathrm{d}x^i \mathrm{d}x^j\right) \,, \qquad 0 < z < 2L \,, \tag{11}$$

$$\gamma = L^2 (1 - \frac{1}{4}(z/L)^2)^2 \mathrm{d}\Omega_d^2 \,. \tag{12}$$

Flat Ambient Metrics

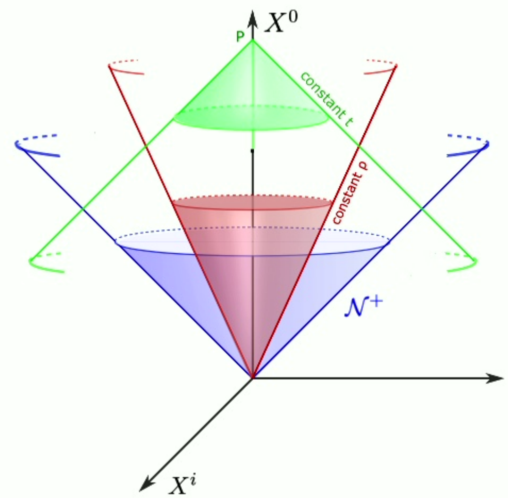
- $\eta = -\mathrm{d}\ell^2 + \frac{\ell^2}{z^2}(\mathrm{d}z^2 + \gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j)$
- Singular when $z \to 0^+$ with ℓ fixed.
- Well-defined when $z \to 0^+, \ell \to 0^+$, with ℓ/z fixed.
- Define $t = \ell/z$, $\rho = -z^2/2$,

$$\eta = 2\rho \mathrm{d}t^2 + 2t\mathrm{d}t\mathrm{d}\rho + t^2\gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j \,.$$



Flat Ambient Metrics

- Any hypersurface $\Sigma : \phi(t, x, \rho) = 0$.
- The cone $\mathcal{N}^+$ at $\rho = 0$ intersects with $\Sigma \Rightarrow t = t(x^i)$.
- Induced metric $\eta|_{\Sigma \cap \mathcal{N}^+} = t(x)^2 L^2 \mathrm{d}\Omega_d^2$.
- Conformally equivalent codim-2 metrics. Pick $t = 1$ as a representative.
- Under diffeomorphism $t = \mathcal{B}(x)^{-1} t'$,
  $\gamma^{(0)} \equiv \eta|_{\rho=0,t=1} = L^2 \mathrm{d}\Omega_d^2$,
  $\gamma'^{(0)} \equiv \eta|_{\rho=0,t'=1} = \mathcal{B}(x)^{-2} L^2 \mathrm{d}\Omega_d^2$,
  $[\gamma^{(0)}]$ gives a conformal class of metrics.
- $\Sigma \cap \mathcal{N}^+$ as the image of embedding $\varphi : M \to \tilde{M}$ gives rise to a conformal manifold $(M, [\gamma^{(0)}])$.



Ricci-Flat Ambient Metrics

- On a $(d+2)$-dim manifold $\tilde{M}$ [Fefferman & Graham (1984, 2011) 0710.0919]

$$\tilde{g} = -\mathrm{d}\ell^2 + \frac{\ell^2}{L^2} g^+_{\mu\nu}(x)\mathrm{d}x^\mu \mathrm{d}x^\nu, \qquad \mu,\nu = 1,\cdots d+1\,. \tag{13}$$

  $g^+(x)$ is an arbitrary $(d+1)$-dim metric independent of ℓ.

- The ambient Ricci tensor can be written as

$$\tilde{R}ic(\tilde{g}) = Ric(g^+) + \frac{d}{L^2} g^+ = G_{\mu\nu}(g^+) + \Lambda g^+_{\mu\nu}\,, \quad \Lambda = -\frac{d(d-1)}{2L^2}\,.$$

- $\tilde{g}$ is Ricci-flat $\Rightarrow$ g^+ is an Einstein metric.
- Fefferman-Graham form of g^+:

$$g^+ = L^2 \frac{\mathrm{d}z^2}{z^2} + \frac{L^2}{z^2}\gamma_{ij}(x,z)\mathrm{d}x^i \mathrm{d}x^j\,, \qquad i,j = 1,\cdots,d\,. \tag{14}$$

Ricci-Flat Ambient Metrics

- Transform the the ambient coordinates $\{\ell, x^i, z\} \to \{t, x^i, \rho\}$

$$\tilde{g} = 2\rho \mathrm{d}t^2 + 2t\mathrm{d}t\mathrm{d}\rho + t^2\gamma_{ij}(x,\rho)\mathrm{d}x^i\mathrm{d}x^j \,. \tag{15}$$

Asymptotic expansion of $\gamma_{ij}(x,\rho)$:

$$\gamma_{ij}(x,\rho) = \gamma_{ij}^{(0)}(x) + \gamma_{ij}^{(1)}(x)\rho + \gamma_{ij}^{(2)}(x)\rho^2 + \cdots \,. \tag{16}$$

- Induced metric at $\rho = 0$, $t = 1$ gives $\tilde{g}|_{t=1,\rho=0} = \gamma^{(0)}$.
- $(\tilde{M}, \tilde{g})$ gives rise to a codim-2 conformal manifold $(M, [\gamma^{(0)}])$.

