Title: Scattering Amplitudes and Tilings of Moduli Spaces

Speakers: Nick Early

Series: Quantum Fields and Strings

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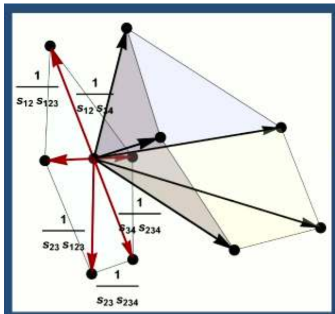
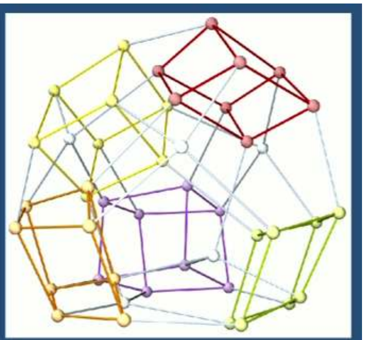
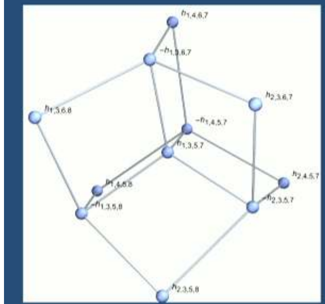
Abstract: In 2013, Cachazo, He and Yuan discovered a remarkable framework for scattering amplitudes in Quantum Field Theory (QFT) which mixes the real, complex and tropical geometry associated to the moduli space of n points on the projective line, $M_{0,n}$. By duality, this moduli space has a twin moduli space of n generic points in P^{n-3}, leading to dual realization of scattering amplitudes, using a generalization of the CHY formalism introduced in 2019 by Cachazo, Early, Guevara and Mizera (CEGM). Any duality begs for an explanation! And, what physical phenomena lie between the twin moduli spaces? CEGM developed a framework to answer the question for moduli spaces of n points in any P^{k-1}, leading to the discovery of rich, recursive structures and novel behaviors which portend an extension of QFT. We discuss recent joint works with Cachazo and Zhang, and with Geiger, Panizzut, Sturmfels, Yun, in which we dig deeper into some of the many mysteries which arise.

Zoom link: https://pitp.zoom.us/j/95706337178?pwd=cnBka3lFWDBoRlFZV1VadXdGZ1JnZz09

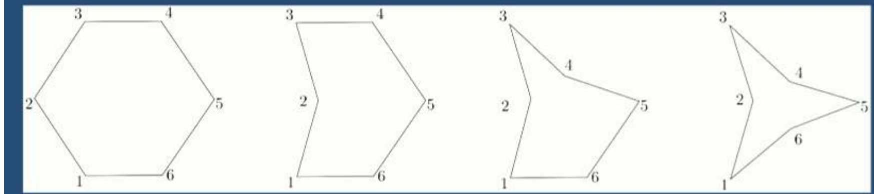


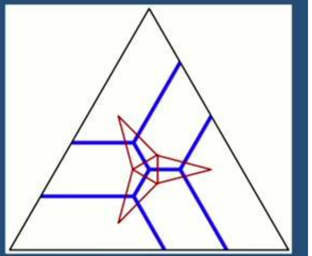


Scattering Amplitudes and Tilings of Moduli Spaces



Nick Early





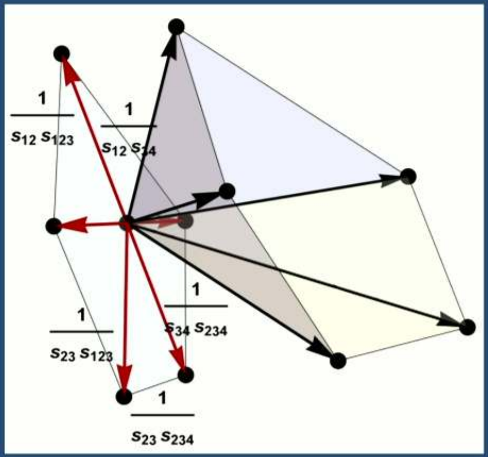
Max Planck Institute for Mathematics in the Sciences, Leipzig

Two parts... in collaboration with:

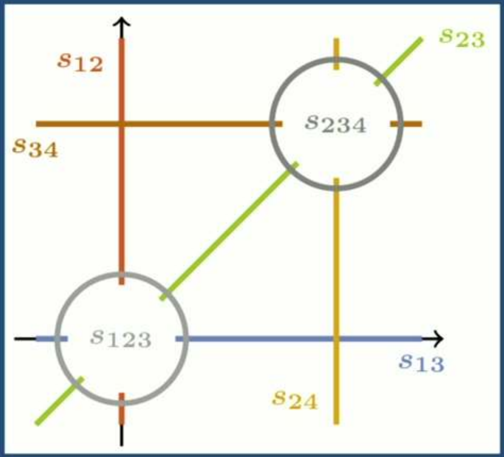
(1) F. Cachazo and Y. Zhang

(2) A. Geiger, M. Panizzut, B. Sturmfels and C. Yun.

Various Computations of the Same Amplitude



Global Schwinger parameterization
(Cachazo-E 2020)

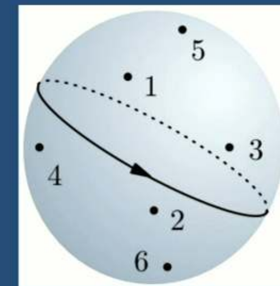


Moduli spaces schematic: grabbed from work of S. Mizera

$$\frac{1}{s_{12}\, s_{34}} + \frac{1}{s_{12}\, s_{123}} + \frac{1}{s_{23}\, s_{123}} + \frac{1}{s_{23}\, s_{234}} + \frac{1}{s_{34}\, s_{234}}$$

Perspective: CHY

$$\begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$



- In 2013, Cachazo-He-Yuan (CHY) discovered a remarkable framework for QFT amplitudes...
  - Architecture:
    - CHY scattering equations on the (auxiliary) moduli space M0n of n generic points on the Riemann sphere.
    - And, a convex polytope known as the associahedron.
  - Note: M0n is also the torus quotient X(2,n) of the Grassmannian G(2,n).

