Title: Ultraslow dynamics, fragile fragmentation, and geometric group theory

Speakers: Ethan Lake

Series: Quantum Matter

Date: September 21, 2023 - 11:00 AM

URL: https://pirsa.org/23090106

Abstract: An ongoing program of work in statistical physics and quantum dynamics is concerned with understanding the character of systems which follow an unconventional approach towards thermal equilibrium. In this talk, I will add to this story by introducing examples of simple 1D systems---both classical and quantum---which thermalize in very unusual ways. These examples have dynamics which is strictly local and translation-invariant, but in spite of this, they: a) can have very long thermalization times, with expectation values of local operators relaxing only over times exponential in the system size; and b) can thermalize only when they are placed in extremely large baths, with the required bath size growing exponentially (or even faster) in system size. Proofs of these results will be given using techniques from geometric group theory, a beautiful area of mathematics concerned with the complexity and geometry of infinite discrete groups. This talk will be based on a paper in preparation with Shankar Balasubramanian, Sarang Golaparakrishnan, and Alexey Khudorozhkov.

Zoom link: https://pitp.zoom.us/j/99430001465?pwd=NENlS1M5UGc5UWM1ekQvRWFrZGYyUT09

Ultraslow dynamics and geometric group theory

Ethan Lake (Berkeley)
PI, Sep 21 2023

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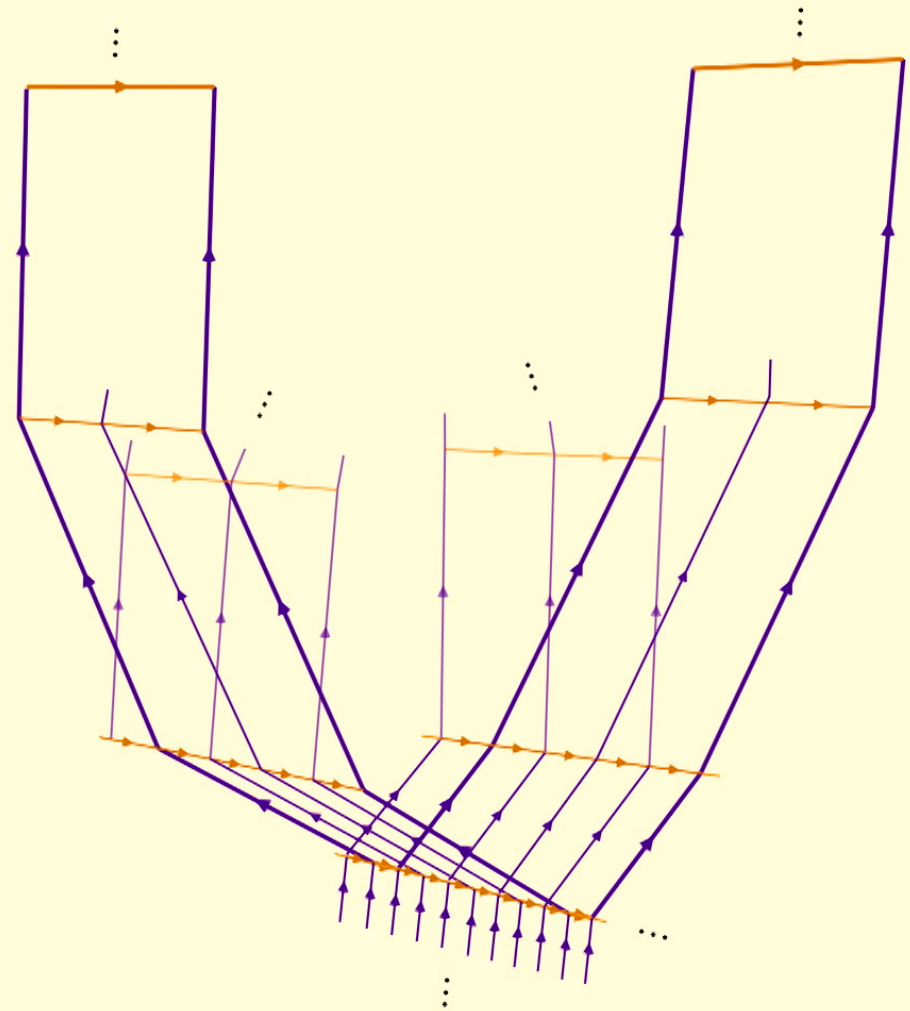
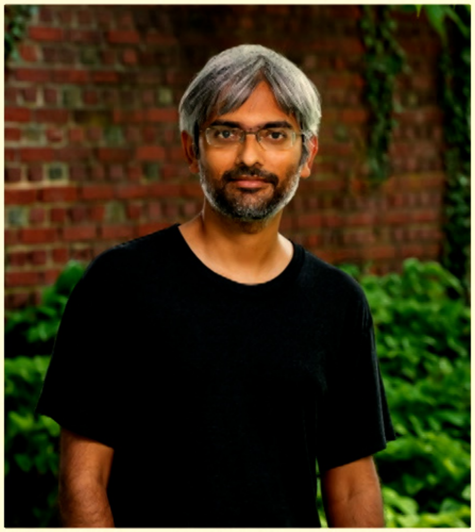


meet the team

Shankar Balasubramanian (MIT)

Alexey Khudorozhkov (BU)



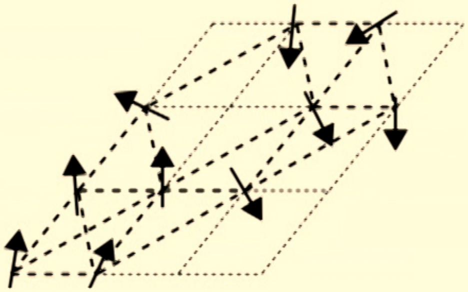
Sarang Gopalakrishnan (Princeton)

fundamental physics question: how fast do systems equilibrate?

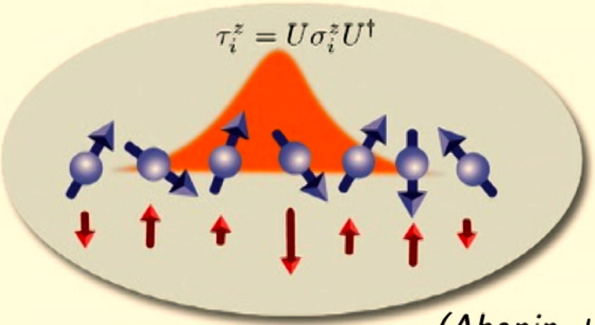
Everyday experience tells us that the answer is "for typical systems, fairly quickly".

Most interesting are systems which equilibrate slowly (or not at all), e.g.

structural glasses



spin glasses



(Abanin + Papic 21)

MBL systems

thermalization game:

Design a physical system whose thermalization time τ is as long as possible.

Rules:

- *must work on a spatial lattice, with a finite dimensional Hilbert space on each site*
- *dynamics must be spatially local*
- *you must be able to* ***prove*** *a bound on the thermalization time*