Fefferman-Graham Expansion

Solving the Ricci-flatness condition $\tilde{R}ic(\tilde{g}) = 0$ order by order yields

$$\gamma_{ij}^{(1)} = 2P_{ij}\,,$$
$$\gamma_{ij}^{(2)} = \Omega_{ij}^{(1)} + P_{ki}P^{k}{}_{j}\,,$$
$$\gamma_{ij}^{(3)} = \frac{1}{3}\Omega_{ij}^{(2)} + \frac{4}{3}\Omega_{k(i}^{(1)}P^{k}{}_{j)}\,.$$

P_{ij} is the Schouten tensor, $\Omega_{ij}^{(k)}$ are the extended obstruction tensors

$$P_{ij} = \frac{1}{d-2}\left(R_{ij}^{(0)} - \frac{R^{(0)}}{2(d-1)}\gamma_{ij}^{(0)}\right)\,.$$
$$\Omega_{ij}^{(1)} = \frac{1}{d-4}\Big(-\nabla_k^{(0)}\nabla_{(0)}^{k}P_{ij} + \nabla_k^{(0)}\nabla_j^{(0)}P_i{}^{k} + W_{kjil}^{(0)}P^{lk}\Big)\,,$$
$$\Omega_{ij}^{(2)} = \frac{1}{d-6}\Big(-\nabla_{(0)}^{k}\nabla_k^{(0)}\Omega_{ij}^{(1)} + 2W_{kjil}^{(0)}\Omega_{(1)}^{lk} + 4P\Omega_{ij}^{(1)} + \cdots\Big)\,.$$

Obstruction Tensors

The pole of $\Omega_{ij}^{(k-1)}$ is the obstruction tensor $\mathcal{O}_{ij}^{(2k)}$

$$\Omega_{ij}^{(k-1)} \propto \frac{1}{d-2k}\mathcal{O}_{ij}^{(2k)} .$$

- The pole of $\gamma_{ij}^{(k)}$ gives rise to the obstruction tensor $\mathcal{O}_{ij}^{(2k)}$.
- For $k = 2$, $\mathcal{O}_{\mu\nu}^{(4)}$ corresponds to the Bach tensor.
- $\mathcal{O}_{\mu\nu}^{(2k)}$ is the only irreducible Weyl-covariant tensor besides Weyl tensor in $2k$-dimension, with Weyl weight $2k-2$.
- Symmetric traceless, $\Omega_{ij}^{(k)}\gamma_{(0)}^{ij} = 0$.
- $\Omega_{ij}^{(k)}$ can be defined through the ambient curvature

$$\Omega_{ij}^{(k)} = \underbrace{\nabla_\rho \cdots \nabla_\rho}_{k-1} \tilde{R}_{\rho ij\rho}|_{t=1,\rho=0} .$$

Outline

Weyl-Fefferman-Graham Gauge

$$g^{+}_{FG} = L^2 \frac{\mathrm{d}z^2}{z^2} + \frac{L^2}{z^2}\gamma_{ij}(x,z)\mathrm{d}x^i\mathrm{d}x^j\,, \qquad i,j = 1,\cdots,d\,. \qquad (17)$$

- FG ansatz is not preserved under a Weyl diffeomorphism

$$z \to z' = z/\mathcal{B}(x)\,, \qquad x^\mu \to x'^\mu = x^\mu\,.$$

- FG gauge breaks the Weyl covariance on the conformal boundary.
- Introduce the WFG gauge [Ciambelli & Leigh (2020) 1905.04339]

$$g^{+}_{WFG} = L^2\left(\frac{dz}{z} - a_i(x,z)dx^i\right)^2 + \frac{L^2}{z^2}\gamma_{ij}(x,z)dx^i dx^j\,.$$

- The WFG ansatz is preserved under a Weyl diffeomorphism, $g^{+}_{WFG}[\gamma, a; x, z] \to g^{+}_{WFG}[\gamma', a'; x', z']$ provided

$$a'_i(x,z) = a_i(x,z) - \partial_i \ln\mathcal{B}(x)\,, \quad \gamma'_{ij}(x',z') = \mathcal{B}(x)^{-2}\gamma_{ij}(x,z)\,.$$

Weyl-Ambient Metrics

$$\tilde{g} = -\mathrm{d}\ell^2 + \frac{\ell^2}{L^2} g^{+}_{WFG} \,. \tag{18}$$

- Transform the the ambient coordinates $\{\ell, x^i, z\} \to \{t, x^i, \rho\}$

$$\tilde{g} = 2\rho \mathrm{d}t^2 + 2t\mathrm{d}\rho(\mathrm{d}t + ta_i \mathrm{d}x^i) + t^2(\gamma_{ij} - 2\rho a_i a_j)\mathrm{d}x^i \mathrm{d}x^j \,. \tag{19}$$

- Asymptotic expansion of $\gamma_{ij}(x,\rho)$ and $a_i(x,\rho)$:

$$\gamma_{ij}(x,\rho) = \gamma^{(0)}_{ij}(x) + \gamma^{(1)}_{ij}(x)\rho + \gamma^{(2)}_{ij}(x)\rho^2 + \cdots \,, \tag{20}$$

$$a_i(x,\rho) = a^{(0)}_i(x) + a^{(1)}_i(x)\rho + a^{(2)}_i(x)\rho^2 + \cdots \,. \tag{21}$$

- At $\rho = 0$, $t = 1$ induces $\gamma^{(0)}_{ij}$ and $a^{(0)}_i$.