$$\sum_{b \neq a} \frac{s_{ab}}{x_a - x_b} = 0$$

$$\mathcal{S} = \sum_{1 \leq a < b \leq n} s_{ab} \log(x_a - x_b)$$

$$\sum_{\substack{b=1 \\ b \neq a}}^{n} s_{ab} = 0 \qquad \forall a$$

Perspective: Dual CHY

- NOTE: there's a **duality** G(2,n) ~ G(n-2,n)! So the moduli space of n points in P1 is isomorphic to the moduli space of n hyperplanes in P^(n-3) and we have defined amplitudes with X(n-2,n) playing the role analogous to M0n!
- What comes in between, e.g. where X(3,n) plays the role of M0n?

$$\begin{pmatrix} 1 & 0 & 1 & x_4 & x_5 & x_6 \\ 0 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 & 1 & x_5 & x_6 \\ 0 & 1 & 0 & 1 & y_5 & y_6 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & x_6 \\ 0 & 1 & 0 & 0 & 1 & y_6 \\ 0 & 0 & 1 & 0 & 1 & z_6 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

$$\begin{matrix} & & & m_4^{(2)} & & & \\ & & m_5^{(2)} & & m_5^{(3)} & & \\ & m_6^{(2)} & & m_6^{(3)} & & m_6^{(4)} & \\ m_7^{(2)} & & m_7^{(3)} & & m_7^{(4)} & & m_7^{(5)} \\ & & & \vdots & & & \end{matrix}$$

Overview

- Configuration space of points in the projective line.
  - Cachazo-He-Yuan (CHY) formula uses the scattering equations to connect G(2,n)/T to scattering amplitudes.
- Configuration spaces of points in projective space.
  - Cachazo-E-Guevara-Mizera (CEGM, 2019) extended the CHY formula to the *positive configuration space*: generic convex point configurations in projective space.
  - Controlled by a very rich object, the positive tropical Grassmannian Trop+G(k,n).
- Extension to non-convex point configurations.
  - Cachazo-E-Zhang (CEZ, 2022;2023) extended the CEGM integral to the nonplanar case: integrands for configurations of non-convex generic point configurations.
  - Introduced a generalization of the positive tropical Grassmannian, called the chirotopal tropical Grassmannian. Provides a combinatorial interpretation of nonplanar CEGM amplitudes.
- del Pezzo moduli spaces.
  - E-Geiger-Panizzut-Sturmfels-Yun (2023). Refine G(3,n)/T to Y(3,n) by removing all coconic point configurations

    → del Pezzo moduli spaces, computed canonical forms, amplitudes...
- The positive del Pezzo moduli space Y+(3,6) is a polytope! Its normal fan embeds into (half of) Trop+G(3,6).

Perspective: biadjoint scalar (and beyond) via CHY

- We study a particularly simple but elegant QFT, the (biadjoint) cubic scalar, whose singularities where P^2=0 correspond to strata in the (real) moduli space of n-pointed stable curves M0n.
  - This theory has a singularity at P^2=0 for the sum P of any subset of momentum vectors.
  - Residues are encoded by a richly structured polytope, the associahedron.
  - ...amplitudes for other theories (e.g. NLSM) have mixed singularities, some at zero and some at infinity, decorated with numerators in Feynman diagrams.

The theory consists of a single massless scalar field $\Phi = \phi_{a\tilde{a}} T^a \widetilde{T}^{\tilde{a}}$ valued in the Lie algebra of $\mathrm{U}(N) \times \mathrm{U}(\widetilde{N})$

$$\mathcal{L}^{\mathrm{BA}} = \frac{1}{2}\phi_{a\tilde{a}}\Box\phi^{a\tilde{a}} - \frac{1}{3} f^{abc}\,\tilde{f}^{\tilde{a}\tilde{b}\tilde{c}}\,\phi_{a\tilde{a}}\,\phi_{b\tilde{b}}\,\phi_{c\tilde{c}}.$$

$$m_4((1234),(1234)) = \frac{1}{(p_1+p_2)^2} + \frac{1}{(p_2+p_3)^2} = \frac{1}{s_{12}} + \frac{1}{s_{23}},$$

$$m_5(12345, 12345) = \frac{1}{s_{12}s_{123}} + \frac{1}{s_{23}s_{123}} + \frac{1}{s_{23}s_{234}} + \frac{1}{s_{34}s_{234}} + \frac{1}{s_{12}s_{34}}$$

Perspective: CEGM

- In 2019, Cachazo-E-Guevara-Mizera (CEGM) developed a framework which provides a novel moduli space generalization of QFT scattering amplitudes (so far, at tree-level).
  - It is based on the moduli space of n points in projective space P(k-1), analogously to CHY when k=2.
  - Poles of amplitudes are new and somewhat mysterious, but those that remain non-spurious in certain limits in momentum space are approximated by standard squared sums of momenta of the form P^2=0.

$$\begin{pmatrix} 1 & 0 & 0 & x_1 & y_1 & z_1 \\ 0 & 1 & 0 & x_2 & y_2 & z_2 \\ 0 & 0 & 1 & x_3 & y_3 & z_3 \end{pmatrix} \Big/ T \longrightarrow \begin{pmatrix} 1 & 0 & 0 & 1 & y_1 & z_1 \\ 0 & 1 & 0 & 1 & y_2 & z_2 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}$$

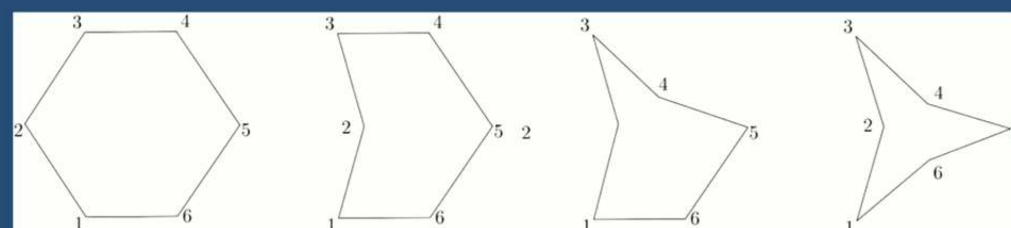
Tiling the Real Grassmannian

The nonnegative part of the real Grassmannian is a beautiful and amazingly well-behaved object which has deep applications in physics, in particular in the study of planar N=4 SYM. Its strata are indexed by certain on-shell diagrams (or in math, planar bicolored graphs, due to Alex Postnikov).

