Title: On unimodularity in the theory of tensor categories

Speakers: Harshit Yadav

Series: Mathematical Physics

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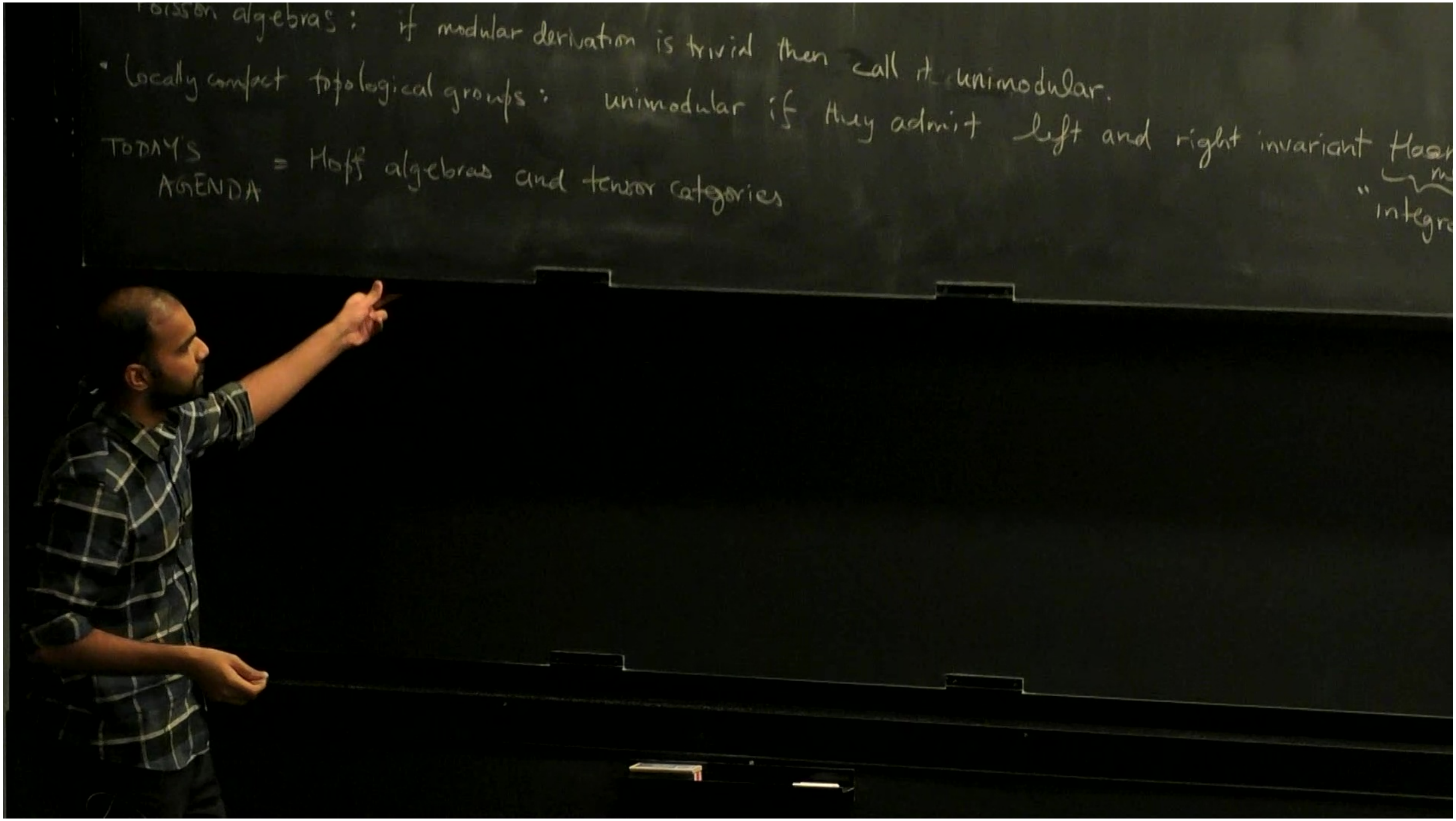
Abstract: Unimodularity is a classical notion shows up in various fields like linear algebra, lattices, Poisson algebras, etc. In this talk, we focus on unimodular Hopf algebras and unimodular tensor categories. We will introduce unimodular module categories and use them to construct Frobenius algebras and unimodular tensor categories. These ideas will be illustrated with examples drawn from Hopf algebras.