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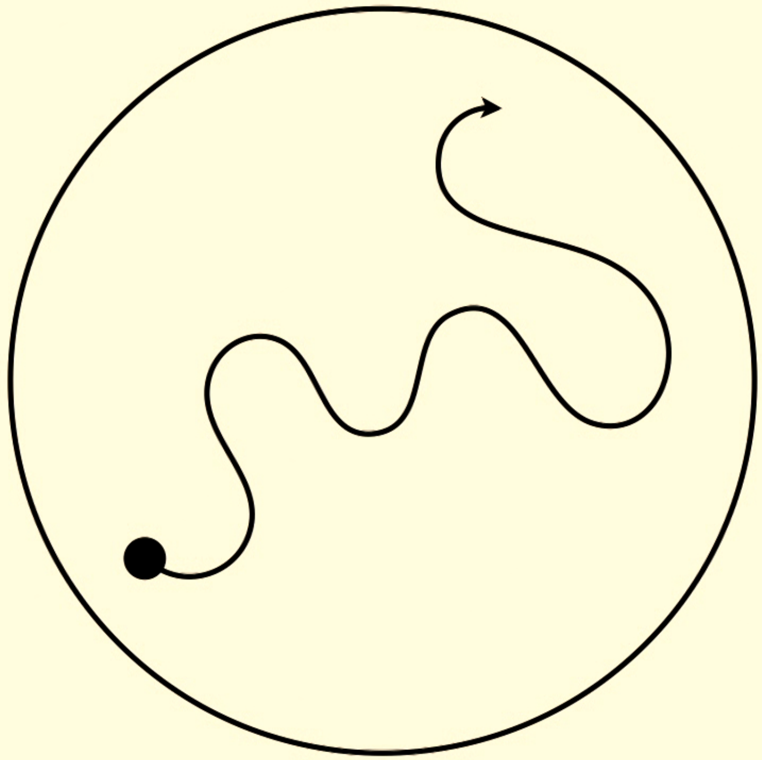
A system cannot thermalize until it is able to explore all of Hilbert space:



every state is usually only $\sim L$ *steps away from every other state.*

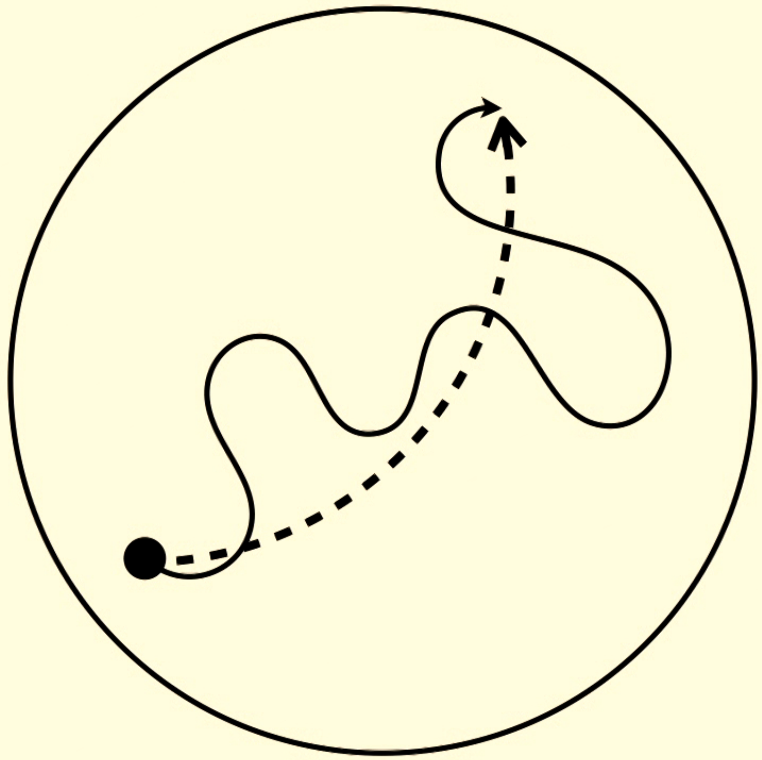
Example: Hamiltonian dynamics. What is the largest power of H needed to connect any two states?

$$H = \sum_i (J Z_i Z_{i+1} + X_i)$$

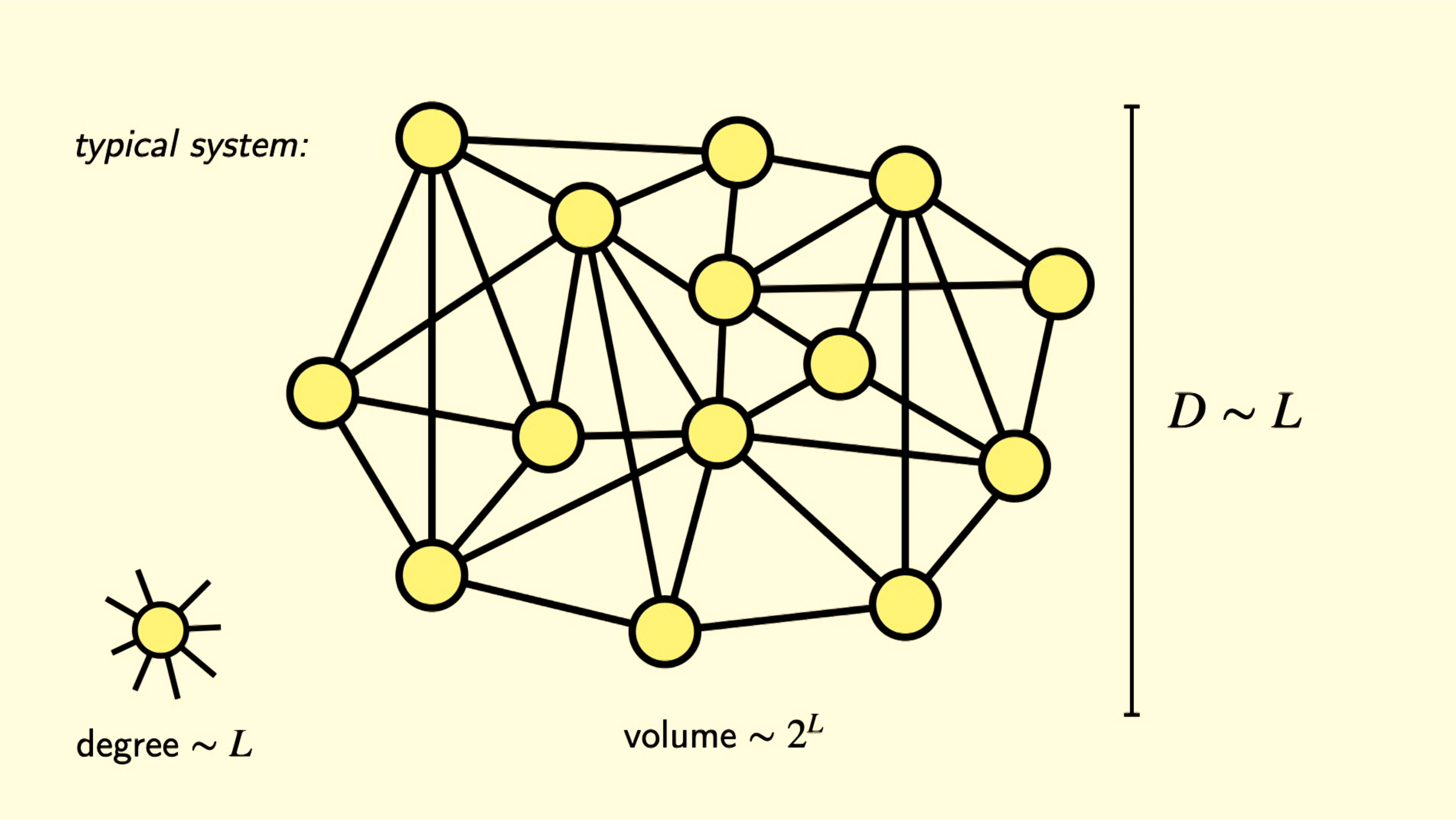
For any two Z-basis product states $|\psi_1\rangle, |\psi_2\rangle$, need *at most* L flips to connect $|\psi_1\rangle$ to $|\psi_2\rangle$:

$$\langle\psi_1|H^L|\psi_2\rangle \neq 0$$

Nope — Hilbert space is usually very highly connected!