The tiling of the full real Grassmannian G(3,n) is not well-behaved in general. But…

Let's take some cautious steps anyway, ``near" the positive Grassmannian…

$$\begin{pmatrix} 1 & 0 & 0 & x_1 & y_1 & z_1 \\ 0 & 1 & 0 & x_2 & y_2 & z_2 \\ 0 & 0 & 1 & x_3 & y_3 & z_3 \end{pmatrix} \Big/ T \longrightarrow \begin{pmatrix} 1 & 0 & 0 & 1 & y_1 & z_1 \\ 0 & 1 & 0 & 1 & y_2 & z_2 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}$$



$p_{123} < 0, \; p_{345} < 0, \; p_{156} < 0$

Scattering equations and amplitudes

$$\mathcal{S} = \sum_{1\le a<b\le n} s_{ab}\log(x_a - x_b)$$

$$\frac{\partial \mathcal{S}}{\partial x_a} = \sum_{\substack{b=1\\ b\neq a}}^{n} \frac{s_{ab}}{x_a - x_b} = 0 \qquad \forall a$$

$$\sum_{\substack{b=1\\ b\neq a}}^{n} s_{ab} = 0 \qquad \forall a$$

$$\begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Base case integrand, the Parke-Taylor factor:

$$PT(12\cdots n) = \frac{1}{(x_1 - x_2)(x_2 - x_3)\cdots(x_n - x_1)}$$

- The following is a number, given by the Cachazo-He-Yuan (CHY) formula (H = Hessian of scattering potential S):

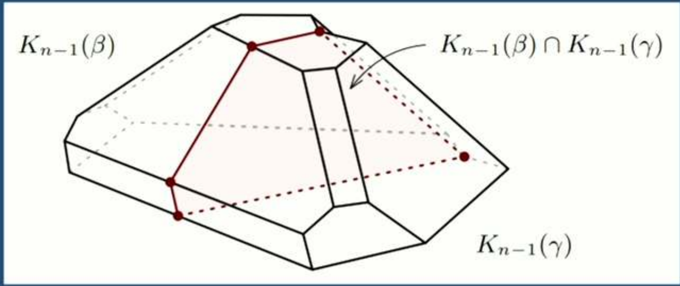
$$m(\alpha,\beta) = \sum_{c\in \mathrm{crit}(\mathcal{S})}^{(n-3)!} \frac{1}{\det H} PT(\alpha)PT(\beta)\Big|_c$$

- That number is also given by a combinatorial formula, summing over *Feynman Diagrams*:

$$m[1234, 1234] = \frac{1}{s_{12}} + \frac{1}{s_{23}}$$

$$m[12345, 12345] = \frac{1}{s_{12}\, s_{34}} + \frac{1}{s_{12}\, s_{123}} + \frac{1}{s_{23}\, s_{123}} + \frac{1}{s_{23}\, s_{234}} + \frac{1}{s_{34}\, s_{234}}$$

When alpha and beta are different, sum over the vertices of the intersection of associahedra:



Credit for visualization: S. Mizera

Next: decomposing the moduli space of points in the real projective line...

Summary of CHY formalism for scattering amplitudes

1. Cachazo-He-Yuan (CHY, 2013) introduced a compact formula for massless gluon tree amplitudes in pure Yang-Mills theory in arbitrary dimension.
2. This revealed a substrate for Quantum Field Theory (QFT) that's based on deep combinatorial ideas.
3. Scattering amplitudes in various theories are related to each other by the CHY formalism and combinatorial and tropical geometry. These include:
   a. Gravity, Born-Infeld, Einstein-Maxwell, Yang-Mills, Nonlinear Sigma Model, biadjoint scalar...
4. Modifying the CHY integrand captures these other theories!
5. CEGM generalized the CHY formulation of the biadjoint scalar.
   a. This theory allows us to study fundamental physical principles that must be satisfied in the new context. They are novel, and highly nontrivial!
6. Other theories and their Lagrangians?
   a. Some encouraging progress, but the computational complexity is enormous beyond the first few cases...

Tiling Moduli Spaces

$$p_{1,4}\, p_{2,3} - p_{1,3}\, p_{2,4} + p_{1,2}\, p_{3,4} = 0$$

1. Consider the subset of the (real) Grassmannian G(2,n) where all Plucker variables are nonzero.
2. Each component is characterized by a chirotope → a vector of signs of Plucker variables.
3. Now mod out by the torus, X(2,n) = G(2,n)/T.
4. Example: components of X(2,5) are characterized by 12 possible sign patterns on certain cross-ratio coordinates (u1,u2,u3,u4,u5).

Next: to each chirotope, we associate a canonical function, the Parke-Taylor factor...

$$u_1 = \frac{p_{25}p_{34}}{p_{35}p_{24}} = \frac{1-y}{1-x},\ u_2 = \frac{p_{13}p_{45}}{p_{14}p_{35}} = y,\ u_3 = \frac{p_{24}p_{15}}{p_{14}p_{25}} = 1-x,\ u_4 = \frac{p_{35}p_{12}}{p_{13}p_{25}} = \frac{x}{y},\ u_5 = \frac{p_{14}p_{23}}{p_{13}p_{24}} = \frac{y-x}{(1-x)y}.$$

$$\begin{matrix} (+++++) & (-++++) & (+-+++) & (++-++) & (+++-+) & (++++-) \\ (-----) & (+-+-+) & (++-+-) & (-++-+) & (+-++-) & (-+-++) \end{matrix}$$

$$M = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & x & y & 1 & 1 \end{bmatrix}$$

Parke-Taylor factor (on M0n=G(2,n)/T)

1. Consider the Parke-Taylor factor, below.
2. Theorem (Kleiss-Kuijff): There are (n-2)! linearly independent functions.
3. Basic linear relation is "U(1) decoupling." In general, *shuffle relations*.
4. Fix a cyclic order (1,2,3) and insert a new point 4 in all possible ways.