Zoom link https://pitp.zoom.us/j/98477599322?pwd=UDJmTklMTGxGODJZTm1Xc1VhL2tDdz09

ON UNIMODULARITY OF TENSOR CATEGORIES (arxiv: 2302.06192)

- Linear algebra: $A \in M_n(\mathbb{Z})$ is called unimodular if $\det(A) = \pm 1$
- Bilinear forms, lattices
- Poisson algebras: if modular derivation is trivial then call it unimodular.


oisson algebras: if modular derivation is trivial then call it unimodular.
• locally compact topological groups: unimodular if they admit left and right invariant
TODAY'S AGENDA = Hopf algebras and tensor categories

AGENDA ... and tensor categories

"integrals"

HOPF ALGEBRA $(\underbrace{H, m, u}_{\text{algebra}}, \underbrace{\Delta, \varepsilon}_{\text{coalgebra}}, S)$

Δ, ε are algebra maps

- $S : H \to H$ antipode

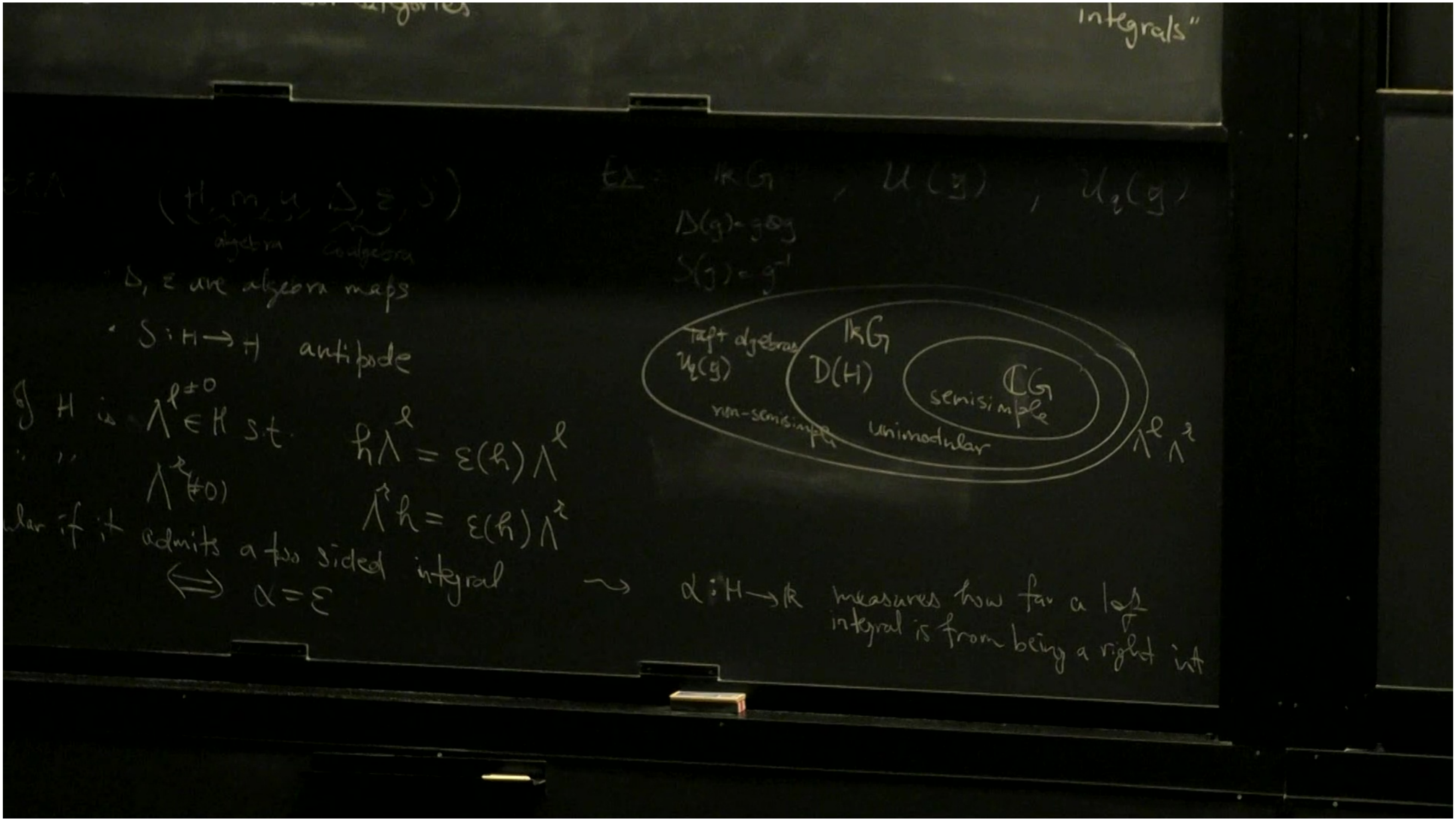
Ex: $\Bbbk G$, $\mathcal{U}(\mathfrak{g})$, $\mathcal{U}_q(\mathfrak{g})$

