Title: One Hundred Years After Heisenberg: Discovering the World of Simultaneous Measurements of Noncommuting Observables

Speakers: Carlton Caves

Series: Quantum Foundations

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Abstract: One hundred years after Heisenberg's Uncertainty Principle, the question of how to make simultaneous measurements of noncommuting observables lingers. I will survey one hundred years of measurement theory, which brings us to the point where we can formulate how to measure any set observables weakly and simultaneously and then concatenate such measurements continuously to determine what is a strong measurement of the same observables. The description of the measurements is independent of quantum states---this we call instrument autonomy---and even independent of Hilbert space---this we call the universal Instrument Manifold Program. But what space, if not Hilbert space? It's a whole new world: the Kraus operators of an instrument live in a Lie-group manifold generated by the measured observables themselves. I will describe measuring position and momentum and measuring the three components of angular momentum, special cases where the instrument approaches asymptotically a phase-space boundary of the instrumental Lie-group manifold populated by coherent states; these special universal instruments structure any Hilbert space in which they are represented. In contrast, for almost all sets of observables other than these special cases, the universal instrument descends into chaos ... literally. This work was done with Christopher S. Jackson, whose genius and vision inform every aspect.

Zoom link https://pitp.zoom.us/j/94135518267?pwd=T2JOL21VaEcrY05KeG1SYTVYdHhxdz09

One Hundred Years After Heisenberg: Discovering the World of Simultaneous Measurements of Noncommuting Observables

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"Well, why not say that all the things which should be handled in theory are just those things which we also can hope to observe somehow." … I remember that when I first saw Einstein I had a talk with him about this. … [H]e said, ``That may be so, but still it's the wrong principle in philosophy." And he explained that it is the theory finally which decides what can be observed and what can not and, therefore, one cannot before the theory, know what is observable and what not.

Werner Heisenberg, recalling a conversation with Einstein in 1926, interviewed by Thomas S. Kuhn, February 15, 1963

This work was carried out with Christopher S. Jackson, whose genius and vision inform every aspect.


A brief glimpse into
a whole new world
Moo Stack and the Villians of Ure
Eshaness, Shetland

A brief glimpse into a whole new world

Any set of Hermitian observables,

$$\vec{X} = \{X_1, \ldots, X_n\},$$

can be measured (differential) weakly and simultaneously in an infinitesimal increment dt, without regard to commutators.

Examples

Single observable X

Position Q and momentum P

Three components of angular momentum, J_x, J_y, and J_z

Two components of angular momentum, J_x and J_y

Two squeezing symplectic generators, $K_x = \frac{1}{4}(Q^2 + P^2)$ and $K_y = -\frac{1}{4}(QP + PQ)$

Concatenating these differential weak measurements continuously should tell one what it means to measure the same observables strongly and simultaneously. And so it does, except that it also leads to . . .

A magic carpet ride
Into a whole new world.
A new, fantastic point of view,
A thrilling chase,
A wondrous space,
And now we bring this whole new world to you.

Big-time apologies to *Aladdin*

A brief glimpse into a whole new world

A magic carpet ride
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A new, fantastic point of view,
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Big-time apologies to *Aladdin*

The details in three papers
C. S. Jackson and C. M. Caves

"Simultaneous measurements of noncommuting observables: Positive transformations and instrumental Lie groups," Entropy 25, 1254 (2023). (a.k.a. 1-2-3), https://doi.org/10.3390/e25091254

"Simultaneous momentum and position measurement and the instrumental Weyl-Heisenberg group," Entropy 25, 1221 (2023). (a.k.a. SPQM), https://doi.org/10.3390/e25081221

"How to perform the coherent measurement of a curved phase space by continuous isotropic measurement. I. Spin and the Kraus-operator geometry of SL(2,C)," Quantum 7, 1085 (2023), arXiv:2107.12396v3. (a.k.a ISM), https://doi.org/10.22331/q-2023-08-16-1085

plus

C. S. Jackson, "The photodetector, the heterodyne instrument, and the principle of instrument autonomy," arXiv:2210.11100.

A brief glimpse into a whole new world

- Any set of observables can be measured simultaneously if measured differential weakly. Commutators can be disregarded for *differential weak measurements*.
- Differential weak measurements define a fundamental incremental Kraus operator, a *differential positive transformation,* which is the positive-operator analogue of infinitesimal unitary transformations and equally fundamental.
- Instrument (Kraus-operator) evolution is *autonomous, temporal,* and *stochastic*.
- *Instrument Manifold Program.* The instrument evolves on the manifold of an *instrumental Lie group*, which is generated by the measured observables.
- Motion of the Kraus operators on the instrumental Lie group is described using the three faces of the *stochastic trinity:* Wiener path integrals, stochastic differential equations, and a diffusion equation for a *Kraus-operator distribution function*.
- *Universal instruments.* The instrumental Lie group is generated universally, detached from and independent of Hilbert space.
- Principal and chaotic instruments. *Principal instruments* (e.g., position and momentum, three components of spin) have a low-dimensional universal instrumental group: they limit to coherent-state POVMs—this is *collapse within irrep*—and thus define a phase space, which is connected to the identity across a symmetric space. *Chaotic instruments* (e.g., two components of spin, two squeezing symplectic generators) have an infinite-dimensional universal instrumental group: these are generic, evolve chaotically, and have no limiting strong measurement.

Pinnacles National Park
Central California

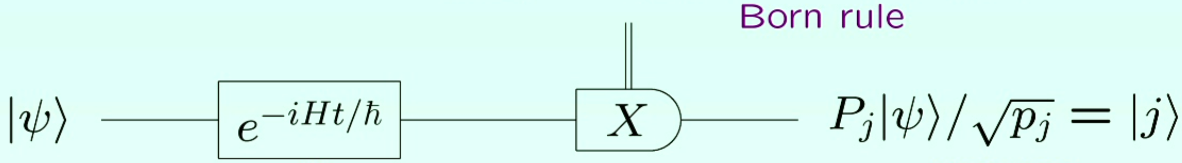
100 years of quantum measurements.
The founding (1925–32)

Matrix mechanics, commutators, and uncertainty principle
Wave mechanics (the Schrödinger equation)
Linear algebra of square-integrable functions
Dirac-Jordan transformation theory
Born probability rule

von Neumann's synthesis: inner products and Hilbert space, unitary transformations (Hamiltonian dynamics), and measurements of Hermitian observables.

$$j,\ p_j = |\langle j|\psi\rangle|^2 = \langle\psi|P_j|\psi\rangle$$

Born rule



unitary Hamiltonian dynamics

Collapse
von Neumann measurement of Hermitian observable

$$X = \sum_j \lambda_j |j\rangle\langle j| = \sum_j \lambda_j P_j$$

Temporal: (continuous) process.

Autonomous: independent of state.

Temporal	Autonomous	Transformation	Group
No	Yes	Yes	No

Drop $|\psi\rangle$ from description.