Next: positive Grassmannian (and slightly beyond)

$$\texttt{parkeTaylor[\{1, 2, 3, 4\}]} = \frac{1}{(x_1 - x_2)(x_2 - x_3)(x_3 - x_4)(-x_1 + x_4)}$$

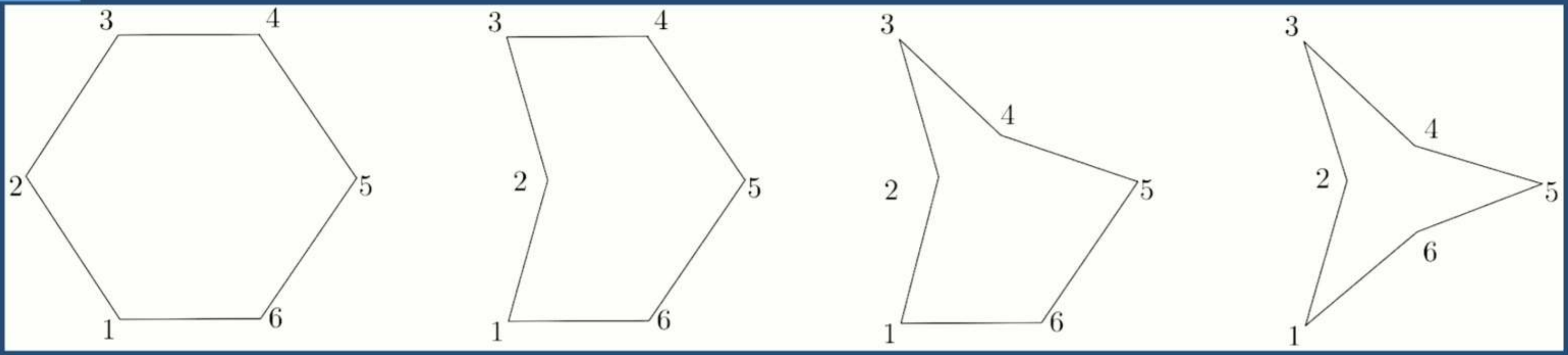
```
parkeTaylor[{1, 2, 3, 4}] + parkeTaylor[{1, 3, 4, 2}] + parkeTaylor[{1, 4, 2, 3}]
```

$$\text{Factor}\left[\frac{1}{(x_1 - x_2)(x_2 - x_3)(x_3 - x_4)(-x_1 + x_4)} + \frac{1}{(x_2 - x_3)(-x_1 + x_3)(x_1 - x_4)(-x_2 + x_4)} + \frac{1}{(-x_1 + x_2)(x_1 - x_3)(x_3 - x_4)(-x_2 + x_4)}\right]$$

0

$$PT_{12345} + PT_{12435} + PT_{12453} + PT_{14235} + PT_{14253} + PT_{14523} = 0$$

Positive Grassmannians (and slightly beyond)



- Cachazo-E-Zhang (CEZ) construction: for each deformation class of 6 generic points in projective plane, a rational function (integrand) on G(3,6).
- 372 connected components of X(3,6) determine a system of 372 "integrands" which are subject to certain gluing conditions.
- Above, p123,p345,p156 change sign, in turn from left to right.

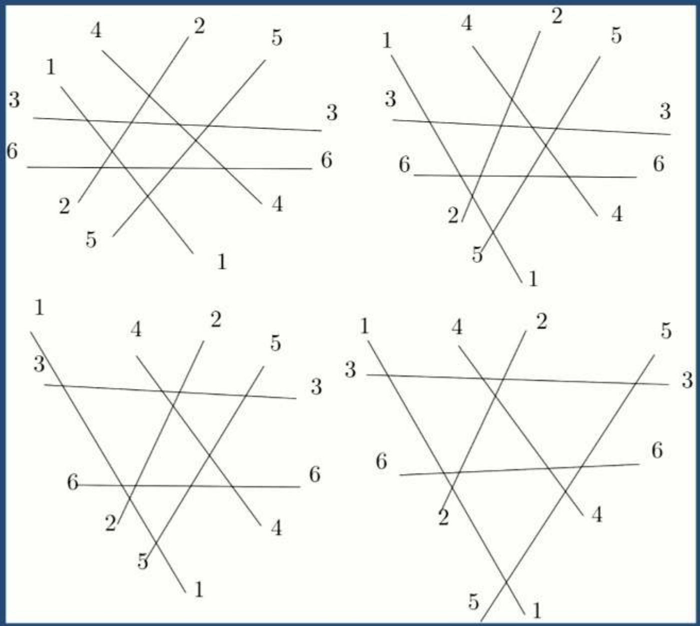
Next: 372 integrands...

CEGM Integrands

1. Four permutation classes of integrands below, corresponding to generic deformation classes of arrangements of 6 lines.

   Each encodes an amplitude, via CEGM (scattering equations formula).

2. Denominators are constructed from <u>triangles</u> in generic line arrangements.



$$\frac{1}{p_{123}\,p_{234}\,p_{345}\,p_{456}\,p_{561}\,p_{612}} \qquad \frac{1}{p_{125}\,p_{126}\,p_{136}\,p_{234}\,p_{345}\,p_{456}}$$

$$\frac{p_{123}}{p_{125}\,p_{126}\,p_{134}\,p_{136}\,p_{234}\,p_{235}\,p_{456}} \qquad \frac{-p_{134}\,p_{156}\,p_{236}\,p_{245}+p_{136}\,p_{145}\,p_{234}\,p_{256}}{p_{125}\,p_{126}\,p_{134}\,p_{136}\,p_{145}\,p_{234}\,p_{235}\,p_{246}\,p_{356}\,p_{456}}$$

1. The four resulting amplitudes are curvy polytopes with 16,14,14,15 facets and 48,41,44,45 vertices, respectively.
2. There are a total of 372 chambers (and CEGM integrands), and together they <u>tile</u> the configuration space of 6 points in the projective plane.