Weyl-Ambient Metrics

Under an ambient Weyl diffeomorphism

$$t' = \mathcal{B}(x)t\,, \qquad x'^i = x^i\,, \qquad \rho' = \mathcal{B}(x)^{-2}\rho\,, \tag{22}$$

The form of the Weyl-ambient metric is preserved

$$\tilde{g}[\gamma, a; t, x, \rho] \to \tilde{g}[\gamma', a'; t', x', \rho']\,. \tag{23}$$

$$\gamma_{ij}^{(k)} \to \mathcal{B}^{2k-2}\gamma_{ij}^{(k)}\,, \quad a_i^{(0)} \to a_i^{(0)} - \partial_i \ln \mathcal{B}\,, \quad a_i^{(k\geqslant 1)} \to \mathcal{B}^{2k} a_i^{(k\geqslant 1)}\,.$$

Define the following dual frame $\{\boldsymbol{e}^P\}$:

$$\boldsymbol{e}^+ = \mathrm{d}t + t a_i \mathrm{d}x^i\,, \quad \boldsymbol{e}^i = \mathrm{d}x^i\,, \quad \boldsymbol{e}^- = t\mathrm{d}\rho + \rho\mathrm{d}t - t\rho a_i \mathrm{d}x^i\,,$$

Weyl-ambient metric has the form

$$\tilde{g} = \boldsymbol{e}^+ \otimes \boldsymbol{e}^- + \boldsymbol{e}^- \otimes \boldsymbol{e}^+ + t^2 \gamma_{ij} \boldsymbol{e}^i \otimes \boldsymbol{e}^j\,. \tag{24}$$

Weyl-Ambient Metrics

The corresponding frame $\{\underline{D}_P\}$ reads

$$\underline{D}_+ = \underline{\partial}_t - \frac{\rho}{t}\underline{\partial}_\rho\,, \quad \underline{D}_i = \underline{\partial}_i - ta_i\underline{\partial}_t + 2\rho a_i\underline{\partial}_\rho\,, \quad \underline{D}_- = \frac{1}{t}\underline{\partial}_\rho\,.$$

- $\underline{D}_+, \underline{D}_-$ are null vectors.
- $\{\underline{D}_i\}$ spans a distribution $C_d \subset T\tilde{M}$.
- C_d is generally non-integrable, may not give a hypersurface

$$[\underline{D}_i, \underline{D}_j] = -tf_{ij}\underline{D}_+ + t\rho f_{ij}\underline{D}_-\,. \tag{25}$$

Integrable only when $f_{ij} \equiv \underline{D}_i a_j - \underline{D}_j a_i = 0$.

Induced Weyl Geometry

Induced metric at $t = 1$, $\rho = 0$

$$\gamma_{ij}^{(0)} = \tilde{g}_{ij}|_{t=1,\rho=0} \,. \tag{26}$$

Under an ambient Weyl diffeomorphism

$$\gamma_{ij}^{\prime(0)} = \tilde{g}'_{ij}|_{t'=1,\rho'=0} = \mathcal{B}(x)^{-2}\gamma_{ij}^{(0)} \,. \tag{27}$$

Ambient LC connection $\tilde{\nabla}$ satisfies $\tilde{\nabla}_{\underline{D}_P}\tilde{g}_{MN} = 0$ $(M, N, P = +, i, -)$.

$$\tilde{\nabla}_{\underline{D}_M}\underline{D}_N = \tilde{\Gamma}^P{}_{MN}\underline{D}_P \tag{28}$$

Connection coefficients $\tilde{\Gamma}^i{}_{jk}$ at $t = 1$, $\rho = 0$ induces

$$\Gamma^i_{(0)jk} = \frac{1}{2}\gamma^{im}_{(0)}(\partial_j\gamma^{(0)}_{mk} + \partial_k\gamma^{(0)}_{jm} - \partial_m\gamma^{(0)}_{jk}) - (a^{(0)}_j\delta^i{}_k + a^{(0)}_k\delta^i{}_j - a^i_{(0)}\gamma^{(0)}_{jk}) \,.$$

Induced Weyl Geometry

$\tilde{\nabla}$ induces a connection $\nabla^{(0)}$ on the codim-2 manifold M, with

$$\nabla^{(0)}_{\underline{\partial}_i}\underline{\partial}_j = \Gamma^{k}_{(0)ij}\underline{\partial}_k \,. \tag{29}$$

$\nabla^{(0)}$ satisfies

$$\nabla^{(0)}_i \gamma^{(0)}_{jk} = 2a^{(0)}_i \gamma^{(0)}_{jk} \,. \tag{30}$$

Define Weyl-LC connection $\hat{\nabla}^{(0)}$

$$\hat{\nabla}^{(0)}_i \gamma^{(0)}_{jk} \equiv \nabla^{(0)}_i \gamma^{(0)}_{jk} - 2a^{(0)}_i \gamma^{(0)}_{jk} = 0 \,. \tag{31}$$

Induced Weyl Geometry

- Structures induced on M

$$\tilde{g} \Rightarrow \gamma^{(0)}, \qquad a_i \Rightarrow a_i^{(0)}, \qquad \tilde{\nabla} \Rightarrow \hat{\nabla}_i^{(0)}. \tag{32}$$

- An ambient Weyl diffeomorphism induces a Weyl transformation

$$\gamma_{ij}^{(0)} \to \mathcal{B}^{-2}\gamma_{ij}^{(0)}, \qquad a_i^{(0)} \to a_i^{(0)} - \partial_i \ln \mathcal{B}. \tag{33}$$

- Equivalence class $[g, a]$ is called a *Weyl class* if

$$(g_{ij}, a_i) \sim (\mathcal{B}(x)^{-2} g_{ij}, a_i - \partial_i \ln \mathcal{B}(x)) \tag{34}$$

- Riemannian manifold $(\tilde{M}, \tilde{g}) \Rightarrow$ Weyl manifold $(M, [\gamma^{(0)}, a^{(0)}])$!