Physics by fiat: no simultaneous measurements of noncommuting observables

100 years of quantum measurements.
The desert (1932–60)

Quantum measurement theory withered under the desert sun, whereas the unitary side of quantum mechanics thrived with constant and well-deserved nurturing.

Everybody used the Born rule, though how to interpret its probabilities remains a source of discussion and debate today. Nobody used and next to nobody bought von Neumann's collapse, because there were no repeated measurements on the same system.

All measurements were actually von Neumann's indirect measurements and analyzed using the Born rule without using von Neumann measurements of Hermitian observables.

Mathematical developments

- Unitary Lie groups: symmetry groups and representation theory
- Functional and harmonic analysis
- Functional (path) integration
- Transformation groups
- Differential geometry of complex Lie groups
- Measures and probability theory, stochastic processes, and stochastic calculus

First three and a bit of the fourth fell on fertile soil in the unitary sector of quantum mechanics and quantum field theory.

None of this got into (or was needed in) the desiccated quantum measurement theory.

We use all these developments.

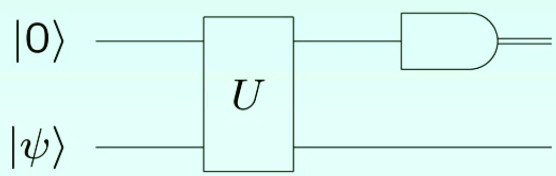
100 years of quantum measurements.
Generalized measurement theory (1960-85)

Overcomplete-basis measurements (measurements of noncommuting observables, coherent states, heterodyne)

Hint of repeated measurements

Generalized measurement theory. Taking advantage of von Neumann's indirect measurements

Wigner
Davies
Ludwig
Kraus



Born rule

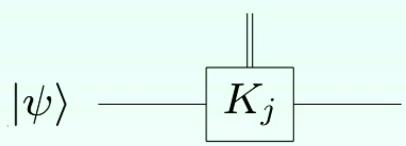
$$j,\ p_j = \mathrm{tr}_S(K_j|\psi\rangle\langle\psi|K_j^\dagger) = \mathrm{tr}_S(E_j|\psi\rangle\langle\psi|)$$

$$\langle j|U|0\rangle|\psi\rangle/\sqrt{p_j} = K_j|\psi\rangle/\sqrt{p_j}$$

"Gentler" collapse

$$j,\ p_j = \langle\psi|E_j|\psi\rangle$$

$\Longleftrightarrow$



$$K_j|\psi\rangle/\sqrt{p_j}$$

100 years of quantum measurements. Generalized measurement theory (1960-85)

Overcomplete-basis measurements (measurements of noncommuting observables, coherent states, heterodyne)

Hint of repeated measurements

Generalized measurement theory. Taking advantage of von Neumann's indirect measurements

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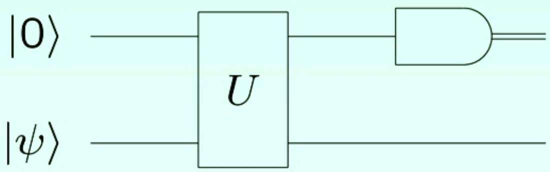
Kraus

Born rule



$$j,\ p_j = \mathrm{tr}_S(K_j|\psi\rangle\langle\psi|K_j^\dagger) = \mathrm{tr}_S(E_j|\psi\rangle\langle\psi|)$$

$$\langle j|U|0\rangle|\psi\rangle/\sqrt{p_j} = K_j|\psi\rangle/\sqrt{p_j}$$

"Gentler" collapse

$$(\text{Kraus operator}) = K_j = \langle j|U|0\rangle$$

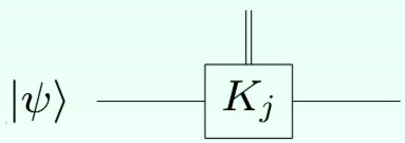
$$(\text{POVM element}) = E_j = K_j^\dagger K_j \geq 0$$

$$(\text{Completeness}):\quad \sum_j E_j = 1$$

Positive

$$j,\ p_j = \langle\psi|E_j|\psi\rangle$$

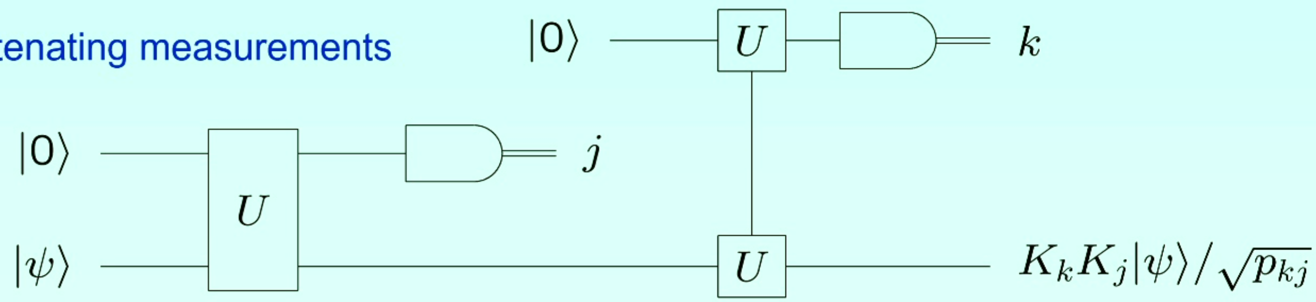
$\Longleftrightarrow$



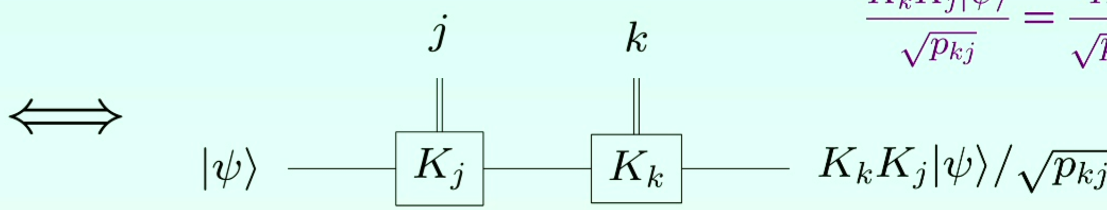
$$K_j|\psi\rangle/\sqrt{p_j}$$

100 years of quantum measurements.
Generalized measurement theory (1960-85)

Concatenating measurements

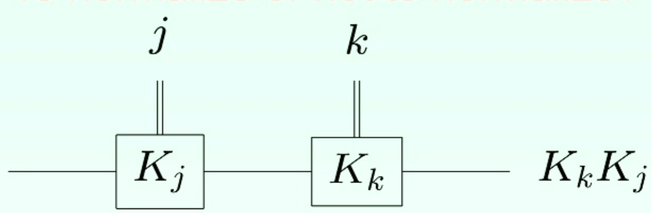


$$\frac{K_k K_j |\psi\rangle}{\sqrt{p_{kj}}} = \frac{K_k}{\sqrt{p_{k|j}}} \frac{K_j}{\sqrt{p_j}} |\psi\rangle$$



Temporal	Autonomous	Transformation	Group
A bit	?	Yes	No

To normalize or not to normalize?