U(1) decoupling and extensions of oriented matroids

1. Fix a generic arrangement of n-1 projective lines. Now add a new generic line in all possible ways.
2. For G(3,6), we get 31 possible arrangements.
3. By analogy with the G(2,n) story, this should give rise to identities among CEGM integrands.
4. But the identities are not irreducible!
5. Cachazo-E-Zhang developed algorithms to produce irreducible ("decoupling") identities.
6. Here is an irreducible identity among those appearing among a set of 31:

$$-\frac{1}{p_{125}\,p_{146}\,p_{156}\,p_{234}\,p_{236}\,p_{345}}-\frac{p_{126}}{p_{123}\,p_{125}\,p_{146}\,p_{156}\,p_{236}\,p_{246}\,p_{345}}+\frac{1}{p_{123}\,p_{146}\,p_{156}\,p_{234}\,p_{256}\,p_{345}}-\frac{1}{p_{123}\,p_{125}\,p_{156}\,p_{246}\,p_{345}\,p_{346}}$$

$$+\frac{p_{235}}{p_{123}\,p_{125}\,p_{146}\,p_{234}\,p_{256}\,p_{345}\,p_{356}}-\frac{1}{p_{123}\,p_{125}\,p_{156}\,p_{234}\,p_{346}\,p_{456}}+\frac{1}{p_{123}\,p_{125}\,p_{146}\,p_{234}\,p_{356}\,p_{456}}=0$$

```
parkeTaylor[{1, 2, 3, 4}] + parkeTaylor[{1, 3, 4, 2}] + parkeTaylor[{1, 4, 2, 3}]
```

$$\text{Factor}\left[\frac{1}{(x_1-x_2)\,(x_2-x_3)\,(x_3-x_4)\,(-x_1+x_4)}+\frac{1}{(x_2-x_3)\,(-x_1+x_3)\,(x_1-x_4)\,(-x_2+x_4)}+\frac{1}{(-x_1+x_2)\,(x_1-x_3)\,(x_3-x_4)\,(-x_2+x_4)}\right]$$

0

Moduli space Y(3,n)

1. Del Pezzo surface: blow up projective space at n points.
2. Y(3,n), n<8: moduli space of del Pezzo surfaces: in addition to no three points on a line, no six on a conic.

$q := p_{123}\, p_{156}\, p_{246}\, p_{345} - p_{126}\, p_{135}\, p_{234}\, p_{456}$

$q = 0 \Rightarrow$ all six points on a conic

Let $Y(3,n)$ denote the moduli space of configurations of n points in general position in the complex projective plane $\mathbb{P}^2$. For us, this means that no three points are collinear and no six lie on a conic. When $n \leq 7$, the space $Y(3,n)$ parametrizes marked del Pezzo surfaces of degree $9-n$ as the blow up of $\mathbb{P}^2$ at the n points. From $Y(3,8)$ one obtains the moduli space by requiring that the eight points are not on a cubic that is singular at one of the points.

del Pezzo Geometry

1. Each of the 60 positive parts in X(3,6) split into two components by removing the coconic locus and we get (60+60) +120+180+12=432.
2. The del Pezzo moduli space Y(3,6) is what results.
3. The E6 Weyl group acts transitively on the 432 components, with stabilizer the symmetric group on 5 elements!
4. Each component is a simple polytope, a pezzotope, with 15 facets and 45 vertices.
5. Ten facets are 3-dimensional associahedra; five facets are cubes.

Cachazo-E (2022) defined a new integrand: $$\frac{1}{p_{123}p_{234}p_{345}p_{456}p_{156}p_{126}}\left[\frac{p_{123}p_{156}p_{246}p_{345}}{p_{126}p_{135}p_{234}p_{456}}-1\right]^{-1} = \frac{p_{135}}{p_{123}p_{345}p_{156}\, q}$$

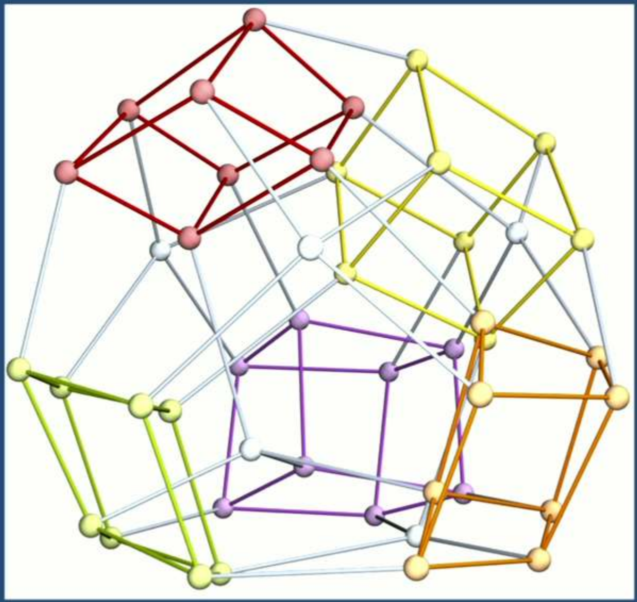
Theorem [CE2022]. Residue of CEGM amplitude corresponding to the q-parameter is the standard n=6 point biadjoint scalar.

Theorem. E-Geiger-Panizzut-Sturmfels-Yun: this integrand has poles on the 15 facets of the pezzotope.

Pezzotopes

Theorem
The moduli space $Y(3,6)$ *has* 432 *regions, all* $W(E_6)$*-equivalent, each homeomorphic to a* 4*-polytope with* $f = (45,90,60,15)$.