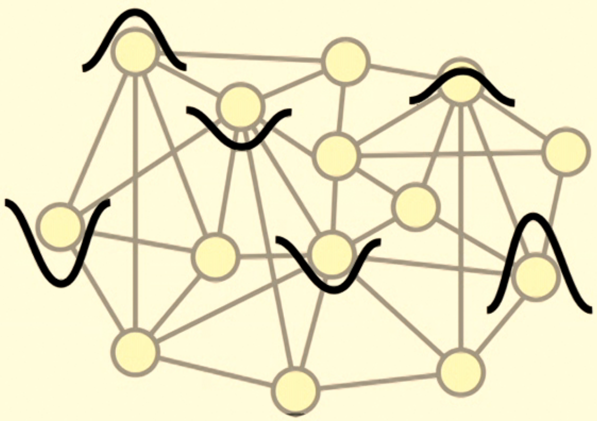
volume $\sim 2^L$


typical system:
$D \sim L$
degree $\sim L$
volume $\sim 2^L$

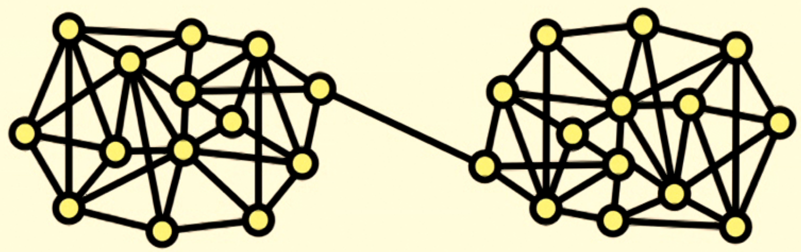
implications

If a system has $\tau > L$, it must be for a nontrivial reason.

Classically, need rough energy landscapes / bottlenecks:



hard to understand...



not very natural...

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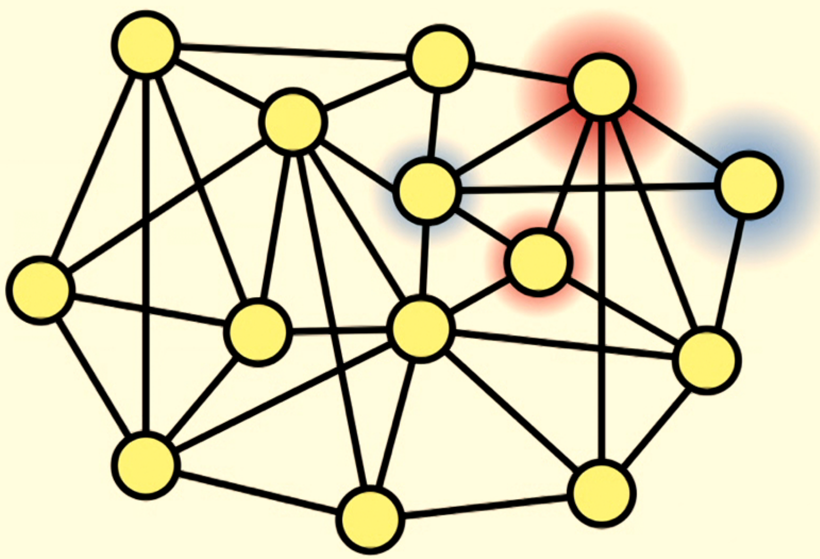
$$\langle \psi_1 | H^L | \psi_2 \rangle \neq 0$$

implications

If a system has $\tau > L$, it must be for a rather special reason!

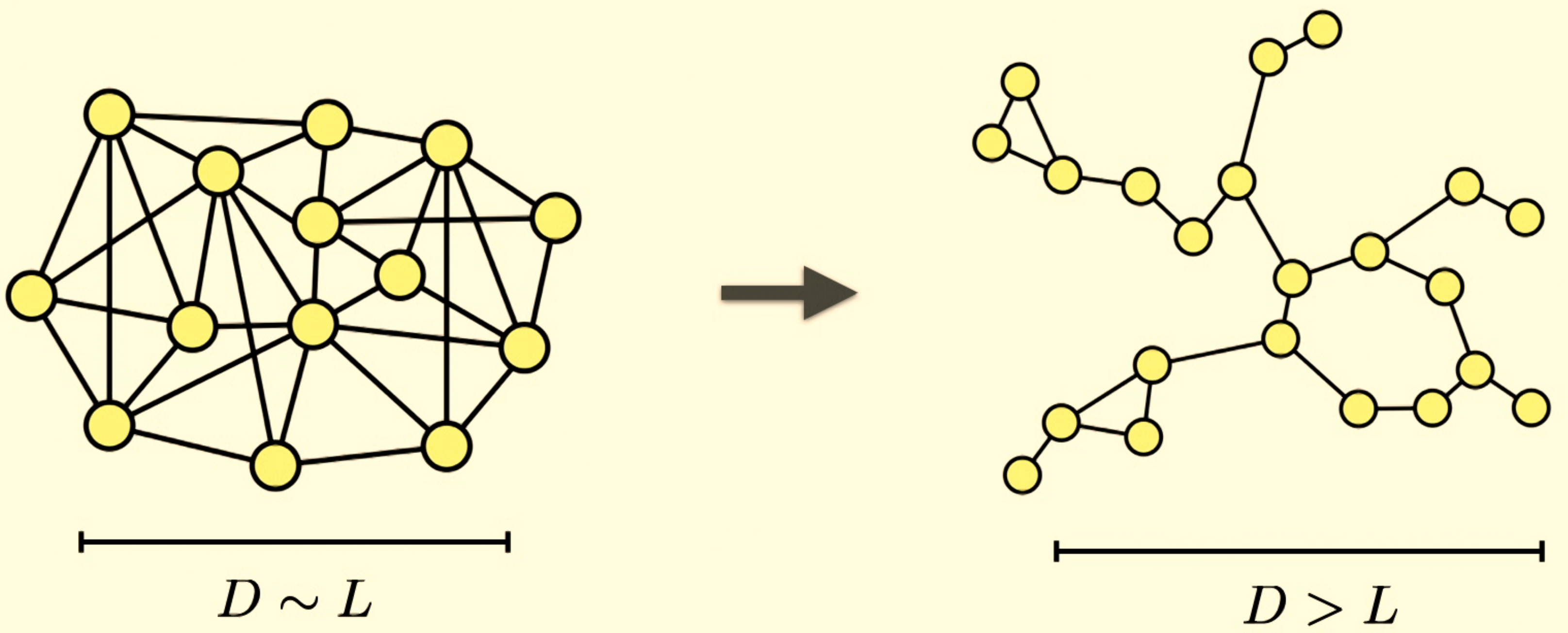
In quantum systems, can use interference:



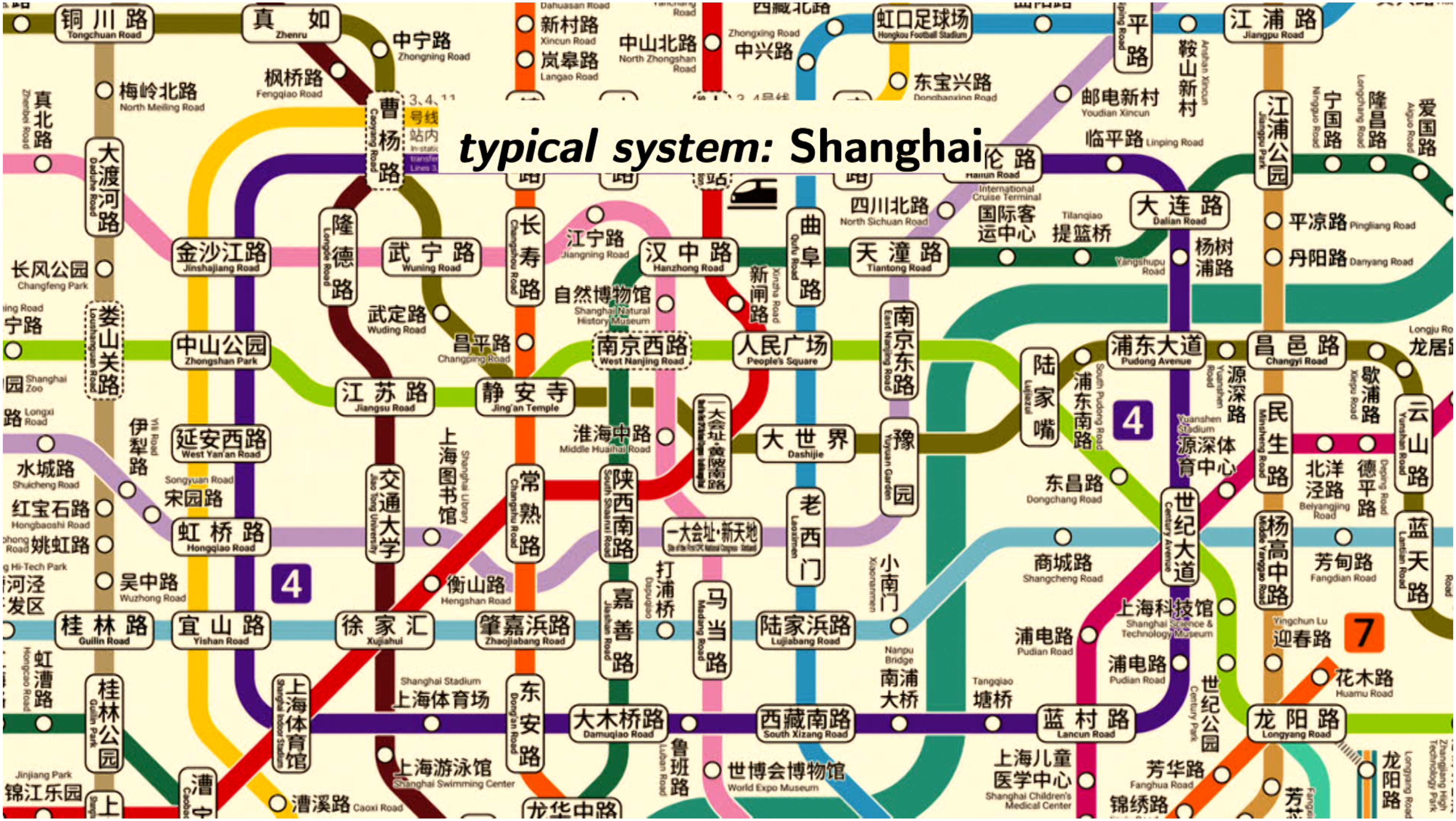
requires rough energy landscapes / very special connectivity, also hard to understand...