$\Delta(g) = g \otimes g$

$S(g) = g^{-1}$

- Left integral of H is $\Lambda^l \in H$ s.t. $h\Lambda^l = \varepsilon(h)\Lambda^l$
- Right " " " Λ^r $\Lambda^r h = \varepsilon(h)\Lambda^r$
- H is unimodular if it admits a two sided integral

$\iff \alpha = \varepsilon$ $\rightsquigarrow$ $\alpha : H \to \Bbbk$ measures how far a left integral is from being a right int


integrals"
(H, m, u, Δ, ε, S)
algebra
coalgebra
Δ, ε are algebra maps
S : H → H antipode
Λ^l ∈ H s.t.
hΛ^l = ε(h) Λ^l
Λ^r (≠0)
Λ^r h = ε(h) Λ^r
admits a two sided integral
⟺ α = ε
Ex: kG , U(g) , U_q(g)
Δ(g) = g⊗g
S(g) = g^-1
Taft algebras
U_q(g)
non-semisimple
kG
D(H)
unimodular
CG
semisimple
Λ^l Λ^r
α : H → k measures how far a left integral is from being a right int

From Hopf algebras to tensor categories

$(H, m, u, \Delta, \varepsilon, S)$

- (H, m, u) $\Bbbk$-alg
- Δ, ε algebra maps
- $S : H \to H$
- $\alpha : H \to \Bbbk$ distinguished character