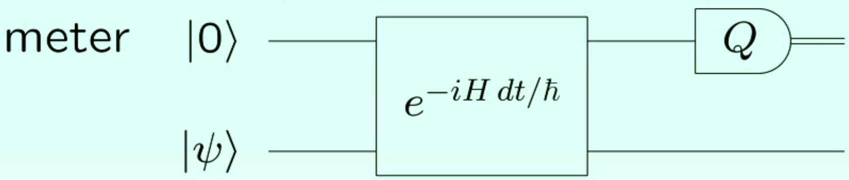
100 years of quantum measurements. Continuous weak measurements (1980–2010)

We are interested in differential weak measurements: Kraus operators close to the identity.

Davies
Barchielli
Carmichael
Milburn
Wiseman
Goetsch/Graham
Doherty
Mabuchi
Jacobs
Brun
Steck

Differential weak measurement of *X* in increment *dt*

$$\langle q|0\rangle = \sqrt{\frac{1}{\sqrt{2\pi\sigma^2}} e^{-q^2/2\sigma^2}}$$



$q,\ dp(q)$

$\sqrt{dq}\,\langle q|e^{-iH\,dt/\hbar}|0\rangle\,|\psi\rangle/\sqrt{dp(q)}$

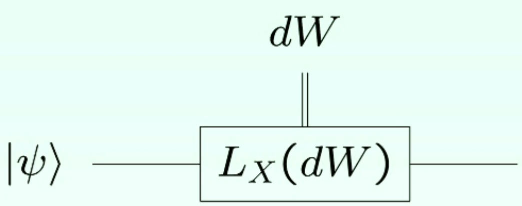
$H\,dt = 2\sqrt{\kappa\,dt}\,\sigma P\otimes X$
Controlled displacement of meter position Q by X

$dW = q\sqrt{dt}/\sigma$ is a *Wiener outcome increment.*
$d\mu(dW)$ = zero-mean Gaussian with $\langle dW^2\rangle = dt$

Kraus operator

$$\sqrt{dq}\,\langle q|e^{-iH\,dt/\hbar}|0\rangle = \sqrt{d\mu(dW)}\,\underbrace{e^{X\sqrt{\kappa}\,dW - X^2\kappa\,dt}}_{=L_X(dW)}$$

$$dp(q) = d\mu(dW)\langle\psi|L_X(dW)^\dagger L_X(dW)|\psi\rangle$$

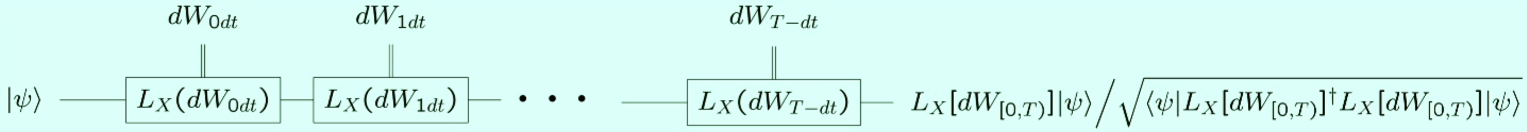


$$L_X(dW)|\psi\rangle/\sqrt{\langle\psi|L_X(dW)^\dagger L_X(dW)|\psi\rangle}$$

100 years of quantum measurements.
Continuous weak measurements (1980–2010)

Concatenating

Continuous, differential weak measurement of *X* over finite time *T.*



$$L_X(dW_t) = e^{\delta_t} = e^{X\sqrt{\kappa}\,dW_t - X^2\kappa\,dt}, \qquad \delta_t \equiv X\sqrt{\kappa}\,dW_t - X^2\kappa\,dt = \text{(forward generator)}$$

$$L\big[dW_{[0,T)}\big] = \mathcal{T}\prod_{k=0}^{T/dt-1} L_X(dW_{kdt}) = \mathcal{T}\exp\left(\int_0^{T-dt} X\sqrt{\kappa}\,dW_t - X^2\kappa\,dt\right)$$

Temporal	Autonomous	Transformation	Group
Yes	?	Yes	?

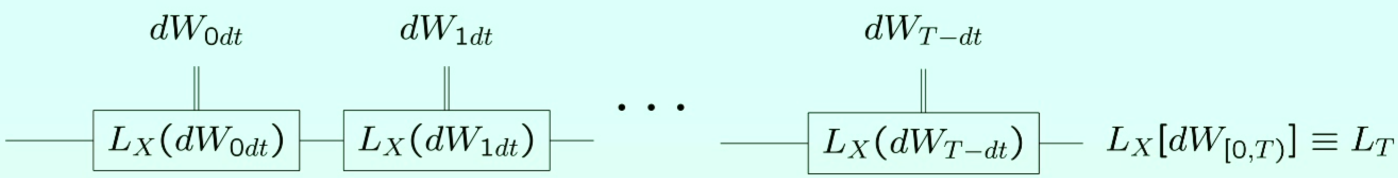
Normalizing at each increment gives a stochastic master equation for an evolving state. Not normalizing (sometimes called a linear quantum trajectory) gives autonomous instrument evolution.

100 years of quantum measurements. Continuous weak measurements (1980–2010)

Concatenating

Continuous, differential weak measurement of *X* over finite time *T*.

Instrument evolution



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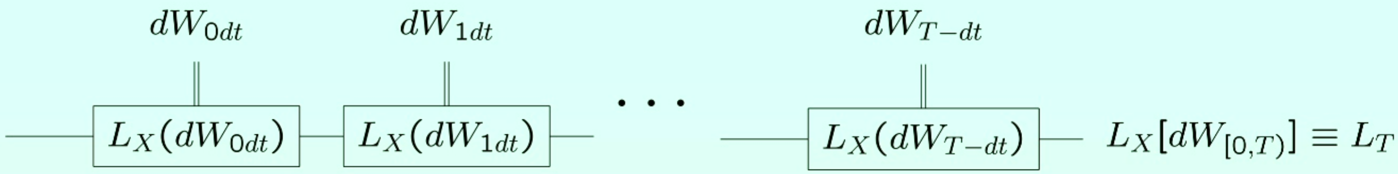
Temporal	Autonomous	Transformation	Group
Yes	Yes	Yes	Yes

100 years of quantum measurements.
Continuous weak measurements (1980–2010)

Concatenating

Continuous, differential weak measurement of X over finite time T.

Instrument evolution



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Temporal	Autonomous	Transformation	Group
Yes	Yes	Yes	Yes

Continuous measurements of a single observable are trivial because everything commutes (time ordering is irrelevant; irreps are 1D). They limit to a strong measurement that is a von Neumann measurement (standard collapse between irreps).