our strategy

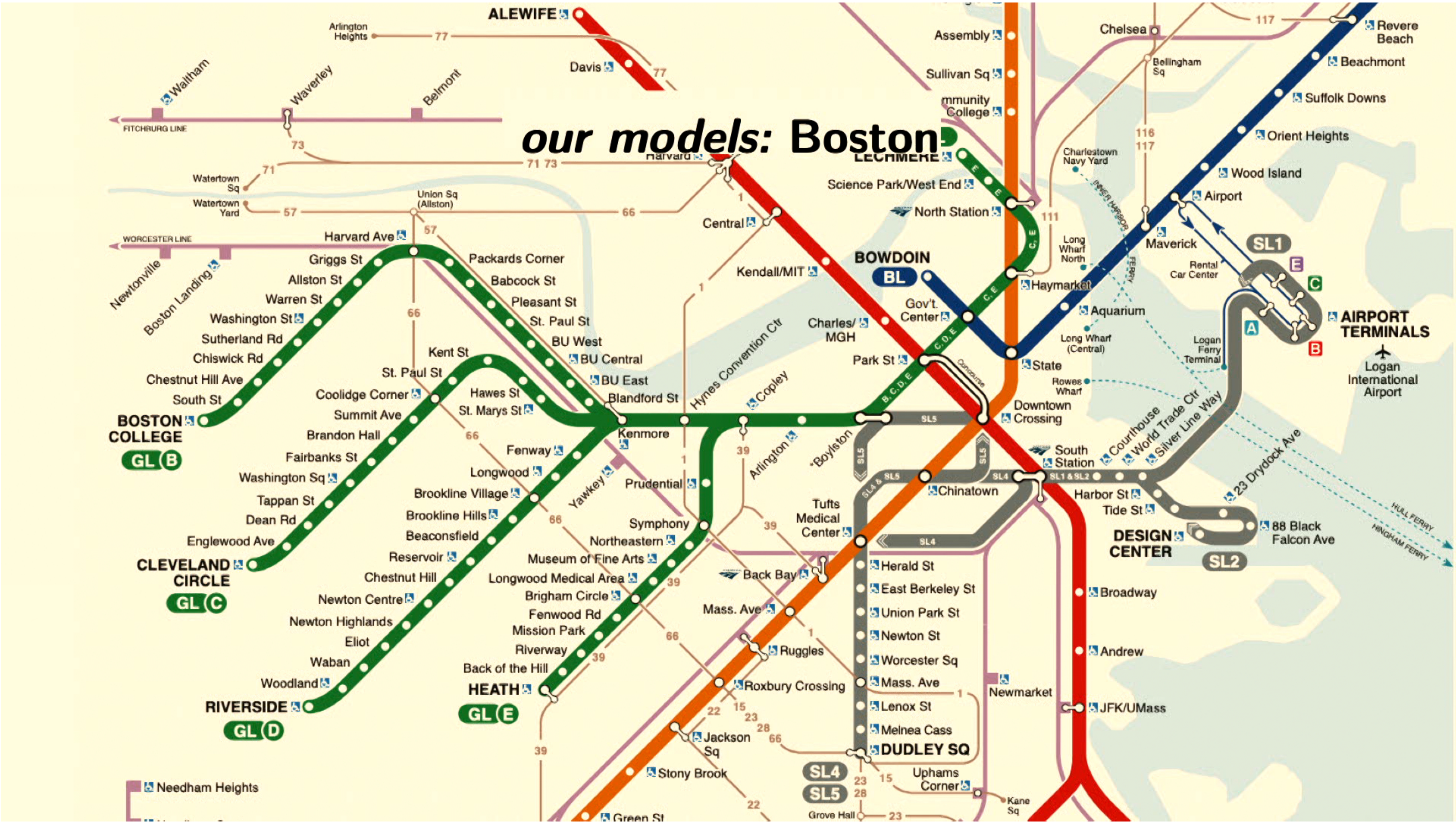
Change the connectivity of Hilbert space:



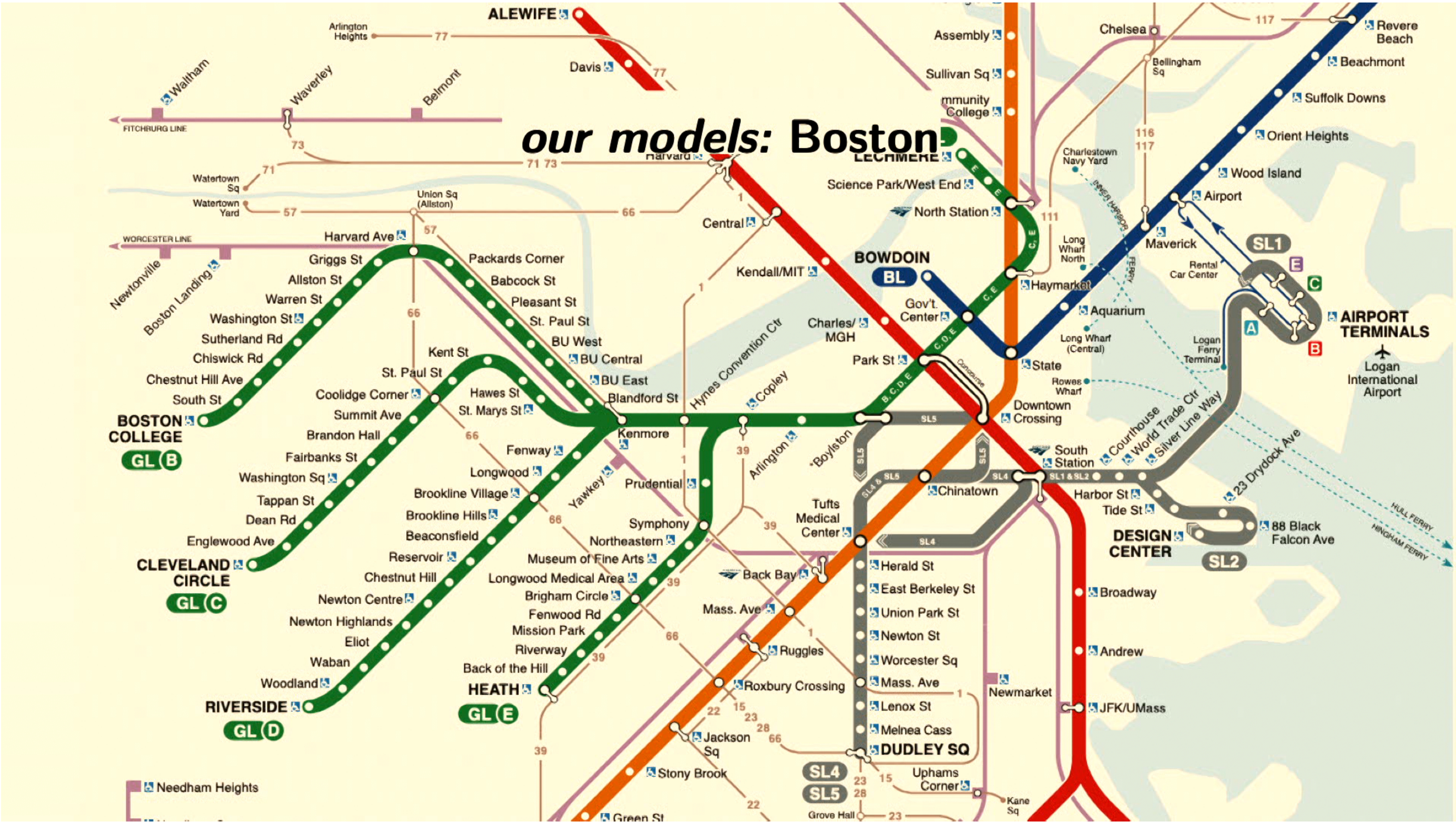
typical system: Shanghai



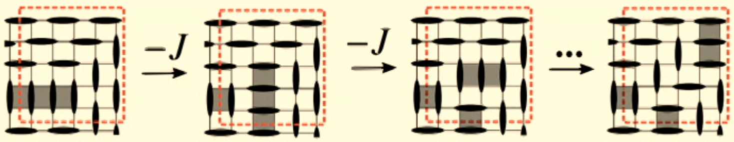
our models: Boston



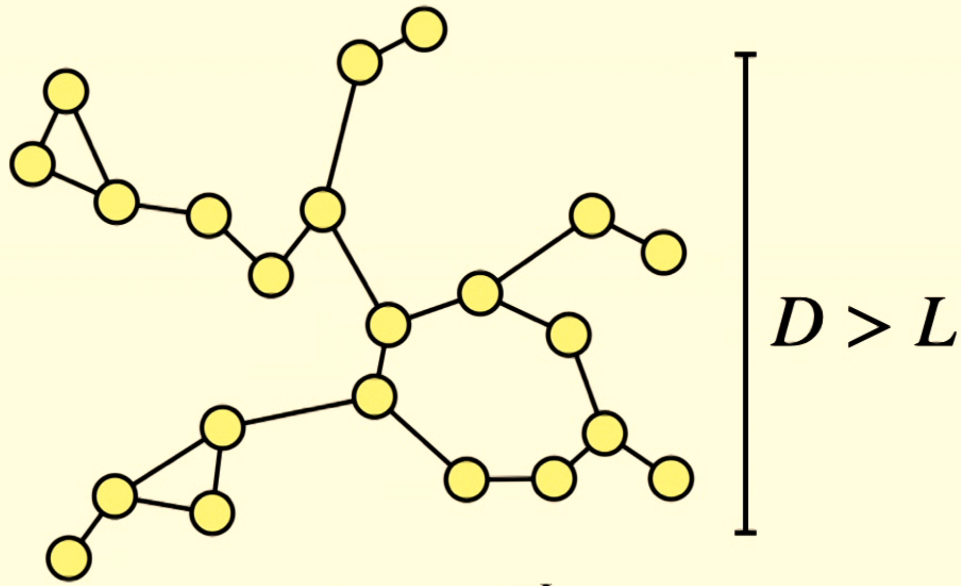
our models: Boston



The slowest examples I know of have $D \sim L^2$ (e.g. Feldmeier + 19)



$D \sim L^2$ also occurs in other models (Motzkin, pair flip, etc.)



And now... a detour into the land of *geometric group theory!*

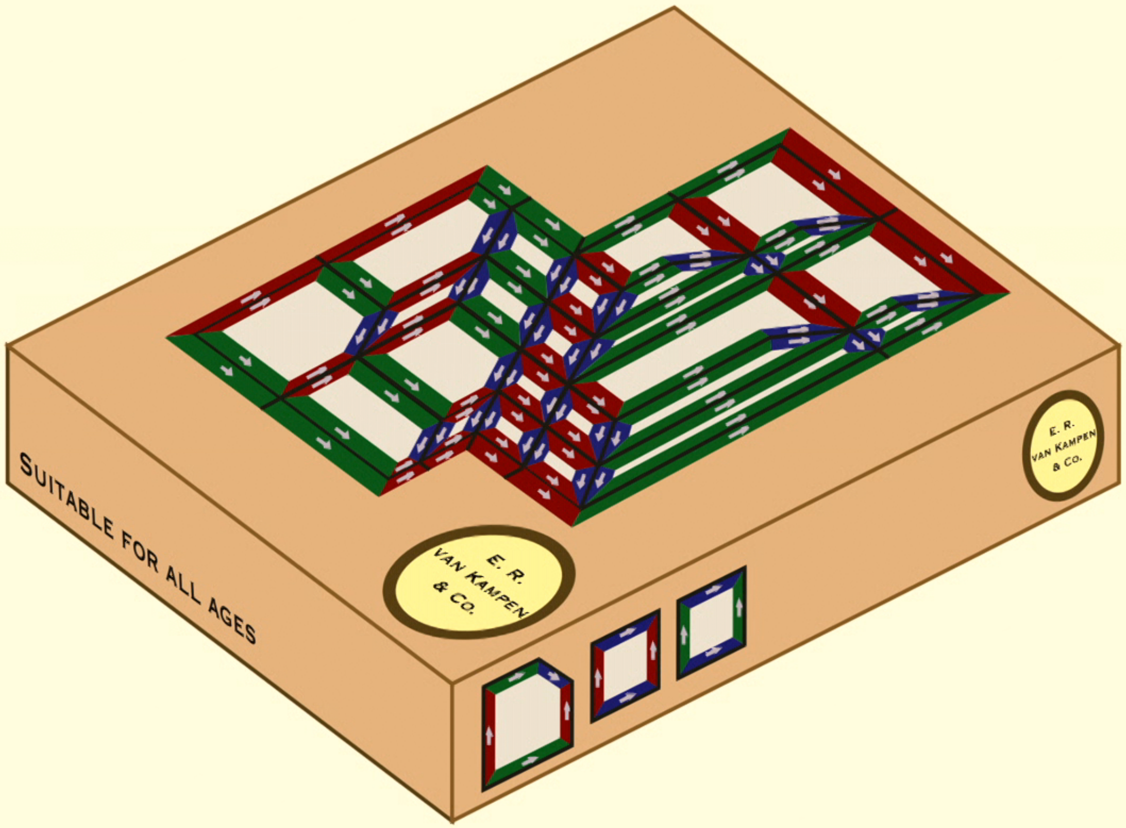


image credit: T. Riley

reminder about discrete groups:

$$G = \langle g_1, \ldots, g_n \,|\, R \,\rangle$$

<u>examples</u>

$\mathbb{Z}$	$\mathbb{Z}_N$	S_3
$\langle\, g \,|\;\; \rangle$	$\langle\, g \,|\, g^N = e \,\rangle$	$\langle\, r,\, f \,|\, r^3 = f^2 = e,\, rf = fr^{-1} \,\rangle$

Definition: a *word* w is a string of group generators, their inverses, and identities.

$$w = g_1\, e\, g_2^{-1}\, g_1\, g_3\, e\, g_1^{-1}$$

the word problem

Given a word w, does w represent the identity element?

$\mathbb{Z}$		$\mathbb{Z}_3$		S_3	
$g\, e\, g^{-1}$	$g\, e\, g$	$g\, g\, g^{-1}$	$g\, g\, g$	$f\, r\, f\, r$	$f\, r\, f\, r^{-1}$
✓	✗	✗	✓	✓	✗

how long does it take to solve the word problem?