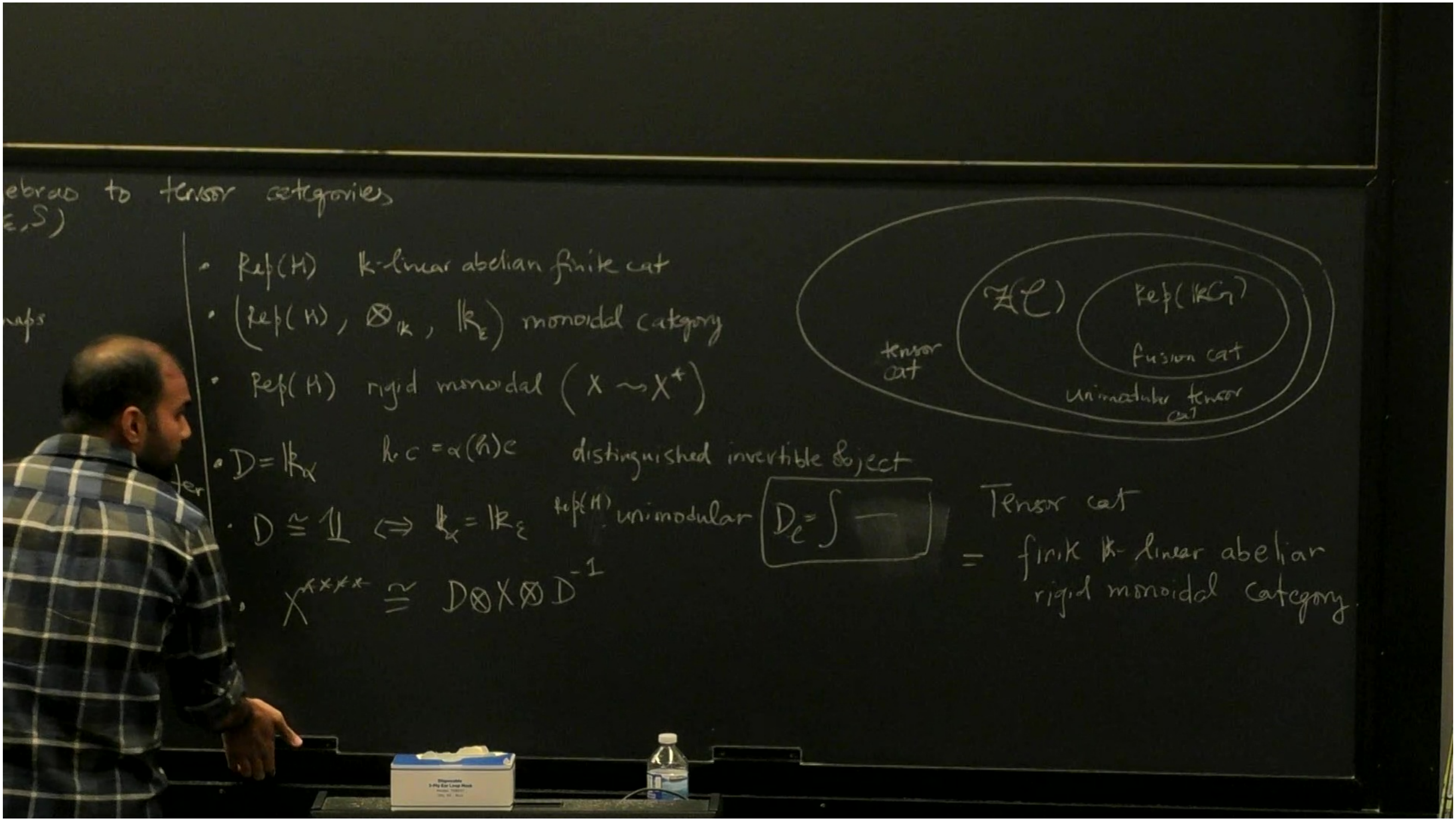
- $\mathrm{Rep}(H)$ $\Bbbk$-linear abelian finite cat
- $(\mathrm{Rep}(H), \otimes_{\Bbbk}, \Bbbk_{\varepsilon})$ monoidal category
- $\mathrm{Rep}(H)$ rigid monoidal $(X \rightsquigarrow X^*)$
- $D = \Bbbk_{\alpha}$ $\quad h.c = \alpha(h)c$ $\quad$ distinguished invertible object
- $D \cong \mathbb{1} \iff \Bbbk_{\alpha} = \Bbbk_{\varepsilon}$ $\quad$ $\mathrm{Rep}(H)$ unimodular