Outline

Weyl-Obstruction Tensors

Solving the Ricci-flatness condition $\tilde{R}ic(\tilde{g}) = 0$ order by order yields

$$
\begin{aligned}
\gamma^{(1)}_{ij} &= 2\hat{P}_{(ij)} = 2\hat{P}_{ij} - f^{(0)}_{ij}\,. \\
\gamma^{(2)}_{ij} &= \hat{\Omega}^{(1)}_{ij} + \hat{P}^k{}_i \hat{P}_{kj} + \hat{\nabla}^{(0)}_{(i} a^{(1)}_{j)}\,, \\
\gamma^{(3)}_{ij} &= \tfrac{1}{3}\hat{\Omega}^{(2)}_{ij} + \tfrac{4}{3}\hat{\Omega}^{(1)}_{k(i}\hat{P}^k{}_{j)} + \tfrac{2}{3}\hat{\nabla}^{(0)}_{(i} a^{(2)}_{j)} + 2a^{(1)}_i a^{(1)}_j - \tfrac{1}{3} a^{(1)} \cdot a^{(1)} \gamma^{(0)}_{ij} + \tfrac{1}{3} P^k{}_{(i} \hat{\nabla}^{(0)}_{j)} a^{(1)}_k \\
&\quad - \tfrac{1}{3} a^k_{(1)} \left(\hat{\nabla}^{(0)}_i \hat{P}_{kj} + \hat{\nabla}^{(0)}_i \hat{P}_{jk} - \hat{\nabla}^{(0)}_k \hat{P}_{ji} + 2\hat{\nabla}^{(0)}_j \hat{P}_{ik} - 2\hat{\nabla}^{(0)}_k \hat{P}_{ij}\right),
\end{aligned}
$$

$\hat{P}_{ij}$ is the Weyl-Schouten tensor, $\hat{\Omega}^{(k)}_{ij}$ will be recognized as the extended Weyl-obstruction tensors

$$
\hat{P}_{ij} = \frac{1}{d-2}\left(\hat{R}^{(0)}_{ij} - \frac{\hat{R}^{(0)}}{2(d-1)}\gamma^{(0)}_{ij}\right), \tag{35}
$$

$$
\hat{\Omega}^{(1)}_{ij} = \frac{1}{d-4}\left(-\hat{\nabla}^{(0)}_k \hat{\nabla}^k_{(0)} \hat{P}_{ij} + \hat{\nabla}^{(0)}_k \hat{\nabla}^{(0)}_j \hat{P}_i{}^k + \hat{W}^{(0)}_{kjil} \hat{P}^{lk}\right), \tag{36}
$$

$$
\hat{\Omega}^{(2)}_{ij} = \frac{1}{d-6}\left(-\hat{\nabla}^k_{(0)} \hat{\nabla}^{(0)}_k \hat{\Omega}^{(1)}_{ij} + 2\hat{W}^{(0)}_{kjil} \hat{\Omega}^{lk}_{(1)} + 4\hat{P}\hat{\Omega}^{(1)}_{ij} + \cdots\right). \tag{37}
$$

Weyl-Obstruction Tensors

- The components of the ambient Riemann tensor induces Weyl-covariant tensors on M

$$\tilde{R}_{-ij-}|_{\rho=0,t=1} = \hat{\Omega}^{(1)}_{ij}\,, \quad \tilde{R}_{-ijk}|_{\rho=0,t=1} = \hat{C}_{ijk}\,, \quad \tilde{R}_{ijkl}|_{\rho=0,t=1} = \hat{W}^{(0)}_{ijkl}\,,$$

where $\hat{C}_{ijk} \equiv \hat{\nabla}_j \hat{P}_{ik} - \hat{\nabla}_k \hat{P}_{ij}$ is the Weyl-Cotton tensor.

- The corresponding curvature 2-form at $\rho = 0, t = 1$:

$$\tilde{\boldsymbol{R}}^M_{(0)N} = \begin{pmatrix} 0 & -\hat{\boldsymbol{C}}_j & 0 \\ 0 & \hat{\boldsymbol{W}}^i_{(0)j} & \hat{\boldsymbol{C}}^i \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \hat{\boldsymbol{\Omega}}^{(1)}_j & 0 \\ 0 & \hat{C}_{kj}{}^i \boldsymbol{e}^k & -\hat{\boldsymbol{\Omega}}^i_{(1)} \\ 0 & 0 & 0 \end{pmatrix} \wedge \boldsymbol{e}^- \,,$$

where $\hat{\boldsymbol{\Omega}}^{(1)}_i = \hat{\Omega}^{(1)}_{ij} \boldsymbol{e}^j$, $\hat{\boldsymbol{C}}_i = \frac{1}{2} \hat{C}_{ikj} \boldsymbol{e}^j \wedge \boldsymbol{e}^k$, $\hat{\boldsymbol{W}}^i_{(0)j} = \hat{W}^i{}_{jkl} \boldsymbol{e}^k \wedge \boldsymbol{e}^l$.