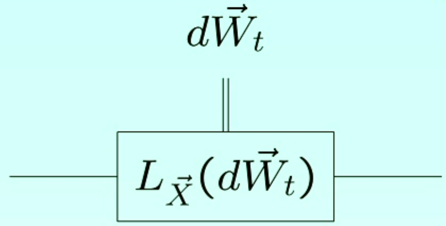
Variegated fairy wren
Oxley Common, Brisbane

Red-backed fairy wren
Oxley Common, Brisbane

Instrument manifold program

Western diamondback rattlesnake
My front yard, Sandia Heights

Instrument manifold program



The incremental Kraus operator for a differential weak measurement of the observables $\vec{X} = \{X_1, \ldots, X_n\}$, beginning at time t for an increment dt, is

$$\sqrt{d\mu(d\vec{W}_t)}\, L_{\vec{X}}(d\vec{W}_t) = \sqrt{d(dW_t^1)\cdots d(dW_t^n)\frac{e^{-d\vec{W}_t\cdot d\vec{W}_t/2dt}}{(2\pi dt)^{n/2}}}\; e^{\vec{X}\cdot\sqrt{\kappa}\,d\vec{W}_t - \vec{X}^2\kappa\,dt},$$

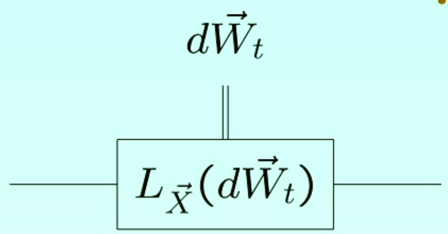
where

$$L_{\vec{X}}(d\vec{W}_t) = e^{\delta_t}, \qquad \delta_t \equiv \vec{X}\cdot\sqrt{\kappa}\,d\vec{W}_t - \vec{X}^2\kappa\,dt = \text{(forward generator)}$$

and

$$\vec{X}\cdot d\vec{W}_t = \sum_\mu X_\mu dW_t^\mu, \quad dW_t^\mu = \text{(Wiener outcome increment for } X_\mu\text{)}.$$

Instrument manifold program



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where

Commutators can be disregarded for *differential weak measurements*.

$$L_{\vec{X}}(d\vec{W}_t) = e^{\delta_t}\,, \qquad \delta_t \equiv \vec{X}\cdot\sqrt{\kappa}\,d\vec{W}_t - \vec{X}^2\kappa\,dt = \text{(forward generator)}$$

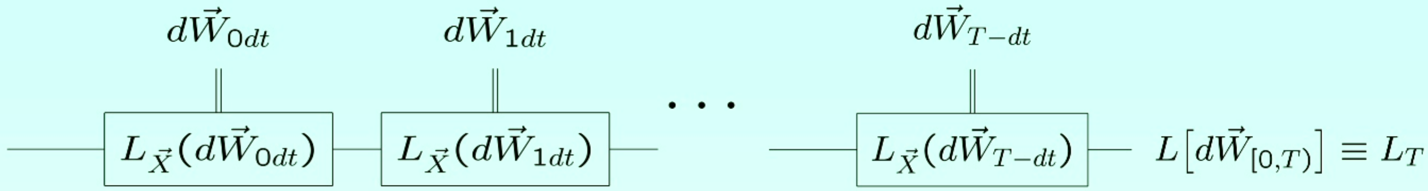
and

Differential weak measurements define a fundamental incremental Kraus operator, a *differential positive transformation*, which is fundamental in the same way as infinitesimal unitary transformations.

$$\vec{X}\cdot d\vec{W}_t = \sum_\mu X_\mu dW_t^\mu\,, \qquad dW_t^\mu = \text{(Wiener outcome increment for } X_\mu\text{)}.$$

Instrument manifold program

Instrument evolution: "piling up" Kraus operators



$$L_{\vec{X}}(d\vec{W}_t) = e^{\delta_t} = e^{\vec{X}\cdot\sqrt{\kappa}\,d\vec{W}_t - \vec{X}^2\kappa\,dt}$$

The differential positive transformations "pile up" as successive incremental measurements are performed; at time T the Kraus operator is

$$L_T = L\big[d\vec{W}_{[0,T)}\big] = \mathcal{T}\exp\left(\int_0^{T-dt}\delta_t\right) = \mathcal{T}\exp\left(\int_0^{T-dt}\vec{X}\cdot\sqrt{\kappa}\,d\vec{W}_t - \vec{X}^2\kappa\,dt\right),$$

where $d\vec{W}_{[0,T)}$ is the Wiener outcome path and $\mathcal{T}$ denotes a time-ordered exponential—commutators do count for finite T! The measuring *instrument* is the collection of these Kraus operators for all Wiener paths (more precisely, the collection of *instrument elements* $L_T \odot L_T^\dagger$).

Instrument evolution is an *autonomous, temporal, stochastic* process.

Instrument manifold program

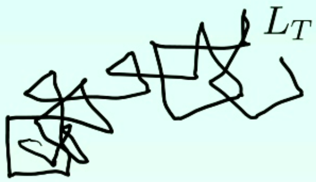
Instrument evolution

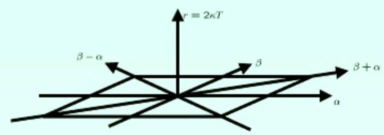
$$d\vec{W}_{0dt} \qquad d\vec{W}_{1dt} \qquad \cdots \qquad d\vec{W}_{T-dt}$$

$$L_{\vec{X}}(d\vec{W}_{0dt}) \; - \; L_{\vec{X}}(d\vec{W}_{1dt}) \; - \cdots - \; L_{\vec{X}}(d\vec{W}_{T-dt}) \; - \; L\big[d\vec{W}_{[0,T)}\big] \equiv L_T$$

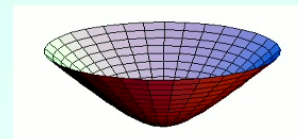
$$L_{\vec{X}}(d\vec{W}_t) = e^{\delta_t} = e^{\vec{X}\cdot\sqrt{\kappa}\,d\vec{W}_t - \vec{X}^2\kappa\,dt}$$

The quantum circuit becomes a stochastic path
on the instrumental Lie group manifold.

1  $L_T = L\big[d\vec{W}_{[0,T)}\big]$



SPQM



ISM

The Kraus operators $L_T = L\big[d\vec{W}_{[0,T)}\big]$ are elements of an *instrumental Lie group* G, which is the group generated by the measured observables, $\vec{X} = \{X_1, \ldots, X_n\}$, and the *quadratic term*, $\vec{X}^2 = \vec{X}\cdot\vec{X} = \sum_\mu X_\mu^2$. The instrument evolves stochastically in the manifold of the instrumental Lie group, which is the natural setting for the measurement.