That is, given a word $w \sim e$, how many relations need to be used to show that $w \sim e$?

$$S_3 \quad \langle r, f \,|\, r^3 = f^2 = e, rf = fr^{-1} \rangle$$

$$\begin{aligned} frfr &= frr^{-1}f \\ &= f^2 \\ &= e \end{aligned}$$

(3 steps)

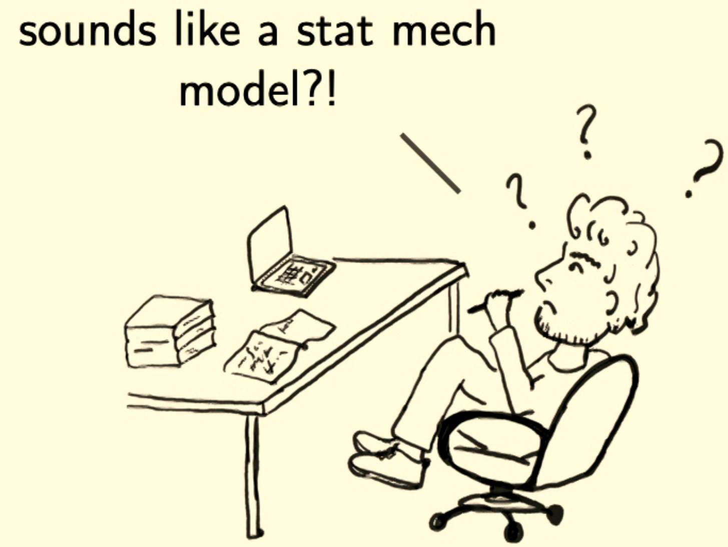
Not always so easy! Consider the following:

$$G = \langle\, a, b, c, d \,|\, ad = bc^2,\ cd^{-1} = a^{-3}b,\ b^5 d = c^{-1} \,\rangle$$

$$w = ab^2c^{-4}da^2c^2b^{-1} \sim e?$$

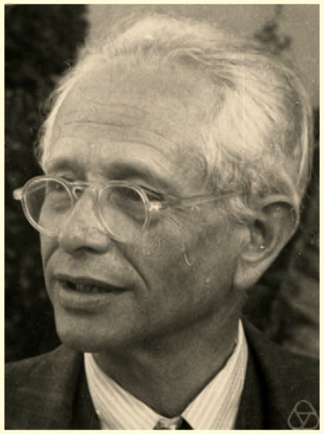
In general, the best strategy for showing that $w \sim e$ is to randomly apply local relations and wait until w is turned into e.

$$f\ e\ r^{-1}\ f\ e\ r\ e\ r$$
$$f\ e\ f\ r\ e\ r\ e\ r$$
$$f\ e\ f\ r\ e\ r\ r\ e$$
$$f\ f\ e\ r\ e\ r\ r\ e$$
$$e\ e\ e\ r\ e\ r\ r\ e$$
$$\cdots$$



Define the *Dehn function* as the time needed to solve the word problem for words of length L:

$$D(L) \equiv \max_{|w|=L} (\#\,\text{steps to show that } w \sim e)$$



Groups with larger $D(L)$ are *more complex*, i.e. have slower "word dynamics".

Even simple-to-write-down groups can be incredibly complex!

good reference: review by T. Riley

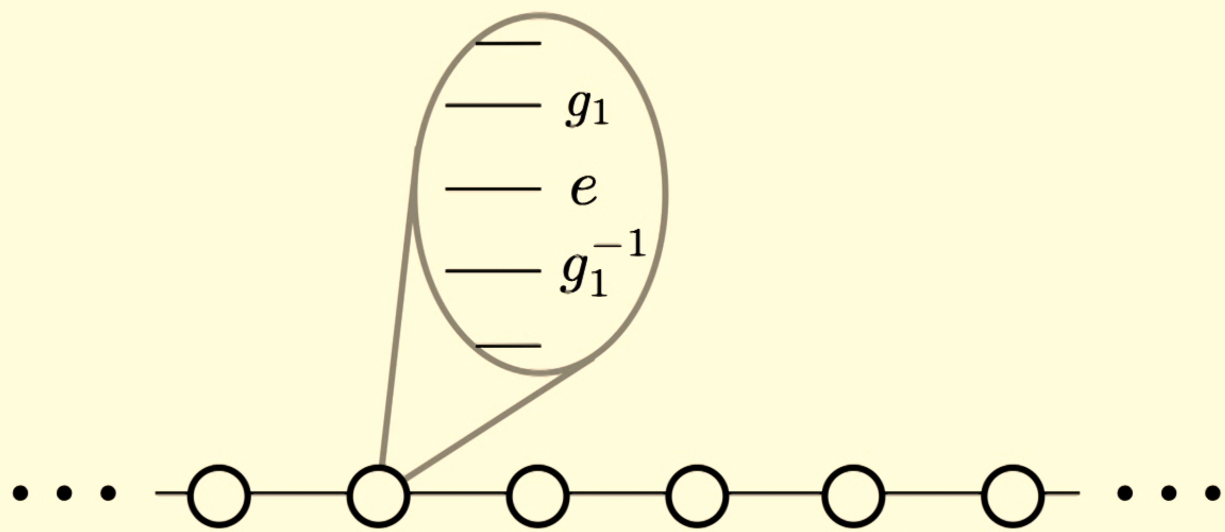
Some simple groups with large complexity: *(all infinite + non-abelian)*

G	$D(L)$
$\langle\, a,\, b \,\vert\, ab = ba^2 \,\rangle$	$\exp(L)$
$\langle\, a,\, b,\, c \,\vert\, ab = ba^2,\, bc = cb^2 \,\rangle$	$\exp(\exp(L))$
$\langle\, a,\, b \,\vert\, b^{-1}a^{-1}bab^{-1}ab = a^2 \,\rangle$	$\exp(\underbrace{\cdots(}_{\ln L}\exp(L))\cdots)$ 😱

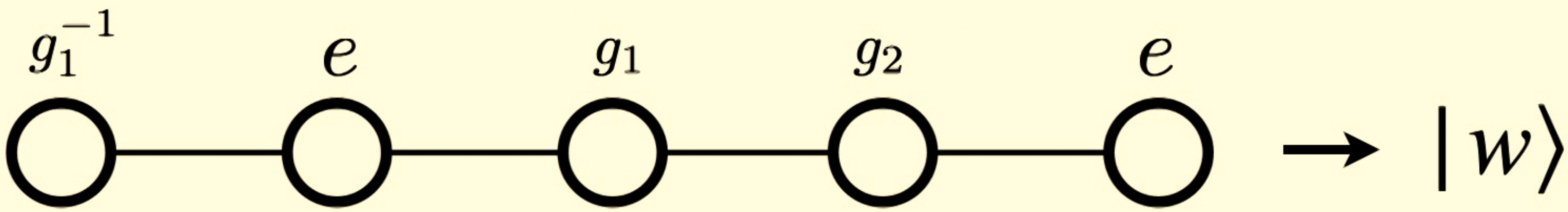
Fix a discrete group G,

$$G = \langle g_1, \ldots, g_n \,|\, R \rangle$$

Consider a spin chain whose onsite Hilbert space has one state for each generator, its inverse, and the identity:



Each product state defines a group word:



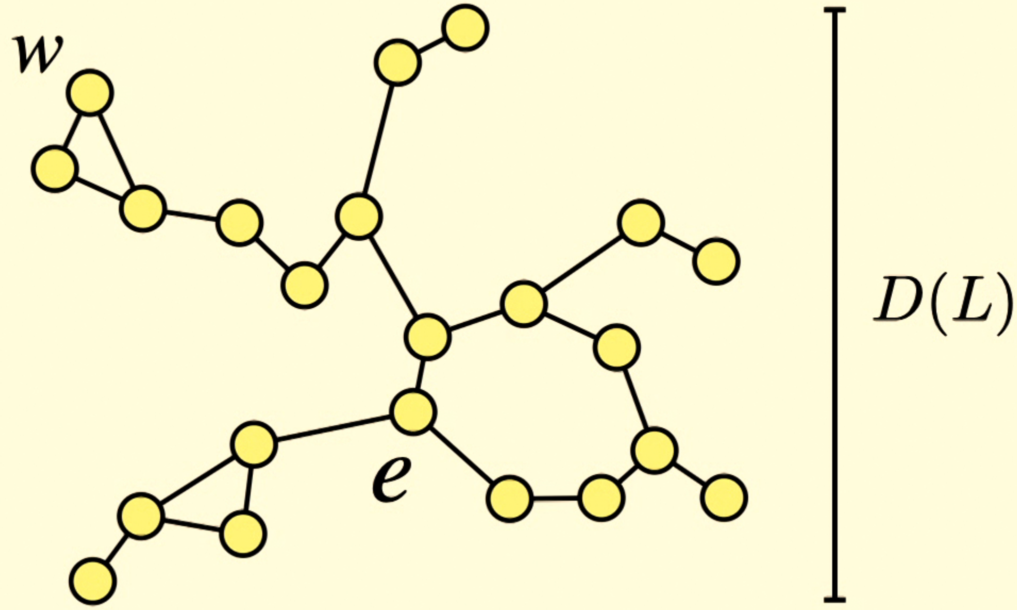
$$|g_1^{-1}, e, g_1, g_2, e\rangle$$

Hilbert space = space of group words

We will consider dynamics consisting of arbitrary local applications of group relations (Hamiltonian, RU circuit, classical Markov chain — all fine)

$$H = \sum_i \sum_{gh=kl} \lambda_i^{ghkl} |g,h\rangle\langle k,l|_{i,i+1}$$

In this system, e^{-iHt} does our "randomly apply relations and see what happens" algorithm.



volume $\sim$ (# generators)^L

thermalization time lower bounded by Dehn function

$$\tau > D(L)$$

meet the BS group

In 1962, Baumslag and Solitar discovered an amazing group:

$$BS = \langle\ a, b \mid ab = ba^2\ \rangle$$

which has $D(L) \sim 2^L$.

— b

— a

— e

— a^{-1}

— b^{-1}

$$b^{-1}ab = a^2$$

$$b^{-2}ab^2 = b^{-1}a^2b = a^4$$

$$b^{-n}ab^n = a^{2^n}$$

as we apply relations, can *exponentially* expand the length of the word!

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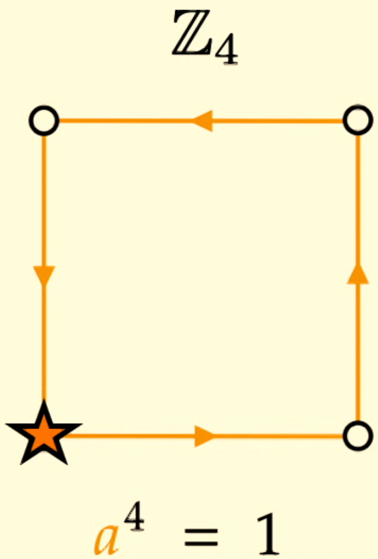
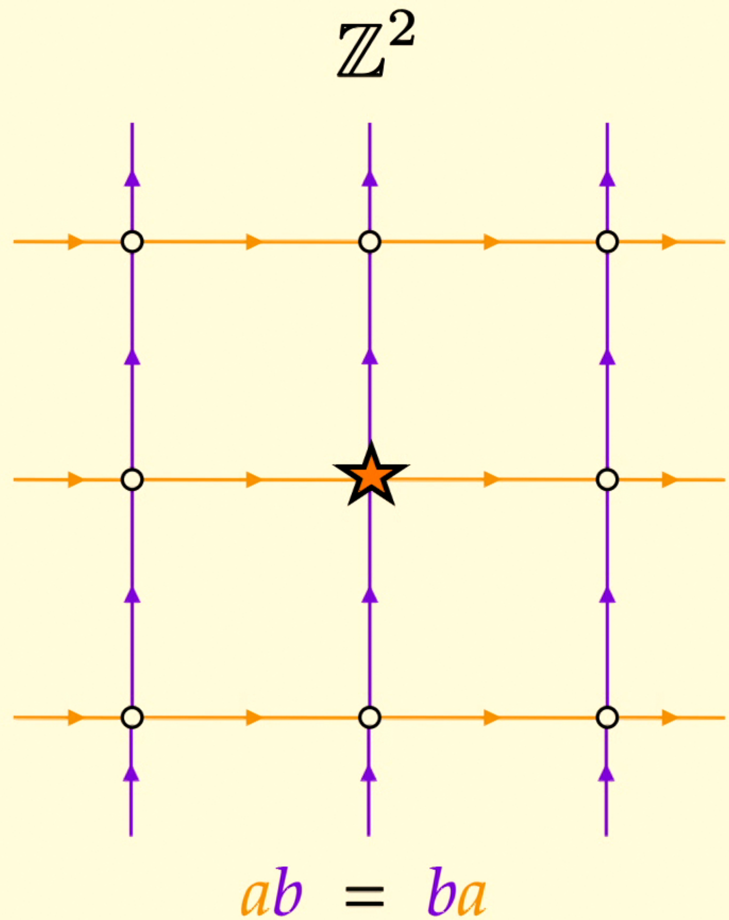
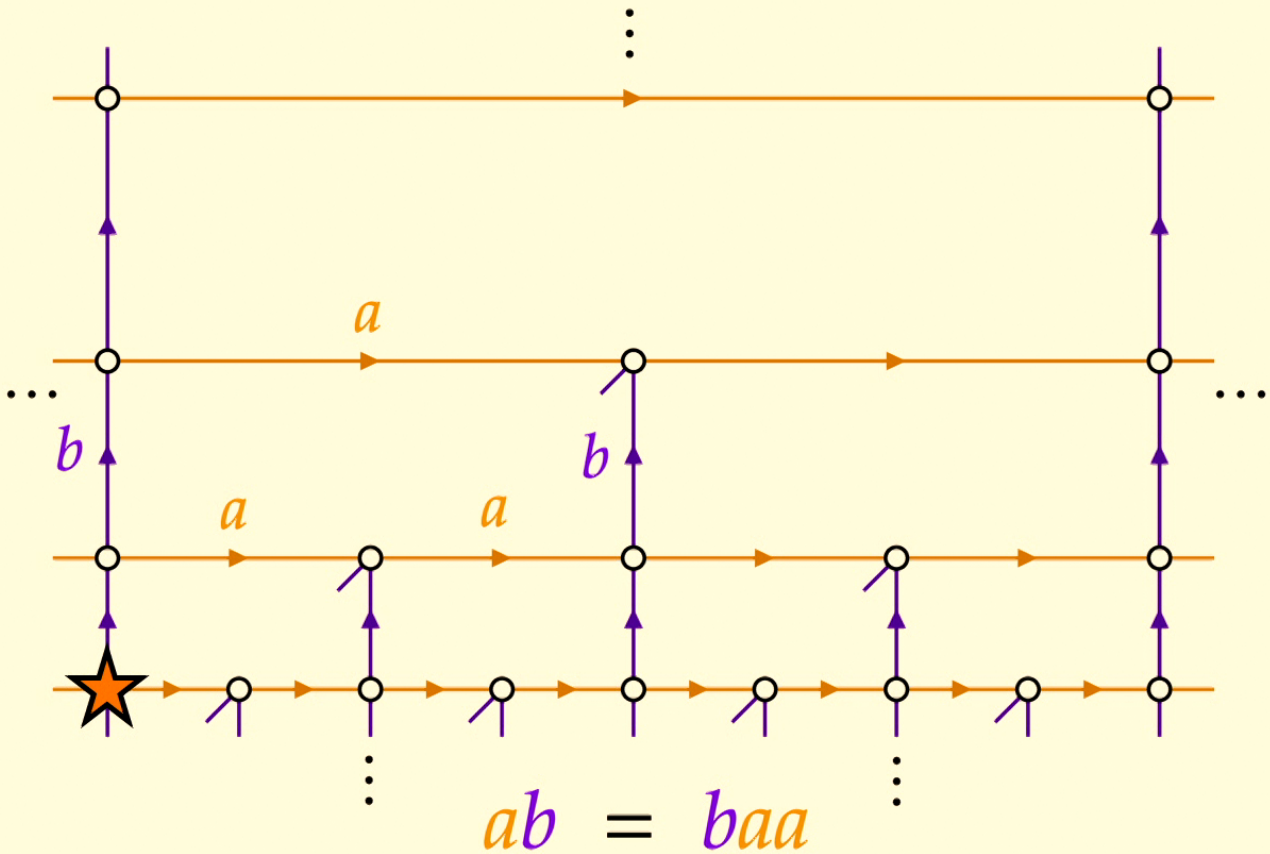
reminder: Cayley graphs