- One of these regions is depicted at right...



u-variables → curvy polytope

- We want a differential form which has simple residues on (and only on) the boundaries of the pezzotope. Every residue of this form should be the canonical form of either a cube or an associahedron.
- Outline of the derivation of the canonical form:
  - Input: collection of 15 poles.
  - Additional data (“ from the universe”): bijection between the 16 rays of the positive tropical Grassmannian and certain generalized cross-ratios w_abc (NOT cluster X-variables!!).
  - Main idea is to find a suitable modification of that bijection…
  - Output: the set of u-variables which satisfy u-equations.

$$w_{346} = \frac{p_{134}\,p_{356}}{p_{135}\,p_{346}}, \quad w_{246} = \frac{p_{124}\,p_{135}\,p_{256}\,p_{346}}{p_{125}\,p_{134}\,p_{246}\,p_{356}}, \quad w_{124} = \frac{p_{125}\,p_{134}}{p_{124}\,p_{135}}, \quad \ldots$$

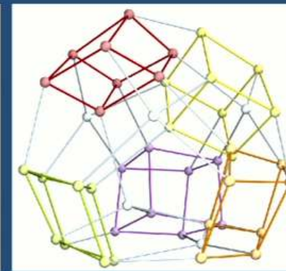
$$u_1 = \frac{-q}{p_{126}p_{135}p_{234}p_{456}}, \quad u_2 = \frac{p_{134}p_{156}p_{235}p_{246}}{p_{135}p_{146}p_{234}p_{256}}, \quad u_3 = \frac{p_{134}p_{356}}{p_{135}p_{346}}, \quad u_4 = \frac{p_{136}p_{145}}{p_{135}p_{146}}, \quad u_5 = \frac{p_{125}p_{136}p_{246}p_{345}}{p_{126}p_{135}p_{245}p_{346}},$$

$$u_6 = \frac{p_{136}p_{235}}{p_{135}p_{236}}, \quad u_7 = \frac{p_{123}p_{145}p_{246}p_{356}}{p_{124}p_{135}p_{236}p_{456}}, \quad u_8 = \frac{p_{125}p_{356}}{p_{135}p_{256}}, \quad u_9 = \frac{p_{125}p_{134}}{p_{124}p_{135}}, \quad u_{10} = \frac{p_{145}p_{235}}{p_{135}p_{245}},$$

$$u_{11} = \frac{p_{135}p_{234}}{p_{134}p_{235}}, \quad u_{12} = \frac{p_{135}p_{456}}{p_{145}p_{356}}, \quad u_{13} = \frac{p_{124}p_{135}p_{256}p_{346}}{p_{125}p_{134}p_{246}p_{356}}, \quad u_{14} = \frac{p_{126}p_{135}}{p_{125}p_{136}}, \quad u_{15} = \frac{p_{135}p_{146}p_{236}p_{245}}{p_{136}p_{145}p_{235}p_{246}}$$

Result: description of the Pezzotope

Theorem 8.1. *The real moduli space $Y(3,6)$ has 432 regions, all $W(E_6)$-equivalent, the closure of each is homeomorphic as cell-complex to a 4-dimensional polytope with f-vector $f = (45, 90, 60, 15)$. The real moduli space $Y(3,7)$ has 60480 regions, all $W(E_7)$-equivalent, the closure of each is homeomorphic as cell-complex to a 6-dimensional curvy polytope with $f = (579, 1737, 2000, 1105, 297, 34)$. Its boundary is a simple homology sphere of dimension 5.*



Theorem 8.2. *The following u-equations define a perfect binary geometry (cf. [5]):*

$$u_1 + u_2u_5u_7u_{13}u_{15} = u_2 + u_1u_4u_8u_{12}u_{14} = u_3 + u_4u_5u_6u_{14}u_{15} = u_4 + u_2u_3u_9u_{11}u_{13} =$$
$$u_5 + u_1u_3u_{10}u_{11}u_{12} = u_6 + u_3u_7u_8u_{12}u_{13} = u_7 + u_1u_6u_9u_{11}u_{14} = u_8 + u_2u_6u_{10}u_{11}u_{15} =$$
$$u_9 + u_4u_7u_{10}u_{12}u_{15} = u_{10} + u_5u_8u_9u_{13}u_{14} = u_{11} + u_4u_5u_7u_8u_{12}u_{13}u_{14}u_{15} =$$
$$u_{12} + u_2u_5u_6u_9u_{11}u_{13}u_{14}u_{15} = u_{13} + u_1u_4u_6u_{10}u_{11}u_{12}u_{14}u_{15} =$$
$$u_{14} + u_2u_3u_7u_{10}u_{11}u_{12}u_{13}u_{15} = u_{15} + u_1u_3u_8u_9u_{11}u_{12}u_{13}u_{14} = 1.$$

Canonical form on Y+(3,6)

$$\Omega = d\log\left(\frac{u_{10}}{u_5u_8u_9u_{13}u_{14}}\right)\wedge d\log\left(\frac{u_9u_{11}}{u_4u_7u_{12}u_{15}}\right)\wedge d\log\left(\frac{u_4u_6u_{14}u_{15}}{u_3u_{13}}\right)\wedge d\log\left(\frac{u_1u_4u_8u_{12}u_{14}}{u_2}\right)$$

- This differential form is proportional to the rational function below:

$$\frac{1}{p_{123}p_{234}p_{345}p_{456}p_{156}p_{126}}\left[\frac{p_{123}p_{156}p_{246}p_{345}}{p_{126}p_{135}p_{234}p_{456}}-1\right]^{-1} = \frac{p_{135}}{p_{123}p_{345}p_{156}\,q}$$

Two combinatorial types of residues: canonical form of a cube, and of an associahedron

$$d\log\left(\frac{u_{10}}{u_9}\right)\wedge d\log\left(\frac{u_6}{u_3}\right)\wedge d\log\left(\frac{u_1}{u_2}\right)$$

$$d\log\left(\frac{u_{10}}{u_8u_9u_{14}}\right)\wedge d\log\left(\frac{u_9u_{11}}{u_4u_{12}}\right)\wedge d\log\left(\frac{u_4u_6u_{14}}{u_3}\right)$$

Embedding and amplitude

$\mathbf{f}_{abcd} = \mathbf{e}_{abc} + \mathbf{e}_{abd} + \mathbf{e}_{acd} + \mathbf{e}_{bcd}$

$$s_2 = \mathbf{e}_{156} \quad s_3 = \mathbf{f}_{1234} \quad s_4 = \mathbf{f}_{1236} \quad s_5 = \mathbf{e}_{345} \quad s_6 = \mathbf{f}_{2345} \quad s_7 = \mathbf{e}_{123} \quad s_8 = \mathbf{f}_{3456}$$
$$s_9 = \mathbf{f}_{1256} \quad s_{10} = \mathbf{f}_{1456} \quad s_{11} = \mathbf{e}_{234}+\mathbf{e}_{156} \quad s_{12} = \mathbf{e}_{123}+\mathbf{e}_{456} \quad s_{13} = \mathbf{g}_{12,34,56} \quad s_{14} = \mathbf{e}_{126}+\mathbf{e}_{345} \quad s_{15} = \mathbf{g}_{16,45,23}$$