Instrument manifold program

The motion of the Kraus operators on the group manifold G is analyzed using the three faces of the *stochastic trinity*. The overall quantum operation is given by a Wiener-like path integral of the measurement record,

$L_T = L[d\vec{W}_{[0,T)}]$

$$\mathcal{Z}_T = \int \mathcal{D}\mu[d\vec{W}_{[0,T)}]\ L[d\vec{W}_{[0,T)}] \odot L[d\vec{W}_{[0,T)}]^\dagger\,;$$

1

it is the solution of a Lindblad equation in which the measured observables are the Lindblad operators. The Kraus-operator paths satisfy a *stochastic differential equation* (SDE),

Maurer-Cartan form (Stratonovich): $\quad dL_t\,L^{-1}_{t+dt/2} = \delta_t = \vec{X}\cdot\sqrt{\kappa}\,d\vec{W} - \vec{X}^2\kappa\,dt\,,$

Modified MC stochastic differential (Itô): $\quad dL_t\,L_t^{-1} - \frac{1}{2}(dL_t\,L_t^{-1})^2 = \delta_t\,.$

A *Kraus-operator distribution function* (KOD) is defined by a Wiener path integral,

$$D_T(L) \equiv \int \mathcal{D}\mu[d\vec{W}_{[0,T)}]\ \delta\big(L, L[d\vec{W}_{[0,T)}]\big)\,,$$

where the δ-function is defined relative to the Haar measure of G. The KOD evolves according to a *Fokker-Planck-Kolmogorov equation* (FPKE),

$$\frac{1}{\kappa}\frac{\partial D_t(L)}{\partial t} = \Delta[D_t](L)\,, \qquad \Delta \equiv \underleftarrow{\vec{X}}^2 + \frac{1}{2}\sum_\mu \underleftarrow{X_\mu}\underleftarrow{X_\mu}\,,$$

where the underarrows denote *right-invariant derivatives*,

$$\underleftarrow{X}[f](L) \equiv \frac{d}{dh} f(e^{hX}L)\bigg|_{h=0} = \lim_{h\to 0}\frac{f(e^{hX}L) - f(L)}{h}\,,$$

these being the natural vector fields on the instrumental Lie-group manifold. Notice that $\underleftarrow{\vec{X}}^2$ describes ballistic motion and

$$\nabla^2 \equiv \sum_\mu \underleftarrow{X_\mu}\underleftarrow{X_\mu}$$

is a Laplacian that describes diffusion.

Instrument manifold program

The motion of the Kraus operators on the group manifold G is analyzed using the three faces of the *stochastic trinity*. The overall quantum operation is given by a Wiener-like path integral of the measurement record,

$L_T = L[d\vec{W}_{[0,T)}]$

$$\mathcal{Z}_T = \int \mathcal{D}\mu[d\vec{W}_{[0,T)}]\; L[d\vec{W}_{[0,T)}] \odot L[d\vec{W}_{[0,T)}]^\dagger\,;$$

1

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Modified MC stochastic differential (Itô): $dL_t\,L_t^{-1} - \frac{1}{2}(dL_t\,L_t^{-1})^2 = \delta_t\,.$

Stochastic differential equation

A *Kraus-operator distribution function* (KOD) is defined by a Wiener path integral,

$$D_T(L) \equiv \int \mathcal{D}\mu[d\vec{W}_{[0,T)}]\;\delta\big(L, L[d\vec{W}_{[0,T)}]\big)\,,$$

Wiener path integral

where the δ-function is defined relative to the Haar measure of G. The KOD evolves according to a *Fokker-Planck-Kolmogorov equation* (FPKE),

$$\frac{1}{\kappa}\frac{\partial D_t(L)}{\partial t} = \Delta[D_t](L)\,, \qquad \Delta \equiv \underset{\leftarrow}{\vec{X}}^2 + \frac{1}{2}\sum_\mu \underset{\leftarrow}{X_\mu}\underset{\leftarrow}{X_\mu}\,,$$

Diffusion equation

where the underarrows denote *right-invariant derivatives*,

$$\underset{\leftarrow}{X}[f](L) \equiv \frac{d}{dh}f(e^{hX}L)\Big|_{h=0} = \lim_{h\to 0}\frac{f(e^{hX}L) - f(L)}{h}\,,$$

these being the natural vector fields on the instrumental Lie-group manifold. Notice that $\underset{\leftarrow}{\vec{X}}^2$ describes ballistic motion and

$$\nabla^2 \equiv \sum_\mu \underset{\leftarrow}{X_\mu}\underset{\leftarrow}{X_\mu}$$

is a Laplacian that describes diffusion.

Stochastic trinity

Instrument manifold program

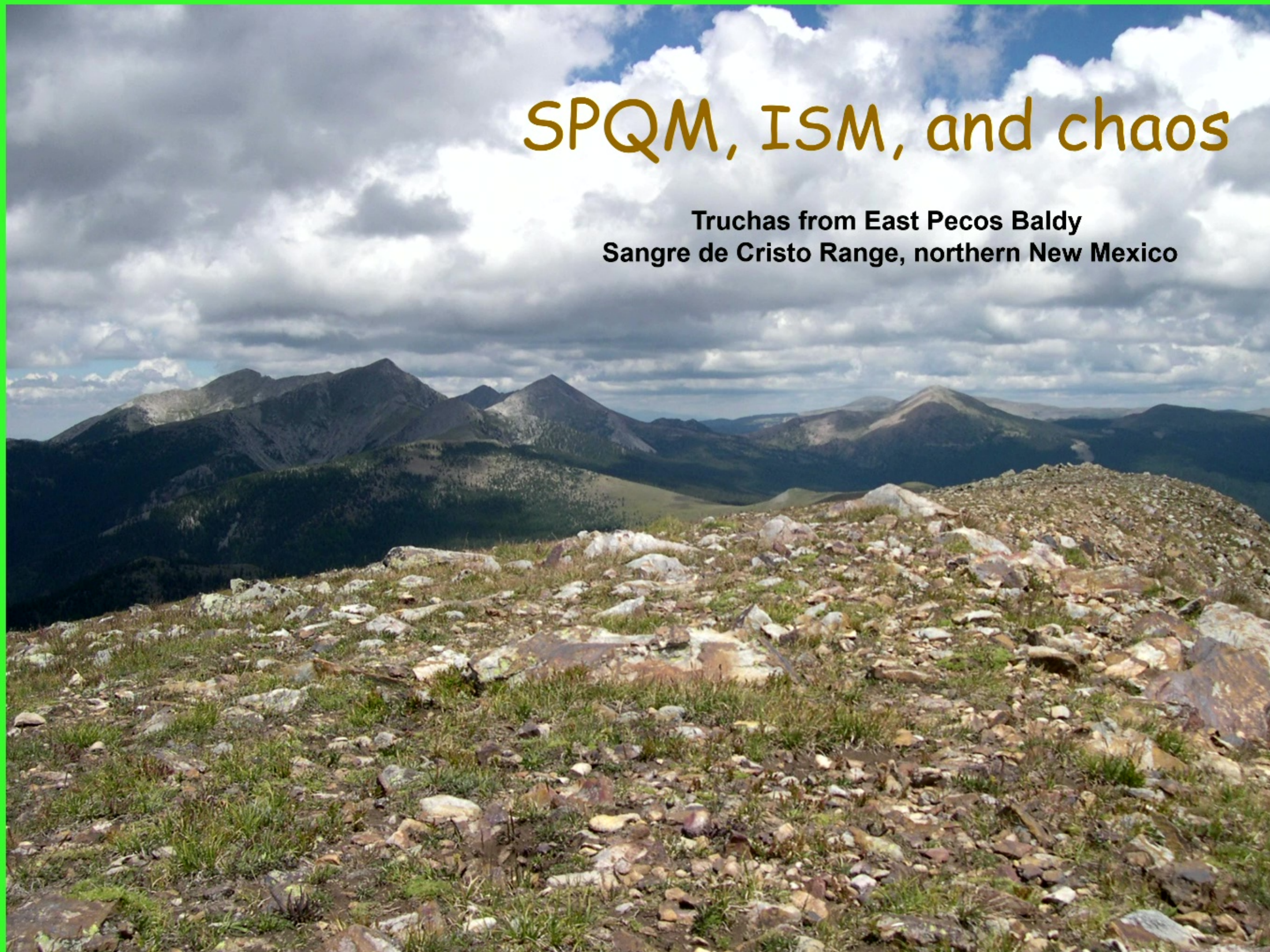
The Lie algebra for the instrumental Lie group can be processed using the matrices of a particular representation or processed *universally*, using only the commutators, within what is called the *universal enveloping algebra*. The *universal instrumental group* is detached from Hilbert space.