Weyl-Obstruction Tensors

Under a Weyl transformation,

$$\hat{\Omega}^{(1)}_{ij} \to \mathcal{B}^2(x)\hat{\Omega}^{(1)}_{ij}\,, \quad \hat{C}_{ijk} \to \hat{C}_{ijk}\,, \quad \hat{W}^{(0)}_{ijkl} \to \mathcal{B}^{-2}(x)\hat{W}^{(0)}_{ijkl}\,.$$

More generally we have the following theorem:

Let $IJKLM_1 \ldots M_r$ be a list of indices, s_+ of which are $+$, s_M of which correspond to x^i, and s_- of which are $-$, then under the ambient Weyl diffeomorphism,

$$\tilde{\nabla}_{M_1} \cdots \tilde{\nabla}_{M_r} \tilde{R}'_{IJKL}|_{\rho'=0,t'=1} = \mathcal{B}(x)^{2s_- - 2} \tilde{\nabla}_{M_1} \cdots \tilde{\nabla}_{M_r} \tilde{R}_{IJKL}|_{\rho=0,t=1}\,.$$

The pullback of an ambient tensor $\tilde{T}_{M_1 \cdots M_k}$ to M:

$$T_{i_1 \cdots i_{s_M}} \equiv \tilde{T}_{M_1 \cdots M_k}|_{\rho=0,t=1}\,, \tag{38}$$

is Weyl-covariant with Weyl weight $2s_- - 2$.

Weyl-Obstruction Tensors

Define the extended Weyl-obstruction tensors as follows:

$$\hat{\Omega}_{ij}^{(k)} = \underbrace{\tilde{\nabla}_- \cdots \tilde{\nabla}_-}_{k-1} \tilde{R}_{-ij-}|_{\rho=0,t=1}. \tag{39}$$

- Weyl-covariant in any dimension, with Weyl weight $2k$.
- Symmetric traceless, $\Omega_{ij}^{(k)} \gamma_{(0)}^{ij} = 0$.
- The pole of $\gamma_{ij}^{(k)}$ gives rise to the obstruction tensor $\mathcal{O}_{ij}^{(2k)}$.

$$\mathrm{Res}_{d=2k} \hat{\Omega}_{ij}^{(k-1)} = \frac{k!}{2} \mathrm{Res}_{d=2k} \gamma_{ij}^{(k)}\,. \tag{40}$$

Weyl-Ambient Space

Top-down perspective:

- Start from a Weyl-ambient space in $(d+2)$-dim with the metric

$$\tilde{g} = 2\rho \mathrm{d}t^2 + 2t\mathrm{d}\rho(\mathrm{d}t + ta_i\mathrm{d}x^i) + t^2(\gamma_{ij} - 2\rho a_i a_j)\mathrm{d}x^i\mathrm{d}x^j \,. \qquad (41)$$

- A Weyl manifold $(M, [g, a])$ is induced at codim-2.

Bottom-up perspective:

- Weyl-ambient space can also be constructed from a d-dim Weyl manifold $(M, [g, a])$.
- Introduce the Weyl structure, i.e., a principal $\mathbb{R}_+$-bundle over M.
- Weyl-ambient metric (41) can be uniquely determined from the Ricci-flatness condition by taking the Weyl structure as the initial surface, provided the data $\gamma_{ij}^{(0)}$ and a_i.

Outline

Weyl-Ward Identity

- d-dim field theory defined on Weyl geometry, Weyl anomaly $\mathcal{A}$ arises in the path integral under a Weyl transformation:

$$Z[\gamma^{(0)}, a^{(0)}] = e^{-\mathcal{A}[\mathcal{B}(x);\gamma^{(0)},a^{(0)}]} Z[\gamma^{(0)}/\mathcal{B}(x)^2, a^{(0)} - \mathrm{d}\ln\mathcal{B}(x)] \,.$$

- In general, $\gamma^{(0)}_{\mu\nu}$ and $a^{(0)}_\mu$ are the sources of $T^{\mu\nu}$ and J^μ

$$\langle T^{\mu\nu}(x)\rangle = \frac{2}{\sqrt{-\gamma^{(0)}}} \frac{\delta S}{\delta\gamma^{(0)}_{\mu\nu}(x)} \,, \quad \langle J^\mu(x)\rangle = \frac{1}{\sqrt{-\gamma^{(0)}}} \frac{\delta S}{\delta a^{(0)}_\mu(x)} \,.$$

- Anomalous Weyl-Ward Identity

$$\frac{1}{\sqrt{-\gamma^{(0)}}} \frac{\delta\mathcal{A}}{\delta\ln\mathcal{B}(x)} = \left\langle T^{\mu\nu}(x)\gamma^{(0)}_{\mu\nu}(x) + \hat{\nabla}^{(0)}_\mu J^\mu(x)\right\rangle .$$

- In a holographic theory, a_μ is pure gauge in the bulk. The current sourced by $a^{(0)}_\mu$ on the boundary can be absorbed into an improved $T^{\mu\nu}$.