The E_6 amplitude is the following sum over 45 terms, one for each vertex of the pezzotope:

$$\frac{1}{s_1 s_3 s_8 s_9} + \frac{1}{s_1 s_3 s_8 s_{12}} + \frac{1}{s_1 s_3 s_9 s_{11}} + \frac{1}{s_1 s_3 s_{10} s_{11}} + \frac{1}{s_1 s_3 s_{10} s_{12}} + \frac{1}{s_1 s_4 s_6 s_{10}} + \frac{1}{s_1 s_4 s_6 s_{14}} + \frac{1}{s_1 s_4 s_8 s_{12}}$$
$$+ \frac{1}{s_1 s_4 s_8 s_{14}} + \frac{1}{s_1 s_4 s_{10} s_{12}} + \frac{1}{s_1 s_6 s_9 s_{11}} + \frac{1}{s_1 s_6 s_9 s_{14}} + \frac{1}{s_1 s_6 s_{10} s_{11}} + \frac{1}{s_1 s_8 s_9 s_{14}} + \frac{1}{s_2 s_3 s_7 s_{10}} + \frac{1}{s_2 s_3 s_7 s_{13}}$$
$$+ \frac{1}{s_2 s_3 s_9 s_{11}} + \frac{1}{s_2 s_3 s_9 s_{13}} + \frac{1}{s_2 s_3 s_{10} s_{11}} + \frac{1}{s_2 s_5 s_6 s_9} + \frac{1}{s_2 s_5 s_6 s_{15}} + \frac{1}{s_2 s_5 s_7 s_{13}} + \frac{1}{s_2 s_5 s_7 s_{15}} + \frac{1}{s_2 s_5 s_9 s_{13}}$$
$$+ \frac{1}{s_2 s_6 s_9 s_{11}} + \frac{1}{s_2 s_6 s_{10} s_{11}} + \frac{1}{s_2 s_6 s_{10} s_{15}} + \frac{1}{s_2 s_7 s_{10} s_{15}} + \frac{1}{s_3 s_7 s_8 s_{12}} + \frac{1}{s_3 s_7 s_8 s_{13}} + \frac{1}{s_3 s_7 s_{10} s_{12}} + \frac{1}{s_3 s_8 s_9 s_{13}}$$
$$+ \frac{1}{s_4 s_5 s_6 s_{14}} + \frac{1}{s_4 s_5 s_6 s_{15}} + \frac{1}{s_4 s_5 s_7 s_8} + \frac{1}{s_4 s_5 s_7 s_{15}} + \frac{1}{s_4 s_5 s_8 s_{14}} + \frac{1}{s_4 s_6 s_{10} s_{15}} + \frac{1}{s_4 s_7 s_8 s_{12}} + \frac{1}{s_4 s_7 s_{10} s_{12}}$$
$$+ \frac{1}{s_4 s_7 s_{10} s_{15}} + \frac{1}{s_5 s_6 s_9 s_{14}} + \frac{1}{s_5 s_7 s_8 s_{13}} + \frac{1}{s_5 s_8 s_9 s_{13}} + \frac{1}{s_5 s_8 s_9 s_{14}}.$$

Minkowski sum decomposition: Newton polytope

1. There's a Minkowski sum decomposition of the pezzotope!
2. This provides a Global Schinger Parameterization (in the sense of CE2020).

$1 + y_1$

$1 + y_2$

$1 + y_2 + y_1 y_2$

$1 + y_3$

$1 + y_3 + y_2 y_3$

$1 + y_4$

$1 + y_4 + y_2 y_4 + y_1 y_2 y_4$

$1 + y_3 + y_2 y_3 + y_4 + y_2 y_4 + y_3 y_4 + y_2 y_3 y_4$

$1 + y_3 + y_2 y_3 + y_4 + y_2 y_4 + y_1 y_2 y_4 + y_3 y_4 + y_2 y_3 y_4$

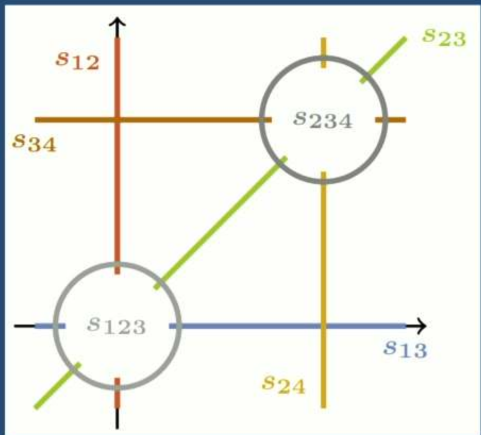
$1 + y_3 + y_2 y_3 + y_4 + y_2 y_4 + y_1 y_2 y_4 + y_3 y_4 + y_2 y_3 y_4 + y_1 y_2 y_3 y_4$

$1 + y_3 + y_2 y_3 + y_1 y_2 y_3 + y_4 + y_2 y_4 + y_1 y_2 y_4 + y_3 y_4 + y_2 y_3 y_4 + y_1 y_2 y_3 y_4$

```
sage: P.f_vector()
(1, 45, 90, 60, 15, 1)
sage: P.is_simple()
True
sage: P.facets()
(A 3-dimensional face of a Polyhedron in ZZ^4 defined as the convex hull of 14 vertices,
 A 3-dimensional face of a Polyhedron in ZZ^4 defined as the convex hull of 8 vertices,
 A 3-dimensional face of a Polyhedron in ZZ^4 defined as the convex hull of 14 vertices,
 A 3-dimensional face of a Polyhedron in ZZ^4 defined as the convex hull of 14 vertices,
 A 3-dimensional face of a Polyhedron in ZZ^4 defined as the convex hull of 14 vertices,
 A 3-dimensional face of a Polyhedron in ZZ^4 defined as the convex hull of 14 vertices,
 A 3-dimensional face of a Polyhedron in ZZ^4 defined as the convex hull of 14 vertices,
 A 3-dimensional face of a Polyhedron in ZZ^4 defined as the convex hull of 8 vertices,
 A 3-dimensional face of a Polyhedron in ZZ^4 defined as the convex hull of 14 vertices,
 A 3-dimensional face of a Polyhedron in ZZ^4 defined as the convex hull of 8 vertices,
 A 3-dimensional face of a Polyhedron in ZZ^4 defined as the convex hull of 8 vertices,
 A 3-dimensional face of a Polyhedron in ZZ^4 defined as the convex hull of 14 vertices,
 A 3-dimensional face of a Polyhedron in ZZ^4 defined as the convex hull of 8 vertices,
 A 3-dimensional face of a Polyhedron in ZZ^4 defined as the convex hull of 14 vertices,
 A 3-dimensional face of a Polyhedron in ZZ^4 defined as the convex hull of 14 vertices)
```