Universal instruments detached from Hilbert space.

Special measurements, such as simultaneous momentum and position measurement (SPQM) and isotropic measurement of the three spin components (ISM), have a low-dimensional universal instrumental group; the instrument's POVM approaches a boundary of coherent states at late times, which is the strong measurement of the observables (we call this collapse within irrep). These instruments we call *principal instruments*. They connect classical phase space to the identity and structure any Hilbert space in which they are represented.

Generic measurements—two components of spin, two squeezing symplectic generators—have an infinite-dimensional universal instrumental group; the instrument's evolution is chaotic, and there is no universal strong measurement of the observables. These instruments we call *chaotic instruments*.

SPQM, ISM, and chaos



Truchas from East Pecos Baldy
Sangre de Cristo Range, northern New Mexico

Simultaneous momentum and position measurement (SPQM): A principal instrument

Measure position Q and momentum P.
Commutator: $[Q,P] = i1$
Quadratic term: $Q^2 + P^2 = 2H_\circ$

7D instrumental Lie algebra: $\text{span}\{1, i1, iQ, -iP, Q, P, H_\circ\}$

7D instrumental Lie group, the *Instrumental Weyl-Heisenberg Group*,
$\text{IWH} = \mathbb{C}\text{WH} \rtimes e^{\mathbb{R}H_\circ}$,

Coördinate manifold with Cartan-like decomposition,
group-theoretic singular-value decomposition,

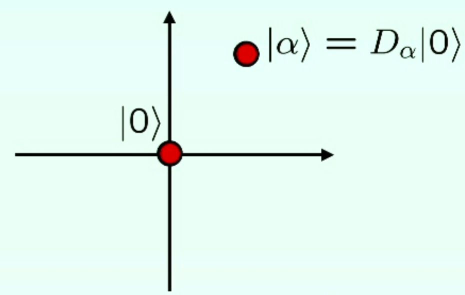
Kraus operator $\quad L = \big(D_\beta e^{i1\phi}\big) e^{-H_\circ r - 1\ell} D_\alpha^\dagger$

D_β and D_α are phase-plane displacement operators.

POVM element $\quad E = L^\dagger L = e^{-2\ell\,1} D_\alpha e^{-H_\circ 2r} D_\alpha^\dagger$

Base manifold of positive transformations

$(\ell,\phi) =$ (center normalization and phase)
$r =$ (ruler/purity), $\quad dr = 2\kappa\, dt\,, \quad r = 2\kappa t$ is ballistic
$\beta =$ (post-measurement phase-plane)
$\alpha =$ (POVM phase-plane)



$r = 0: \quad E = L^\dagger L = e^{-2\ell} 1$
$r \to \infty: \quad E = L^\dagger L = e^{-2\ell} |\alpha\rangle\langle\alpha|$

SPQM: A principal instrument

Kraus operator $\quad L = \big(D_\beta e^{i1\phi}\big)e^{-H_\circ r - 1\ell}D_\alpha^\dagger$

POVM element $\quad E = L^\dagger L = e^{-2\ell\,1}D_\alpha e^{-H_\circ 2r}D_\alpha^\dagger$

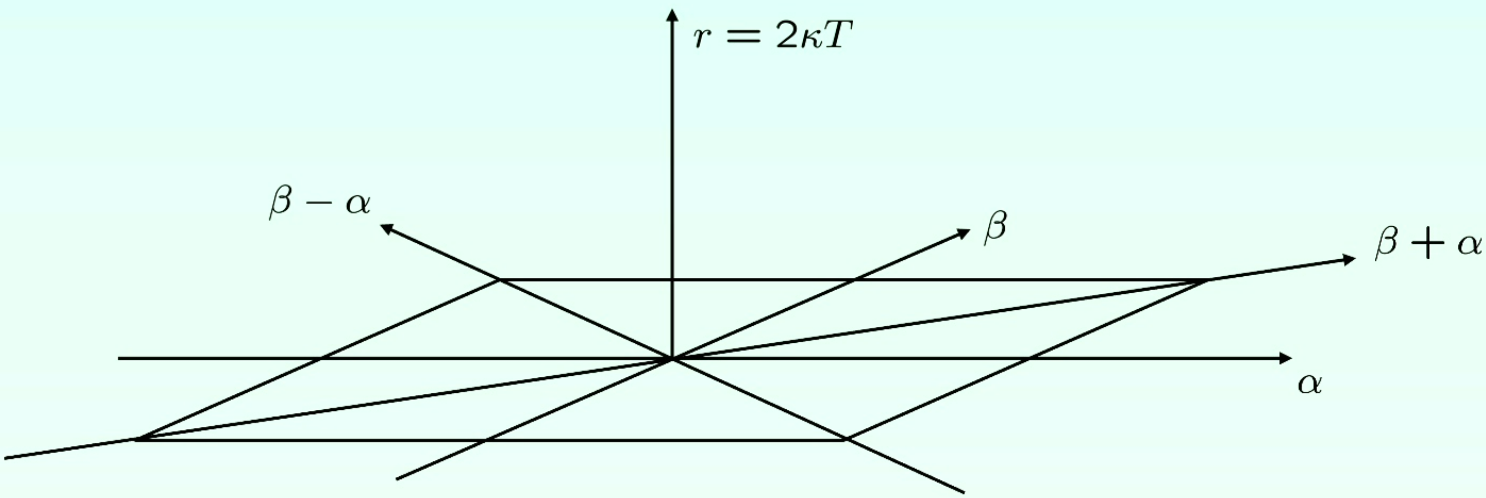
$(\ell,\phi) =$ (center normalization and phase)

$r =$ (ruler/purity), $\quad dr = 2\kappa\,dt\,,\quad r = 2\kappa t$ is ballistic