Holographic Weyl Anomaly

Holographic dictionary [Witten (1998) hep-th/9802150]

$$\exp\left(-S_{bulk}[g;\gamma_{(0)},a_{(0)}]\right) = \exp\left(-S_{bdr}[\gamma_{(0)},a_{(0)}]\right) ,$$

Bulk Einstein-Hilbert action

$$S_{bulk} = \frac{1}{2\kappa^2}\int_M \sqrt{-g}\,(R-2\Lambda)\boldsymbol{e}^z \wedge \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^d ,$$

where $\boldsymbol{e}^z = L\frac{dz}{z} - La_\mu(z,x)dx^\mu$. On-shell expansion near the boundary:

$$S^{o.s.}_{bulk} = -\frac{L^{-2}}{\kappa^2}\int_M \frac{L^d}{z^d}\left(d + \frac{d}{2}\frac{z^2}{L^2}X^{(1)} + \frac{d}{2}\frac{z^4}{L^4}X^{(2)} + \cdots\right)\boldsymbol{e}^z \wedge vol_\Sigma ,$$

where $X^{(k)}$ comes from the expansion of $\sqrt{-g}$

$$\sqrt{-g} = \frac{L^d}{z^d}\sqrt{-\gamma^{(0)}}\left(1 + \frac{1}{2}\frac{z^2}{L^2}X^{(1)} + \frac{1}{2}\frac{z^4}{L^4}X^{(2)} + \cdots\right) .$$

Holographic Weyl Anomaly

- In the WFG gauge, boundary may not be part of a foliation. Cutoff regularization is not available.
- In dim reg, $X^{(k)}$ encounter a pole at $d = 2k$, which corresponds to the log divergence in the cutoff scheme.
- The difference of the pole term in S_{bulk} under a Weyl diffeomorphism gives rise to the Weyl anomaly in $2k$-dim. [Ciambelli & Leigh (2020) 1905.04339]

$$\begin{aligned}\mathcal{A}_k &= \lim_{d\to 2k^-} S_{pole}[z', x] - S_{pole}[z, x] \\ &= \frac{k}{\kappa^2 L}\int \ln \mathcal{B} X^{(k)}_{d=2k} vol_\Sigma \,.\end{aligned}$$

Weyl Anomaly in $2d$

Find $X^{(1)}$ from the bulk equation of motion:

$$X^{(1)} = -\frac{L^2}{2(d-1)}\hat{R} = -L^2\hat{P}\,, \qquad \hat{P} \equiv \hat{P}_{\mu\nu}\gamma^{\mu\nu}_{(0)}\,.$$

The Weyl anomaly in $2d$ reads

$$\mathcal{A}_1 = \frac{1}{\kappa^2 L}\int \ln \mathcal{B} X^{(1)}_{d=2} vol_\Sigma = -\frac{L}{\kappa^2}\int \ln \mathcal{B}\hat{P}\sqrt{-\gamma^{(0)}}\mathrm{d}^2x\,.$$

The Weyl-Ward identity

$$\langle T^\mu{}_\mu \rangle = -\frac{L}{8\pi G}\hat{P} = -\frac{L}{16\pi G}\hat{R}\,.$$

Weyl Anomaly in $4d$

Find $X^{(2)}$ from the bulk equation of motion:

$$X^{(2)} = -\frac{L^4}{4}\mathrm{tr}(\hat{P}^2) + \frac{L^4}{4}\hat{P}^2 - \frac{L^2}{2}\hat{\nabla}\cdot a^{(2)}$$

The Weyl anomaly in $4d$ reads

$$\begin{aligned}\mathcal{A}_2 &= -\frac{L^3}{\kappa^2}\int \mathrm{d}^4x\sqrt{-\gamma^{(0)}}\ln\mathcal{B}\left(\frac{1}{2}\mathrm{tr}(\hat{P}^2) - \frac{1}{2}\hat{P}^2 + \frac{1}{L^2}\hat{\nabla}\cdot a^{(2)}\right) \\ &= -\frac{L}{\kappa^2}\int\left[\frac{L^2}{16}\left(\hat{W}^2 - \hat{E}^{(4)}\right) + \hat{\nabla}\cdot a^{(2)}\right]\ln\mathcal{B}\sqrt{-\gamma^{(0)}}\mathrm{d}^4x\,,\end{aligned}$$

where $\hat{W}^2 = \hat{W}_{\mu\nu\rho\sigma}\hat{W}^{\rho\sigma\mu\nu}$, $\hat{E}^{(4)} = \hat{R}_{\mu\nu\rho\sigma}\hat{R}^{\rho\sigma\mu\nu} - 4\hat{R}_{\mu\nu}\hat{R}^{\nu\mu} + \hat{R}^2$.
$a^{(2)}_\mu$ only appears through a cohomologically trivial term.

Weyl Anomaly in $6d$ and $8d$

$$
\begin{aligned}
\mathcal{A}_3 &= \frac{3}{\kappa^2 L}\int \ln \mathcal{B} X^{(3)}_{d=6} vol_\Sigma \\
&= -\frac{L^5}{\kappa^2}\int d^6x\sqrt{-\gamma^{(0)}}\bigg(\frac{1}{4}\mathrm{tr}(\hat{P}^3) - \frac{3}{8}\mathrm{tr}(\hat{P}^2)\hat{P} + \frac{1}{8}\hat{P}^3 - \frac{1}{4}\mathrm{tr}(\hat{\Omega}^{(1)}\hat{P}) \\
&\quad + \frac{1}{L^4}\hat{\nabla}\cdot a^{(4)} + \frac{1}{4L^2}\hat{\nabla}_\mu\big[a^{(2)}_\nu(3\hat{P}^{\mu\nu} + \hat{P}^{\nu\mu} - 3\hat{P}\gamma^{\mu\nu}_{(0)})\big]\bigg)\ln \mathcal{B}\,, \\
\mathcal{A}_4 &= -\frac{L^7}{\kappa^2}\int d^8x\sqrt{-\gamma^{(0)}}\bigg(\frac{1}{8}\mathrm{tr}(\hat{P}^4) - \frac{1}{6}\mathrm{tr}(\hat{P}^3)P - \frac{1}{16}(\mathrm{tr}(\hat{P}^2))^2 \\
&\quad + \frac{1}{8}\mathrm{tr}(\hat{P}^2)\hat{P}^2 - \frac{1}{48}\hat{P}^4 + \frac{1}{6}\mathrm{tr}(\hat{\Omega}^{(1)}\hat{P})\hat{P} - \frac{1}{6}\mathrm{tr}(\hat{\Omega}^{(1)}\hat{P}^2) \\
&\quad + \frac{1}{24}\mathrm{tr}(\hat{\Omega}^{(1)}\hat{\Omega}^{(1)}) + \frac{1}{24}\mathrm{tr}(\hat{\Omega}^{(2)}\hat{P}) + \text{total derivatives}\bigg)\ln \mathcal{B}\,.
\end{aligned}
$$

