

Title: On unimodularity in the theory of tensor categories

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Abstract: Unimodularity is a classical notion shows up in various fields like linear algebra, lattices, Poisson algebras, etc. In this talk, we focus on unimodular Hopf algebras and unimodular tensor categories. We will introduce unimodular module categories and use them to construct Frobenius algebras and unimodular tensor categories. These ideas will be illustrated with examples drawn from Hopf algebras.

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Zoom link <https://pitp.zoom.us/j/98477599322?pwd=UDJmTkIiMTGxGODJZTm1Xc1VhL2tDdz09>

# ON UNIMODULARITY OF TENSOR CATEGORIES

(arxiv: 2302.06192)

- Linear algebra:  $A \in M_n(\mathbb{Z})$  is called unimodular if  $\det(A) = \pm 1$
- Bilinear forms, lattices
- Poisson algebras: if modular derivation is trivial then call it unimodular.

Poisson algebras: if modular derivation is trivial then call it unimodular.  
• locally compact topological groups: unimodular if they admit left and right invariant Haar  
"integrals"

TODAY'S AGENDA = Hopf algebras and tensor categories



Hopf Algebras

$$(H, m, u, \Delta, \varepsilon, S)$$

algebra      coalgebra

$\Delta, \varepsilon$  are algebra maps

$S: H \rightarrow H$  antipode

- Left integral of  $H$  is  $\lambda^l \in H$  st.  $h\lambda^l = \varepsilon(h)\lambda^l$
- Right " " " "  $\lambda^r$   $\lambda^r h = \varepsilon(h)\lambda^r$

$H$  is unimodular if it admits a two sided integral  $\iff \alpha = \varepsilon$

ex:  $\mathbb{K}G, U(\mathfrak{g}), U_2(\mathfrak{g})$

$$\Delta(g) = g \otimes g$$

$$S(g) = g^{-1}$$

$\alpha: H \rightarrow \mathbb{K}$  measures how far a left integral is from being a right int

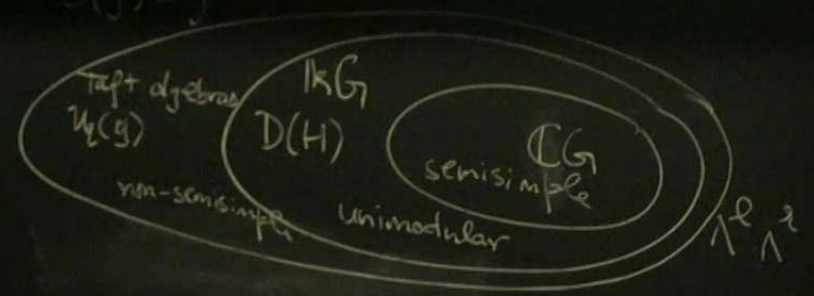
integrals"

$(H, m, \mu, \Delta, \varepsilon, S)$   
 algebra coalgebra

$\Delta, \varepsilon$  are algebra maps  
 $S: H \rightarrow H$  antipode

$\exists \lambda \in H$  st.  $h\lambda = \varepsilon(h)\lambda$   
 $\lambda^2 \neq 0$   
 $\lambda^2 h = \varepsilon(h)\lambda^2$   
 iff it admits a two sided integral  
 $\iff \alpha = \varepsilon$

EX:  $KG, U(\mathfrak{g}), U_q(\mathfrak{g})$   
 $\Delta(g) = g \otimes g$   
 $S(g) = g^{-1}$



$\alpha: H \rightarrow \mathbb{R}$  measures how far a left integral is from being a right int