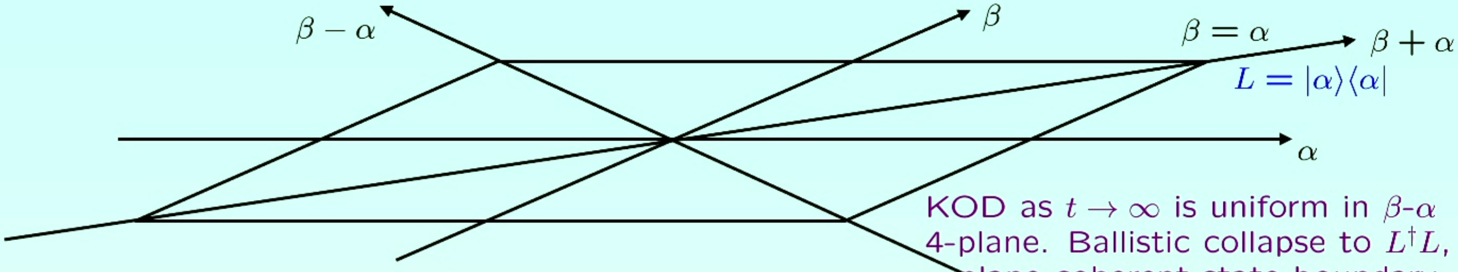
$\beta =$ (post-measurement phase-plane)

$\alpha =$ (POVM phase-plane)

Work in the coset of the center $Z = \big\{e^{1(-\ell+i\phi)}\big\}$;
i.e., identify points with different center coördinates ℓ and ϕ.

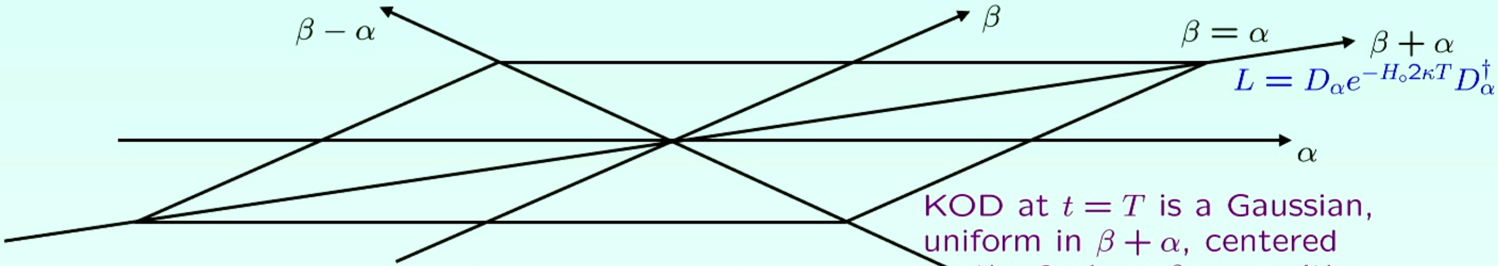


SPQM: A principal instrument



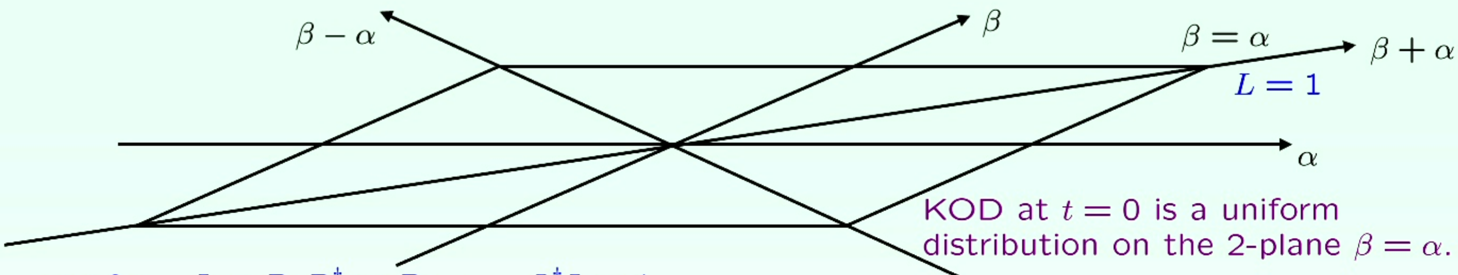
$r \to \infty: \quad L = |\beta\rangle\langle\alpha|\,, \quad L^\dagger L = |\alpha\rangle\langle\alpha|$

KOD as $t \to \infty$ is uniform in β-α 4-plane. Ballistic collapse to $L^\dagger L$, α-plane coherent-state boundary at $r = \infty$.



$r = 2\kappa T: \quad L = D_\beta e^{-H_\circ 2\kappa T} D_\alpha^\dagger\,, \quad L^\dagger L = D_\alpha e^{-H_\circ 4\kappa T} D_\alpha^\dagger$

KOD at $t = T$ is a Gaussian, uniform in $\beta + \alpha$, centered on the 2-plane $\beta = \alpha$, with mean-square distance between β and α given by $\Sigma_T = \kappa T - \tanh \kappa T$.



$r = 0: \quad L = D_\beta D_\alpha^\dagger = D_{\beta-\alpha}\,, \quad L^\dagger L = 1$

KOD at $t = 0$ is a uniform distribution on the 2-plane $\beta = \alpha$.

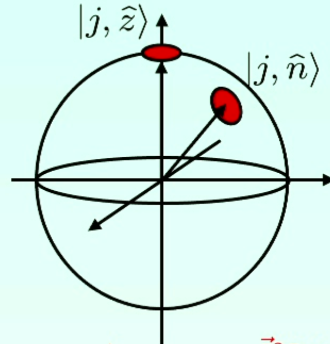
Isotropic spin measurement (ISM): A principal instrument

Measure J_x, J_y, and J_z.
Commutators: $[J_\mu, J_\nu] = i\epsilon_{\mu\nu\eta}J_\eta$
Quadratic term: $J_x^2 + J_y^2 + J_z^2 = \vec{J}^2 =$ Casimir invariant $= 1_j j(j+1)$

7D instrumental Lie algebra: span$\{-iJ_x, -iJ_y, -iJ_z, J_x, J_y, J_z, \vec{J}^2\}$

7D instrumental Lie group, the *Instrumental Spin Group*, ISpin(3) $=$ SL(2, $\mathbb{C}$) $\times\, e^{\mathbb{R}\vec{J}^2}$

Coördinate manifold with Cartan decomposition,
group-theoretic singular-value decomposition,



Kraus operator $\quad L = (D_{\hat{m}} e^{-iJ_z\psi}) e^{-\vec{J}^2\ell + J_z a} D_{\hat{n}}^\dagger$

$D_{\hat{m}}$ and $D_{\hat{n}}$ are spherical displacement operators.