$\mathcal{A}_k$ can also be written as a linear combination of $\hat{E}^{(2k)}$ and the conformal invariants in $2k$-dim. [Bonora et al. (1985); Deser & Schwimmer (1993) hep-th/9302047; Henningson & Skenderis (1998) hep-th/9806087; Boulanger & Erdmenger (2004) hep-th/0405228]

Weyl Anomaly in higher dimensions

Denote the polynomial terms of $X^{(k)}/L^{2k}$ (without total derivatives) by $\bar{X}^{(k)}$

$$
\begin{aligned}
\bar{X}^{(1)} &= -\delta^{\mu}_{\nu}\hat{P}^{\nu}{}_{\mu}\,, \\
2\bar{X}^{(2)} &= \frac{1}{2}\delta^{\mu_1\mu_2}_{\nu_1\nu_2}\hat{P}^{\nu_1}{}_{\mu_1}\hat{P}^{\nu_2}{}_{\mu_2}\,, \\
6\bar{X}^{(3)} &= -\frac{1}{4}\delta^{\mu_1\mu_2\mu_3}_{\nu_1\nu_2\nu_3}\hat{P}^{\nu_1}{}_{\mu_1}\hat{P}^{\nu_2}{}_{\mu_2}\hat{P}^{\nu_3}{}_{\mu_3} - \frac{1}{2}\delta^{\mu_1\mu_2}_{\nu_1\nu_2}\hat{\Omega}^{\nu_1}_{(1)\mu_1}\hat{P}^{\nu_2}{}_{\mu_2}\,, \\
24\bar{X}^{(4)} &= \frac{1}{8}\delta^{\mu_1\mu_2\mu_3\mu_4}_{\nu_1\nu_2\nu_3\nu_4}\hat{P}^{\nu_1}{}_{\mu_1}\hat{P}^{\nu_2}{}_{\mu_2}\hat{P}^{\nu_3}{}_{\mu_3}\hat{P}^{\nu_4}{}_{\mu_4} + \frac{1}{2}\delta^{\mu_1\mu_2\mu_3}_{\nu_1\nu_2\nu_3}\hat{\Omega}^{\nu_1}_{(1)\mu_1}\hat{P}^{\nu_2}{}_{\mu_2}\hat{P}^{\nu_3}{}_{\mu_3} \\
&\quad + \frac{1}{4}\delta^{\mu_1\mu_2}_{\nu_1\nu_2}\hat{\Omega}^{\nu_1}_{(1)\mu_1}\hat{\Omega}^{\nu_2}_{(1)\mu_2} + \frac{1}{4}\delta^{\mu_1\mu_2}_{\nu_1\nu_2}\hat{\Omega}^{\nu_1}_{(2)\mu_1}\hat{P}^{\nu_2}{}_{\mu_2}\,.
\end{aligned}
$$

where $\delta^{\mu_1\cdots\mu_s}_{\nu_1\cdots\nu_s} = s!\delta^{\mu_1}{}_{[\nu_1}\cdots\delta^{\mu_s}{}_{\nu_s]}$.

Weyl Anomaly in higher dimensions

- a_μ only appears as total derivatives in $X^{(k)}$, and hence represents cohomologically trivial contributions in Weyl anomaly.
- Weyl anomaly still has type A and type B in WFG gauge, both are Weyl covariant due to $a_\mu^{(0)}$.
- Weyl-Schouten tensor $\hat{P}_{\mu\nu}$ and extended Weyl-obstruction $\hat{\Omega}_{\mu\nu}^{(j)}$ tensors are natural building blocks.
- $X^{(k)}$ contains all kinds of possible combinations of $\hat{P}_{\mu\nu}$ and $\hat{\Omega}_{\mu\nu}^{(j<k-1)}$ whose Weyl weights add up to be $2k$.