From Hopf algebras to tensor categories  
 $(H, m, \mu, \Delta, \varepsilon, S)$

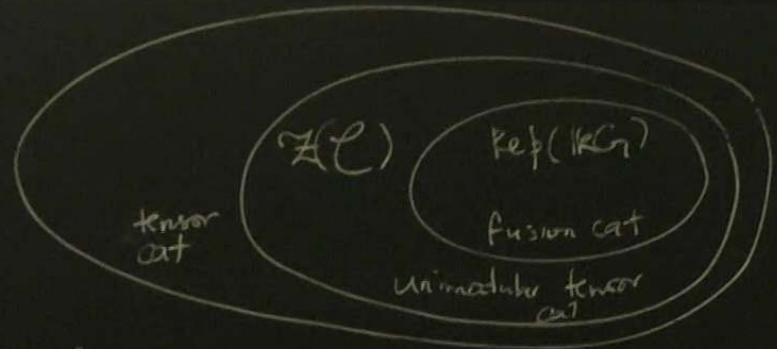
- $(H, m, \mu)$   $k$ -alg
  - $\Delta, \varepsilon$  algebra maps
  - $S: H \rightarrow H$
  - $\alpha: H \rightarrow k$  distinguished character
- $\text{Rep}(H)$   $k$ -linear abelian finite cat
  - $(\text{Rep}(H), \otimes_k, k_\varepsilon)$  monoidal category
  - $\text{Rep}(H)$  rigid monoidal ( $X \rightsquigarrow X^*$ )
  - $D = k_\alpha$   $hc = \alpha(h)c$  distinguished invertible object
  - $D \cong \mathbb{1} \Leftrightarrow k_\alpha = k_\varepsilon$   $\text{Rep}(H)$  unimodular

From Hopf algebras to tensor categories  
 $(H, m, u, \Delta, \varepsilon, S)$

- $(H, m, u)$   $k$ -alg
  - $\Delta, \varepsilon$  algebra maps
  - $S: H \rightarrow H$
  - $\alpha: H \rightarrow k$  distinguished character
  - $\alpha = \varepsilon$
  - $S^4(h) = g_H^{-1}(\alpha \rightarrow h \leftarrow \alpha^{-1}) \leftarrow g_H^{-1}$
- $\text{Rep}(H)$   $k$ -linear abelian finite cat
  - $(\text{Rep}(H), \otimes_k, k_\varepsilon)$  monoidal category
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  - $D = k_\alpha$   $h \circ c = \alpha(h)c$  distinguished invertible object
  - $D \cong \mathbb{1} \Leftrightarrow k_\alpha = k_\varepsilon$   $\text{Rep}(H)$  unimodular
  - $X^{\otimes 4} \cong D \otimes X \otimes D^{-1}$

embed to tensor categories  
( $\mathcal{C}, S$ )

- $\text{Rep}(H)$   $k$ -linear abelian finite cat
- $(\text{Rep}(H), \otimes_k, \mathbb{1}_k)$  monoidal category
- $\text{Rep}(H)$  rigid monoidal ( $X \rightsquigarrow X^*$ )
- $D = \mathbb{1}_k$   $h.c = \alpha(h)c$  distinguished invertible object
- $D \cong \mathbb{1} \Leftrightarrow \mathbb{1}_k = \mathbb{1}_\varepsilon$   $\text{Rep}(H)$  unimodular  $D_c = \int -$
- $X^{***} \cong D \otimes X \otimes D^{-1}$



Tensor cat  
= finite  $k$ -linear abelian rigid monoidal category

H-comodule algebras

Rep(H)-module categories

•  $\mathcal{L}$ -Module categories =  $(\mathcal{M}, \triangleright: \mathcal{L} \times \mathcal{M} \rightarrow \mathcal{M})$

$\mathcal{M}$  = abelian,  $k$ -linear

•  $(X \otimes Y) \triangleright M \cong X \triangleright (Y \triangleright M)$

•  $\mathbb{1} \triangleright M \cong M$

• We call  $\mathcal{M}$  exact,  $\forall X$  projective in  $\mathcal{L}$  and  $M \in \mathcal{M}$   
want  $X \triangleright M$  to be projective in  $\mathcal{M}$ .

Theorem (Etingof-Ostrik)

$\mathcal{L}$  = finite tensor cat,  $\mathcal{M}$  = exact  $\mathcal{L}$ -module then  $\text{End}_{\mathcal{L}}(\mathcal{M})$  is

•  $H$ -comodule algebras

•  $\text{Rep}(H)$ -module categories

•  $\mathcal{E}$ -Module categories =  $(\mathcal{M}, \triangleright: \mathcal{E} \times \mathcal{M} \rightarrow \mathcal{M})$

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Theorem (Etingof-Ostrik)

$\mathcal{E}$  = finite tensor cat,  $\mathcal{M}$  = exact  $\mathcal{E}$ -module then

$\text{End}_{\mathcal{E}}^{\text{rex}}(\mathcal{M})$  is

Q When is  $\text{End}_{\mathcal{E}}^{\text{rex}}(\mathcal{M})$  unimodular?