From Hopf algebras to tensor categories

$(H, m, u, \Delta, \varepsilon, S)$

- (H, m, u) $\Bbbk$-alg
- Δ, ε algebra maps
- $S : H \to H$
- $\alpha : H \to \Bbbk$ distinguished character
- $\alpha = \varepsilon$
- $S^4(h) = g_H \rightharpoonup (\alpha \rightharpoonup h \leftharpoonup \alpha^{-1}) \leftharpoonup g_H^{-1}$

- $\mathrm{Rep}(H)$ $\Bbbk$-linear abelian finite cat
- $(\mathrm{Rep}(H), \otimes_{\Bbbk}, \Bbbk_\varepsilon)$ monoidal category
- $\mathrm{Rep}(H)$ rigid monoidal $(X \rightsquigarrow X^*)$
- $D = \Bbbk_\alpha$ $\quad h \cdot c = \alpha(h) c$ $\quad$ distinguished invertible object
- $D \cong \mathbb{1} \iff \Bbbk_\alpha = \Bbbk_\varepsilon$ $\quad \mathrm{Rep}(H)$ unimodular
- $X^{****} \cong D \otimes X \otimes D^{-1}$


ebras to tensor categories
ε, S)
maps
• Rep(H) k-linear abelian finite cat
• (Rep(H), ⊗k, kε) monoidal category
• Rep(H) rigid monoidal (X ⇝ X*)
• D = kα   h.c = α(h)c   distinguished invertible object
• D ≅ 1 ⟺ kα = kε   Rep(H) unimodular
Dℭ = ∫ —
• X**** ≅ D⊗X⊗D⁻¹
Z(ℭ)
Rep(kG)
fusion cat
unimodular tensor cat
tensor cat
Tensor cat
= finite k-linear abelian rigid monoidal category.

• H-comodule algebras

• Rep(H)-module categories

• $\mathcal{C}$-Module categories $= (\mathcal{M}, \triangleright : \mathcal{C}\times\mathcal{M}\to\mathcal{M})$

$\mathcal{M}$ = abelian, $\Bbbk$-linear

• $(X\otimes Y)\triangleright M \cong X\triangleright(Y\triangleright M)$

• $\mathbb{1}\triangleright M \cong M$

• We call $\mathcal{M}$ <u>exact</u>, $\forall$ X projective in $\mathcal{C}$ and any $M\in\mathcal{M}$ want $X\triangleright M$ to be projective in $\mathcal{M}$.

<u>Theorem</u> (Etingof-Ostrik) $\mathcal{C}$ = finite tensor cat, $\mathcal{M}$ = exact $\mathcal{C}$-module then $\mathrm{End}_{\mathcal{C}}(\mathcal{M})$ is

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<u>Theorem</u> (Etingof-Ostrik) $\mathcal{C}$ = finite tensor cat, $\mathcal{M}$ = exact $\mathcal{C}$-module then $\mathrm{End}^{\mathrm{rex}}_{\mathcal{C}}(\mathcal{M})$ is

<u>Q</u> When is $\mathrm{End}^{\mathrm{rex}}_{\mathcal{C}}(\mathcal{M})$ unimodular?

algebras A

- $\mathrm{Rep}(H)$-module categories $= \mathrm{Rep}(A)$
- $\mathrm{End}_{\mathrm{Rep}(H)}(\mathrm{Rep}(A)) = {}^H_A\mathcal{M}_A$

categories $= (\mathcal{M}, \triangleright : \mathcal{C} \times \mathcal{M} \to \mathcal{M})$

$\Bbbk$-linear

- $(X \otimes Y) \triangleright M \cong X \triangleright (Y \triangleright M)$
- $\mathbb{1} \triangleright M \cong M$

$\mathcal{C}$-module functor: F $\mathcal{M}$-
$F(X \triangleright M) \cong X \triangleright F(M$

exact, $\forall$ X projective in $\mathcal{C}$ and any $M \in \mathcal{M}$
want $X \triangleright M$ to be projective in $\mathcal{M}$.