Summary

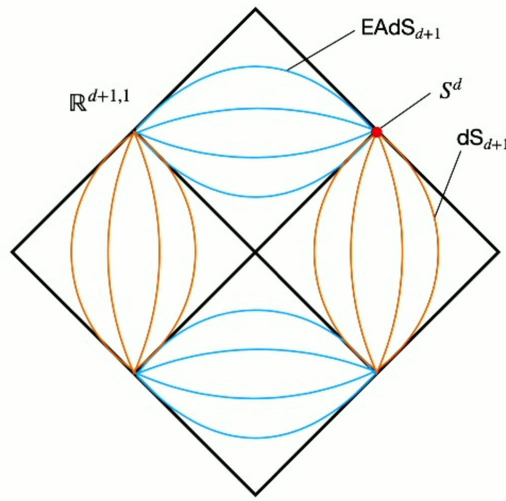
	Riemannian manifold	Weyl manifold
Structures	$(\tilde{M}, \tilde{g})$	$(M, [g, a])$
Dimensions	$d+2$	d
Symmetries	$\text{Diff}(\tilde{M})$	$\text{Diff}(M) \ltimes \text{Weyl}$
Covariant quantities	$R, \nabla R,$ $\nabla\nabla R, \cdots$	$\hat{R}, \hat{\nabla}\hat{R}$ $\hat{\nabla}\hat{\nabla}\hat{R}, \cdots$

Summary

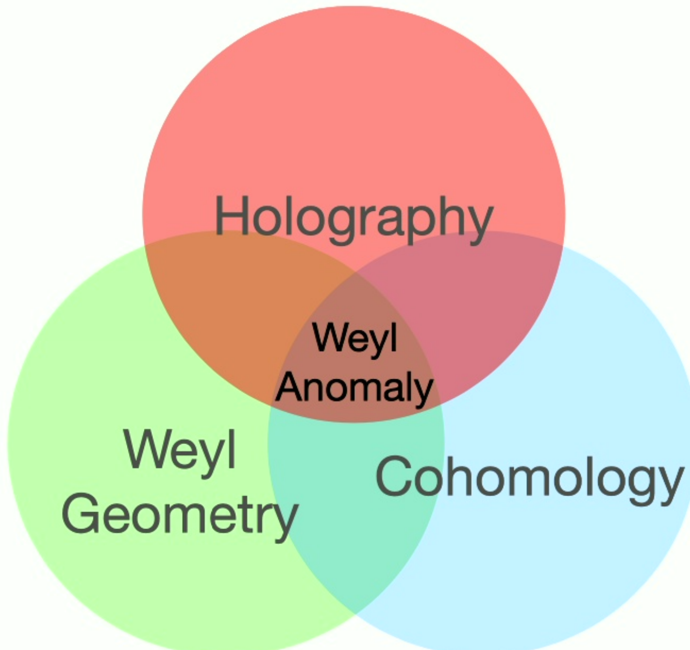
- Weyl ambient space induces a Weyl geometry at codimension-2 and Weyl-covariantizes the boundary geometric quantities.
- Weyl-obstruction tensor $\mathcal{O}_{\mu\nu}$ is the pole of the AlAdS bulk metric at $d = 2k$ in the WFG gauge.
- Weyl-Schouten tensor and Weyl-obstruction tensors can be used as building blocks of the holographic Weyl anomaly.
- Weyl connection $a_\mu^{(0)}$ covariantizes the Weyl anomaly, subleading $a_\mu^{(k>0)}$ only have cohomologically trivial contributions.

Outlook

- Weyl-Schouten tensor and Weyl-obstruction tensor in the Weyl anomaly of general (non-holographic) theories.
- Geometric interpretation for the Weyl anomaly from cohomology.
  [WJ, Klinger & Leigh (2023) 2303.05540]
- Realizing corner symmetries
  [Ciambelli & Leigh (2021, 2023) 2104.07643, 2207.06441]
  $\mathrm{Diff}(M) \ltimes \mathrm{Weyl} \subset$ Weyl-BMS.
  [Freidel, Oliveri, Pranzetti & Speziale (2021) 2104.05793]



- Application of the Weyl-ambient construction in celestial holography.
  [Strominger (2014) 1312.2229; Pasterski, Pate & Raclariu (2021) 2111.11392 ......]
- Charges in the WFG gauge [Ciambelli, Delfante, Ruzziconi & Zwikel (2023) 2308.15480].
- Generalize CFT to WFT (Weyl field theory) defined on a non-conformally flat background ⇒ AlAdS/WFT correspondence.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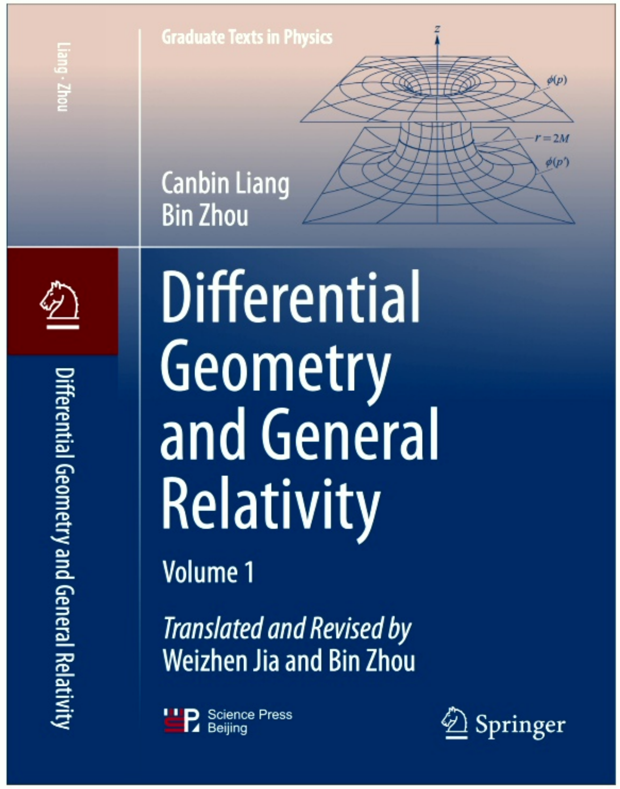
Weyl
Anomaly
Weyl
Geometry
Cohomology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