algebras  $A$

- $\text{Rep}(H)$ -module categories =  $\text{Rep}(A)$
- $\text{End}_{\text{Rep}(H)}(\text{Rep}(A)) = {}^H M_A$

categories =  $(\mathcal{M}, \triangleright: \mathcal{E} \times \mathcal{M} \rightarrow \mathcal{M})$

$k$ -linear

- $(X \boxtimes Y) \triangleright M \cong X \triangleright (Y \triangleright M)$
- $\mathbb{1} \triangleright M \cong M$

$\mathcal{E}$ -module functor:  $F: \mathcal{M} \rightarrow \mathcal{M}$   
 $F(X \triangleright M) \cong X \triangleright F(M)$

exact,  $\forall X$  projective in  $\mathcal{E}$  and  $M \in \mathcal{M}$   
 want  $X \triangleright M$  to be projective in  $\mathcal{M}$ .

Ostrik)  $\mathcal{E}$  = finite tensor cat,  $\mathcal{M}$  = exact  $\mathcal{E}$ -module then  $\text{End}_{\mathcal{E}}^{\text{rex}}(\mathcal{M})$  is a tensor category.

$\text{End}_{\mathcal{E}}^{\text{rex}}(\mathcal{M})$  unimodular?

is exact as a  $\text{Rep}(H)$ -module category  $\mathcal{F}$ :  
 $\mathcal{F} \cong k$  •  $A$  is  $H$ -simple from the right (Skryabin) }  $\Rightarrow A$  is Frobenius algebra.  
 $\chi: A \rightarrow k$

# ON UNIMODULARITY OF TENSOR CATEGORIES

(arXiv: 2302.06192)

Theorem (Shimizu, FGJS, Y.)

The following are equivalent

i)  $\text{End}_{\mathcal{C}}^{\text{rex}}(\mathcal{M})$  is unimodular

ii)  $\mathcal{N} \cong_{\mathcal{C}} \text{id}_{\mathcal{M}}$

•  $\mathcal{C} \curvearrowright \mathcal{M}$

•  $\mathcal{M} = A\text{-mod}$

$\xrightarrow{\quad} A^* \otimes - : A\text{-mod} \rightarrow A\text{-mod}$

$\mathcal{N}_{A\text{-mod}}$  (Nakayama functor)

$\text{Hom}_{\mathcal{M}}(M, N) \quad M, N \in \mathcal{M}$

$\text{Hom}_X(X \otimes M, N) \cong \text{Hom}_{\mathcal{C}}(X, \text{Hom}_{\mathcal{M}}(M, N))$

$\mathcal{S}$  relative Serre functor  $\phi: \text{Hom}_{\mathcal{M}}(M, N)^* \rightarrow \text{Hom}_{\mathcal{M}}(N, \mathcal{S}(M))$

exists  $\iff \mathcal{M}$  is exact

right integral  $\iff \mathcal{S} = \text{id}$

$\alpha: H \rightarrow \mathbb{R}$  measures how far a left integral is from being a right int

# ON UNIMODULARITY OF TENSOR CATEGORIES

(arxiv: 2302.0619)

Theorem (Shimizu, FGJS, Y.)

The following are equivalent

i)  $\text{End}_e^{\text{rex}}(\mathcal{M})$  is unimodular

ii)  $\mathbb{N} \cong_e \text{id}_{\mathcal{M}}$  (commutative)

iii)  $\psi^{\text{ra}}(\text{id}_{\mathcal{M}})$  is a Frobenius algebra in  $Z(\mathcal{C})$ .

iv)  $\psi^{\text{ra}}$  is a

$$\Psi: Z(\mathcal{C}) \rightarrow \text{End}_e(\mathcal{M})$$

$$(c, \sigma) \mapsto (c \bar{\sigma}, \bar{\sigma})$$

$$\Psi \vdash \Psi^{\text{ra}}$$

$$H_0$$

$$e \curvearrowright \mathcal{M}$$

$$\mathcal{M} = A\text{-mod} \rightarrow A^* \otimes - : A\text{-mod} \rightarrow A\text{-mod}$$

$\mathbb{N}_{A\text{-mod}}$  (Nakayama functor)

$$\text{Hom}(M, N) \quad M, N \in \mathcal{M}$$

$$\text{Hom}_X(X \otimes M, N) \cong \text{Hom}_e(X, \text{Hom}(M, N))$$

$$\mathbb{S} \text{ relative Serre functor } \phi: \text{Hom}(M, N)^* \rightarrow \text{Hom}(N, M)$$

exists  $\iff \mathcal{M}$  is exact

unimodular if  $\mathcal{C}$  admits a two-sided integral  $\iff \Lambda^2 k = \varepsilon(k) \Lambda^2$

in  $Z(\mathcal{C})$ .