-Ostrik) $\mathcal{C}$ = finite tensor cat, $\mathcal{M}$ = exact $\mathcal{C}$-module then $\mathrm{End}^{\mathrm{rex}}_{\mathcal{C}}(\mathcal{M})$ is a tensor category.

$\mathrm{End}^{\mathrm{rex}}_{\mathcal{C}}(\mathcal{M})$ unimodular?

is exact as a $\mathrm{Rep}(H)$-module category if:
$\cong \Bbbk$ coH
- A is H-simple from the right (Skryabin)

$\Rightarrow$ A is Frobenius algebra.
$\lambda : A \to \Bbbk$

ON UNIMODULARITY OF TENSOR CATEGORIES (arxiv: 2302.06192)

Theorem (Shimizu, FGJS, Y.)

The following are equivalent

i) $\mathrm{End}^{rex}_{\mathcal{C}}(\mathcal{M})$ is unimodular

ii) $\mathbb{S}\,\mathbb{N} \cong_{\mathcal{C}} \mathrm{id}_{\mathcal{M}}$

- $\mathcal{C} \curvearrowright \mathcal{M}$
- $\mathcal{M} = A\text{-mod}$

  $\mathbb{N}_{A\text{-mod}} \longrightarrow A^* \otimes - : A\text{-mod} \to A\text{-mod}$

  (Nakayama functor)
- $\underline{\mathrm{Hom}}(M,N)$ $\quad M, N \in \mathcal{M}$

  $\mathrm{Hom}_{\mathcal{M}}(X \triangleright M, N) \cong \mathrm{Hom}_{\mathcal{C}}(X, \underline{\mathrm{Hom}}(M,N))$

  $\mathbb{S}$ relative Serre functor $\quad \phi : \underline{\mathrm{Hom}}(M,N)^* \longrightarrow \underline{\mathrm{Hom}}(N, \mathbb{S}(M))$

  exists $\Longleftrightarrow$ $\mathcal{M}$ is exact

exists a two sided integral $\Longleftrightarrow$ $\alpha = \varepsilon$ $\quad \rightsquigarrow \quad \alpha : H \to k$ measures how far a left integral is from being a right int

ON UNIMODULARITY OF TENSOR CATEGORIES (arxiv: 2302.0619

Theorem (Shimizu, FGJS, Y.)

The following are equivalent

i) $\mathrm{End}^{rex}_{\mathcal{C}}(\mathcal{M})$ is unimodular

ii) $\exists\, \mathbb{N} \cong_{\mathcal{C}} \mathrm{id}_{\mathcal{M}}$

iii) $\psi^{ra}(\mathrm{id}_{\mathcal{M}})$ is a (commutative) Frobenius algebra in $Z(\mathcal{C})$.

iv) ψ^{ra} is a

$$\left(\begin{array}{l}\psi : Z(\mathcal{C}) \to \mathrm{End}_{\mathcal{C}}(\mathcal{M}) \\ \quad (c,\sigma) \mapsto (c \triangleright -, \mathfrak{s}^{\sigma})\end{array}\right)$$

$\psi \dashv \psi^{ra}$

Ho

- $\mathcal{C} \curvearrowright \mathcal{M}$
- $\mathcal{M} = A\text{-mod}$

  $\mathbb{N}_{A\text{-mod}} \to A^* \otimes - : A\text{-mod} \to$

  (Nakayama functor)

- $\underline{\mathrm{Hom}}(M,N) \qquad M, N \in \mathcal{M}$

  $\mathrm{Hom}_{\mathcal{M}}(X \triangleright M, N) \cong \mathrm{Hom}_{\mathcal{C}}(X, \underline{\mathrm{Hom}}$

  $\exists\, \mathbb{S}$ relative Serre functor $\quad \phi : \underline{\mathrm{Hom}}(M,N)^* \longrightarrow \underline{\mathrm{Hom}}(N, \mathbb{S}$

  exists $\Leftrightarrow$ $\mathcal{M}$ is exact

$\Lambda h = \varepsilon(h)\Lambda$

H is unimodular if it admits a two sided integral

$\Leftrightarrow \alpha = \varepsilon \quad \rightsquigarrow \quad \alpha : H \to \mathbb{k}$ measures how f in $\mathcal{Z}(\mathcal{C})$.

iv) ψ^{ra} is a Frobenius monoidal functor

In this case, we call $\mathcal{M}$ unimodular.