Summary

1. Connected components of X(2,n)=M(0,n) =G(2,n)/T are (curvy) associahedra.
2. Connected components of X(3,n) = G(3,n)/T via chirotopes (aka oriented matroids), n=5,6,7,8.
3. [CEZ 2022;2023]: nonplanar amplitudes, one for each connected component of X(3,n) for n=6,7,8.
   - Theorem. For n=6,7,8: explicit combinatorial formula for the CEGM integral in terms of a certain chirotopal tropical Grassmannian. **This generalizes the positive tropical Grassmannian (to other oriented matroids)!**
4. Del Pezzo moduli space: Y(3,n) (no 3 points collinear, no 6 on a conic)
   - Thm. (E-Geiger-Panizzut-Sturmfels-Yun) 2023)) Y+(3,6) is a positive geometry; it is a *pezzotope* (a simple polytope, f=(45,60,90,15)).


s12
s23
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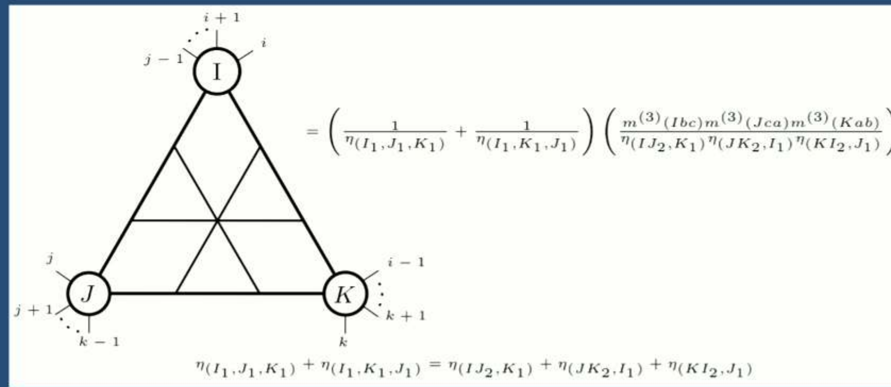
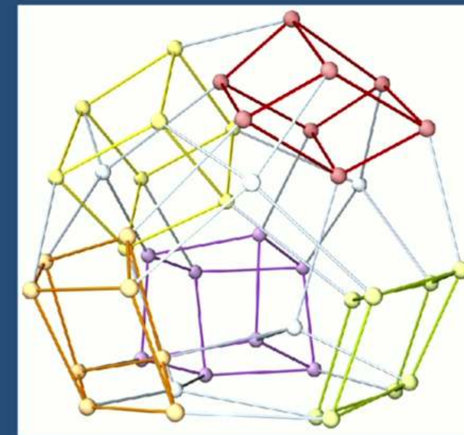
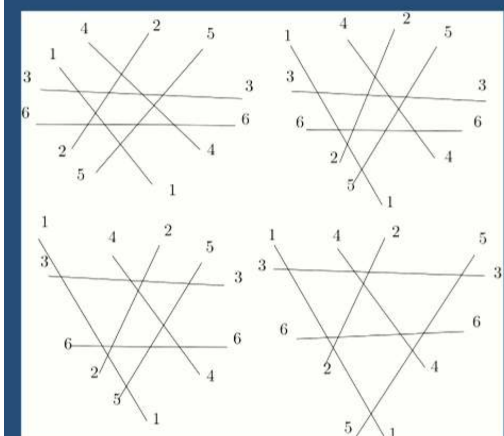
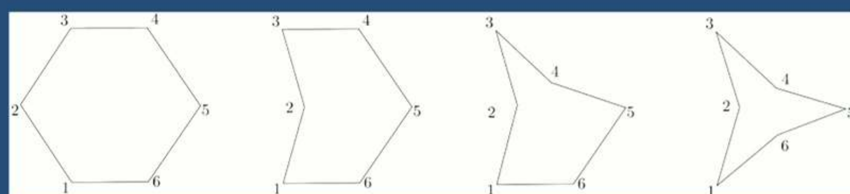
s24

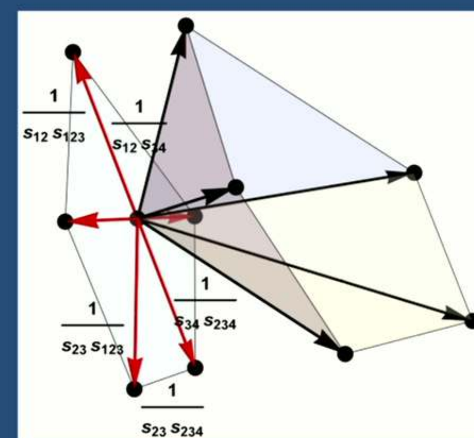

i + 1
j − 1
i
I
j
J
j + 1
k − 1
K
k
i − 1
k + 1
= (1/η(I1,J1,K1) + 1/η(I1,K1,J1)) (m(3)(Ibc) m(3)(Jca) m(3)(Kab) / η(IJ2,K1) η(JK2,I1) η(KI2,J1))
η(I1,J1,K1) + η(I1,K1,J1) = η(IJ2,K1) + η(JK2,I1) + η(KI2,J1)



Thanks!






1/s12 s123
1/s12 s34
1/s34 s234
1/s23 s123
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