$(c, 0) \mapsto (c^*, 0)$

$\text{Hom}_{\mathcal{C}}(X, \text{Hom}(M, N))$

$\mathcal{B}$  relative Serre functor  $\phi: \text{Hom}(M, N)^* \rightarrow \text{Hom}(N, \mathcal{B}(M))$   
exists  $\iff M$  is exact

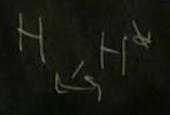
iv)  $\psi^{\text{ra}}$  is a Frobenius monoidal functor  
In this case, recall  $M$  unimodular.

Theorem  $\mathcal{C}$  = pivotal finite tensor category  $(\tau) \cong \text{id}_{\mathcal{C}}$   
 $M = \text{unimodular, pivotal } (\mathcal{B} \cong \text{id}_{\mathcal{C}})$

$\implies \psi^{\text{ra}}$  is a pivotal Frobenius monoidal functor

eg:  $\mathcal{C} \xrightarrow{M} \mathcal{C}$  unimodular  $\iff \mathcal{C}$  is a unimodular tensor cat

$\mathcal{C} = \text{Rep}(H) \rightarrow \text{Vec}$ ,  $\text{End}_{\text{Rep}(H)}(\text{Vec}) \cong \text{Rep}(H^*)$



- $\text{Rep}(H)$ -module categories =  $\text{Rep}(A)$
- $\text{End}_{\text{Rep}(H)}(\text{Rep}(A)) = {}_A^H \mathcal{M}_A$

dual basis  $\{a_i, b_j\}$  of  $A$

$$\langle \lambda, a_i b_j \rangle = \delta_{i,j}$$

$$\langle \lambda, ab \rangle = \langle \lambda, v(b)a \rangle$$

modular  $\Leftrightarrow A$  admits an invertible element  $\hat{g}$  st

$$i) \quad \hat{g} a \hat{g}^{-1} = \langle \alpha, S(a_i) \rangle v^2(a_o)$$

$$ii) \quad 1_H \otimes \hat{g} = \tau \cdot p(\hat{g})$$

$\tau =$  complicated expression

does not admit a  
unimodular module category.

$$p: A \rightarrow H \otimes A$$

$$a \mapsto a_{-1} \otimes a_0$$

$$T = \frac{|k(x, g)|}{\langle g^2=1, x^2=0, gx=-xg \rangle}$$

• H-comodule algebras  $A$

$\left( \begin{array}{c} \text{when} \\ \text{Rep}(A) \text{ unimodular?} \end{array} \right)$

•  $\text{Rep}(H)$ -module categories =  $\text{Rep}(A)$

•  $\text{End}_{\text{Rep}(H)}(\text{Rep}(A)) = {}^H M_A$

$\Rightarrow A$  is Frobenius

dual basis  $(a_i, b_j)$  of  $A$

$\langle \lambda, a_i b_j \rangle = \delta_{ij}$

$\langle \lambda, a_i \rangle$

Theorem:  $\text{Rep}(A)$  is unimodular  $\Leftrightarrow A$  admits an invertible element  $\hat{g}$  st

i)  $\hat{g} a \hat{g}^{-1} = \langle \alpha, S(a) \rangle \tau(a)$

ii)  $1_H \otimes \hat{g} = \tau$

$\tau = \text{complication}$

Theorem:  $\mathcal{C} = \text{Rep}(\text{Taft alg})$  does not admit a unimodular module category.

(Morita equivalence class of  $\mathcal{C}$  does not contain a unimodular tensor cat)

$T =$

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- $S^4(h) = g_H^{-1}(\alpha \rightarrow h \leftarrow \alpha^{-1}) \leftarrow g_H^{-1}$
- $H$ -comodule algebras  $A$

$\underline{\underline{\text{Q}}}$  when  $\text{Rep}(A)$  unimodular?

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