$\mathrm{Hom}_{\mathcal{C}}(X, \underline{\mathrm{Hom}}(M,N))$

$\mathbb{S}$ relative Serre functor $\quad \phi: \underline{\mathrm{Hom}}(M,N)^* \longrightarrow \underline{\mathrm{Hom}}(N, \mathbb{S}(M))$

exists $\Leftrightarrow$ $\mathcal{M}$ is exact

<u>Theorem</u> $\mathcal{C}$ = pivotal finite tensor category $\left((-)^{**} \overset{p}{\cong} \mathrm{id}_{\mathcal{C}} \right)$

$\mathcal{M}$ = unimodular, pivotal $\left(\mathbb{S} \cong_{\mathcal{C}} \mathrm{id}_{\mathcal{M}} \right)$

$\Rightarrow$ ψ^{ra} is a pivotal Frobenius monoidal functor

<u>Eg</u>: • $\mathcal{C} \curvearrowright \overset{\mathcal{M}}{\mathcal{C}}$ unimodular $\Leftrightarrow$ $\mathcal{C}$ is a unimodular tensor cat

• $\mathcal{C} = \mathrm{Rep}(H) \to \mathrm{Vec}$, $\quad \mathrm{End}_{\mathrm{Rep}(H)}(\mathrm{Vec}) \simeq \mathrm{Rep}(H^*)$

$H \leftrightarrow H^*$

- Rep(H)-module categories = Rep(A)
- $\mathrm{End}_{\mathrm{Rep}(H)}(\mathrm{Rep}(A)) = {}^{H}_{A}\mathcal{M}_{A}$