POVM element $\quad E = L^\dagger L = e^{-\vec{J}^2 2\ell} D_{\hat{n}} e^{J_z 2a} D_{\hat{n}}^\dagger$

Base manifold (symmetric space)
of positive transformations.

$a = 0: \; L^\dagger L = e^{-\vec{J}^2 2\kappa t}$

$a \to \infty: \; L^\dagger L \propto e^{-\vec{J}^2 2\kappa t} |j, \hat{n}\rangle\langle j, \hat{n}|$

$\ell =$ (center normalization), $\quad d\ell = \kappa\, dt$,

$\ell = \kappa t$ is ballistic

$\psi =$ (geodesic curvature between past and future)

$a =$ (radial/purity), $\quad da_t = \kappa\, dt\, \coth a_t + \sqrt{\kappa}\, dY_t^z$,

a is ballistic and diffusive

$\hat{m} =$ (post-measurement Bloch sphere)

$\hat{n} =$ (POVM Bloch sphere)

ISM: A principal instrument

Coördinate manifold with Cartan decomposition, group-theoretic singular-value decomposition,

Kraus operator $L = (D_{\hat{m}} e^{-iJ_z\psi}) e^{-\vec{J}^2\ell + J_z a} D_{\hat{n}}^\dagger$

POVM element $E = L^\dagger L = e^{-\vec{J}^2 2\ell} D_{\hat{n}} e^{J_z 2a} D_{\hat{n}}^\dagger$

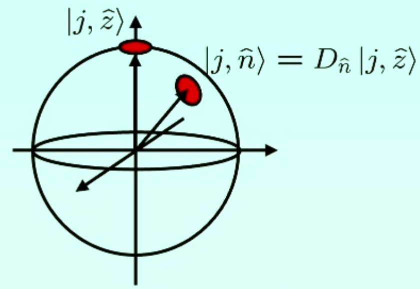
$\ell =$ (center normalization), $d\ell = \kappa\, dt\,$,
$\ell = \kappa t$ is ballistic

$\psi =$ (geodesic curvature between past and future)

$a =$ (radial/purity), $da_t = \kappa\, dt\, \coth a_t + \sqrt{\kappa}\, dY_t^z\,$,
a is ballistic and diffusive

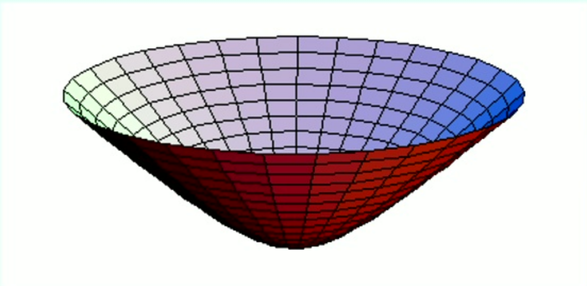
$\hat{m} =$ (post-measurement Bloch sphere)

$\hat{n} =$ (POVM Bloch sphere)



$a = 0$: $L^\dagger L = e^{-\vec{J}^2 2\kappa t}$

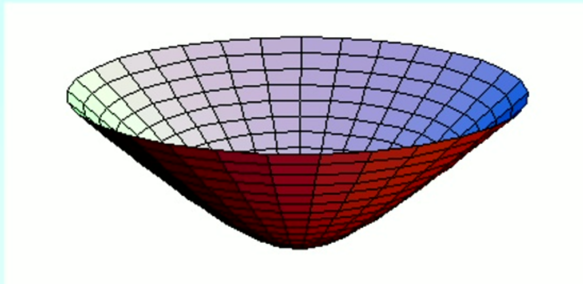
$a \to \infty$: $L^\dagger L \propto e^{-\vec{J}^2 2\kappa t} |j, \hat{n}\rangle\langle j, \hat{n}|$



The base manifold of positive transformations (POVM elements) is a symmetric space, in this case a 3-hyperboloid of constant negative curvature, coördinated by a and $\hat{n}$.

To get the induced geometry right, one must embed the hyperboloid in Minkowski space, not Euclidean space.

ISM: A principal instrument

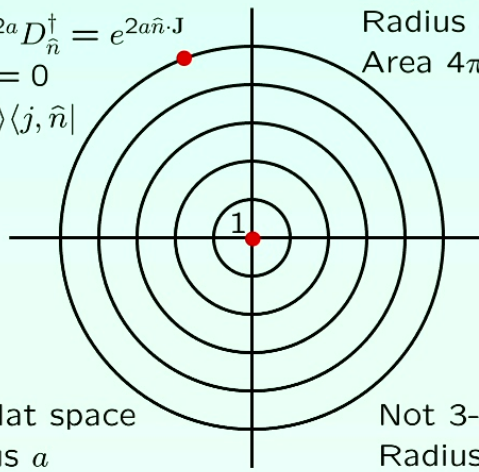


3-hyperboloid
Radius a
Area $4\pi \sinh^2 a$

$E = D_{\hat{n}} e^{J_z 2a} D_{\hat{n}}^\dagger = e^{2a\hat{n}\cdot\mathbf{J}}$

$E = 1$ at $a = 0$

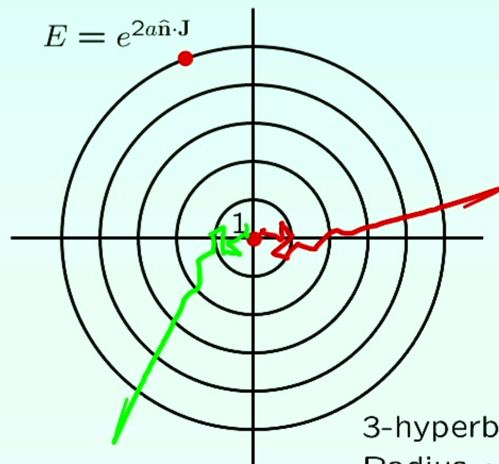
$E \underset{a\to\infty}{\longrightarrow} e^{2aj} |j, \hat{n}\rangle\langle j, \hat{n}|$



Not flat space
Radius a
Area $4\pi a^2$

Not 3-sphere
Radius a
Area $4\pi \sin^2 a$

$E = e^{2a\hat{\mathbf{n}}\cdot\mathbf{J}}$



3-hyperboloid
Radius a
Area $4\pi \sinh^2 a$

Principal vs. chaotic instruments

SPQM and ISM are principal universal instruments, for which the universal instrumental group is finite-dimensional. Principal instruments are very special: they approach a strong measurement of coherent states asymptotically; they thus structure any Hilbert space in which they are represented.

Generic measurements—two components of spin, two squeezing symplectic generators—have an infinite-dimensional universal instrumental group because of the nonlinear quadratic term. They do not have a representation-independent strong measurement, and the evolution of the instrument devolves into

Concluding paragraph of SPQM: Suffice it to say that we think we have found something: the home of quantum dynamics, the Lie-group manifold that supports all three faces of the trinity. The exhausted reader who has survived to read this concluding sentence of a very long paper might be pleased—or so we hope—to learn that our ambition is larger than was evident at the beginning.

Cable Beach
Western Australia