dual basis $\{a^i\}, \{b_i\}$ of A

$\langle \lambda, a^i b_j \rangle = \delta_{ij}$

$\langle \lambda, ab \rangle = \langle \lambda, \nu(b) a \rangle$

modular $\iff$ A admits an invertible element $\hat{g}$ st

$\rho: A \to H \otimes A$

$a \mapsto a_{-1} \otimes a_0$

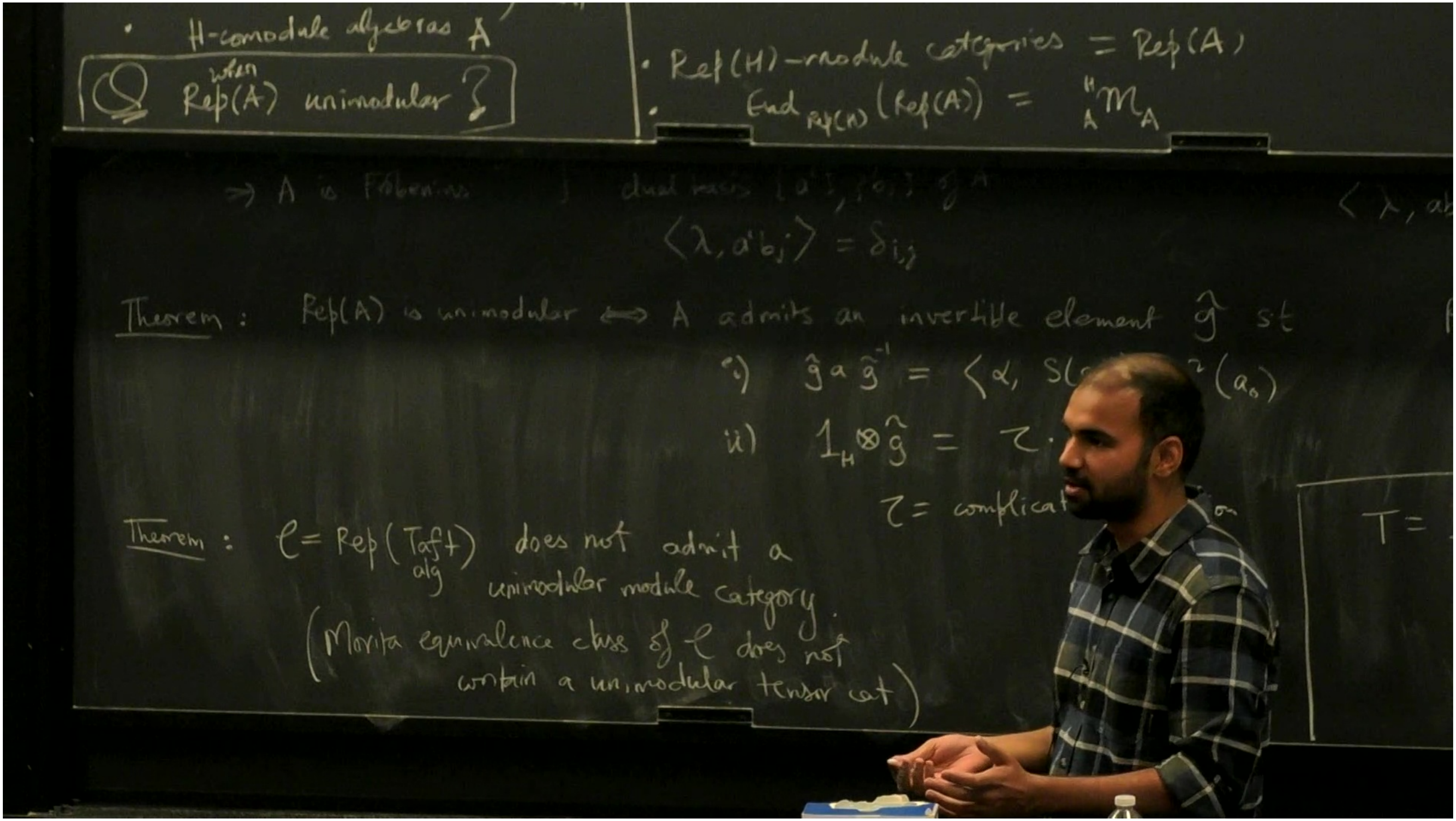
i) $\hat{g} a \hat{g}^{-1} = \langle \alpha, S(a_{-1}) \rangle \nu^2(a_0)$

ii) $1_H \otimes \hat{g} = \tau \cdot \rho(\tilde{g})$

τ = complicated expression

does not admit a unimodular module category.

$$T = \frac{\Bbbk(x, g)}{\langle g^2 = 1, x^2 = 0, gx = -xg \rangle}$$


H-comodule algebras A
Q when Rep(A) unimodular ?
Rep(H)-module categories = Rep(A)
End Rep(H) (Rep(A)) = H A M A
⇒ A is Frobenius
dual basis
Theorem : Rep(A) is unimodular ⟺ A admits an invertible element ĝ s.t
i)
ii)
Theorem : C = Rep(Taft alg) does not admit a unimodular module category.
(Morita equivalence class of C does not contain a unimodular tensor cat)

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$\boxed{\text{Q when } \mathrm{Rep}(A) \text{ unimodular ?}}$

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- $D \cong \mathbb{1} \iff \Bbbk_{\alpha} = \Bbbk_{\varepsilon}$ $\quad$ $\mathrm{Rep}(H)$ unimodular $\boxed{D_{\mathcal{C}} = \int \text{—}}$
- $X^{****} \cong D \otimes X \otimes D^{-1}$
- $\mathrm{Rep}(H)$-module categories $= \mathrm{Rep}(A)$
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