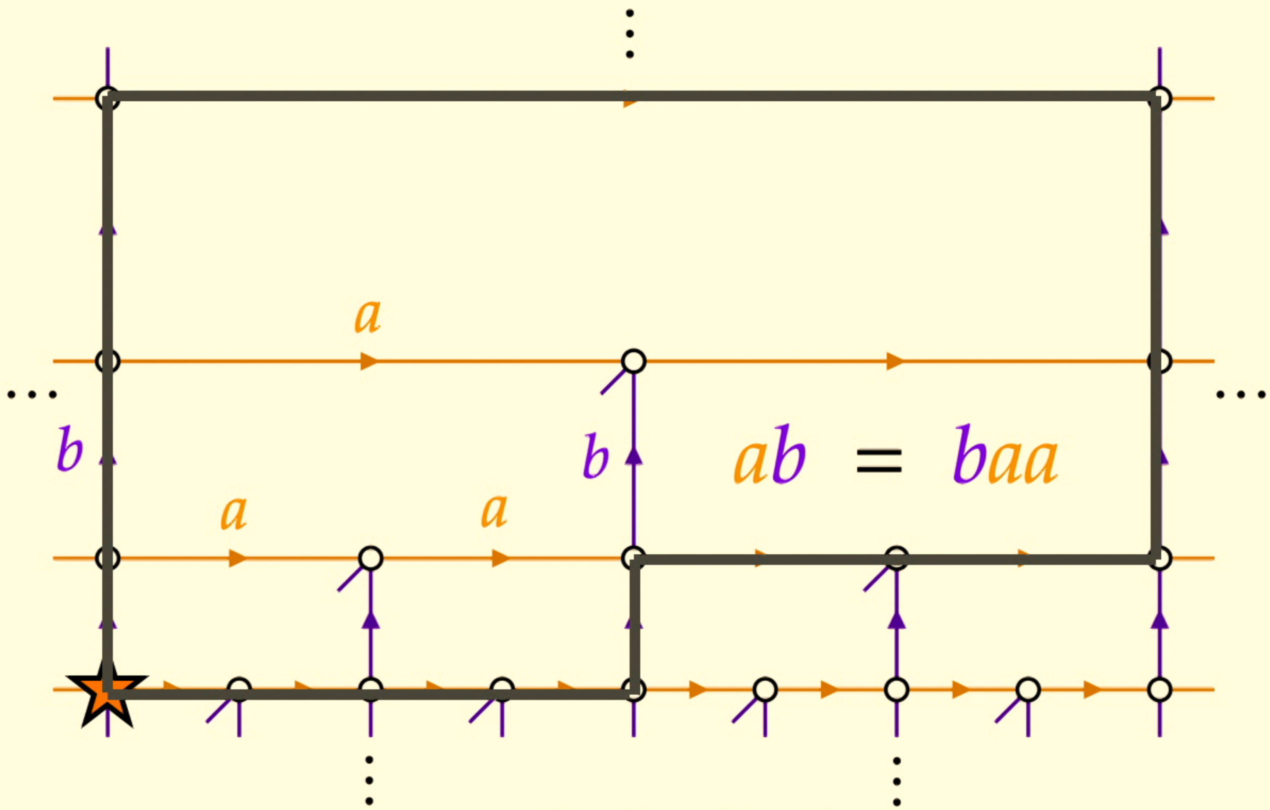




BS Cayley graph:



geometric picture of thermalization:

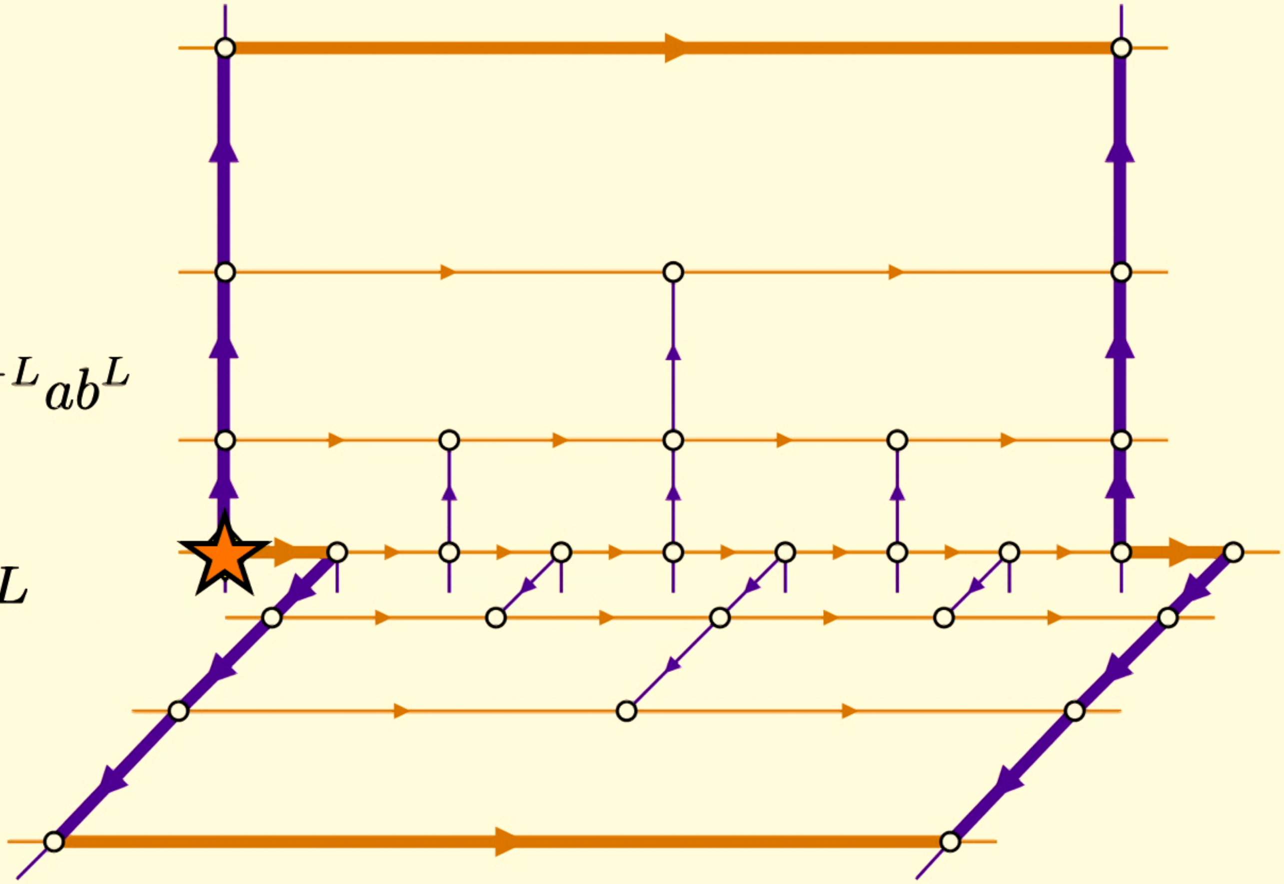


$$w = a^{-4}b^{-1}a^{-2}b^{-2}ab^3$$

a word with large area:

$$w_{large} = a^{-1}b^{-L}a^{-1}b^{L}ab^{-L}ab^{L}$$

$$\text{Area}(w_{large}) \sim 2^{L}$$

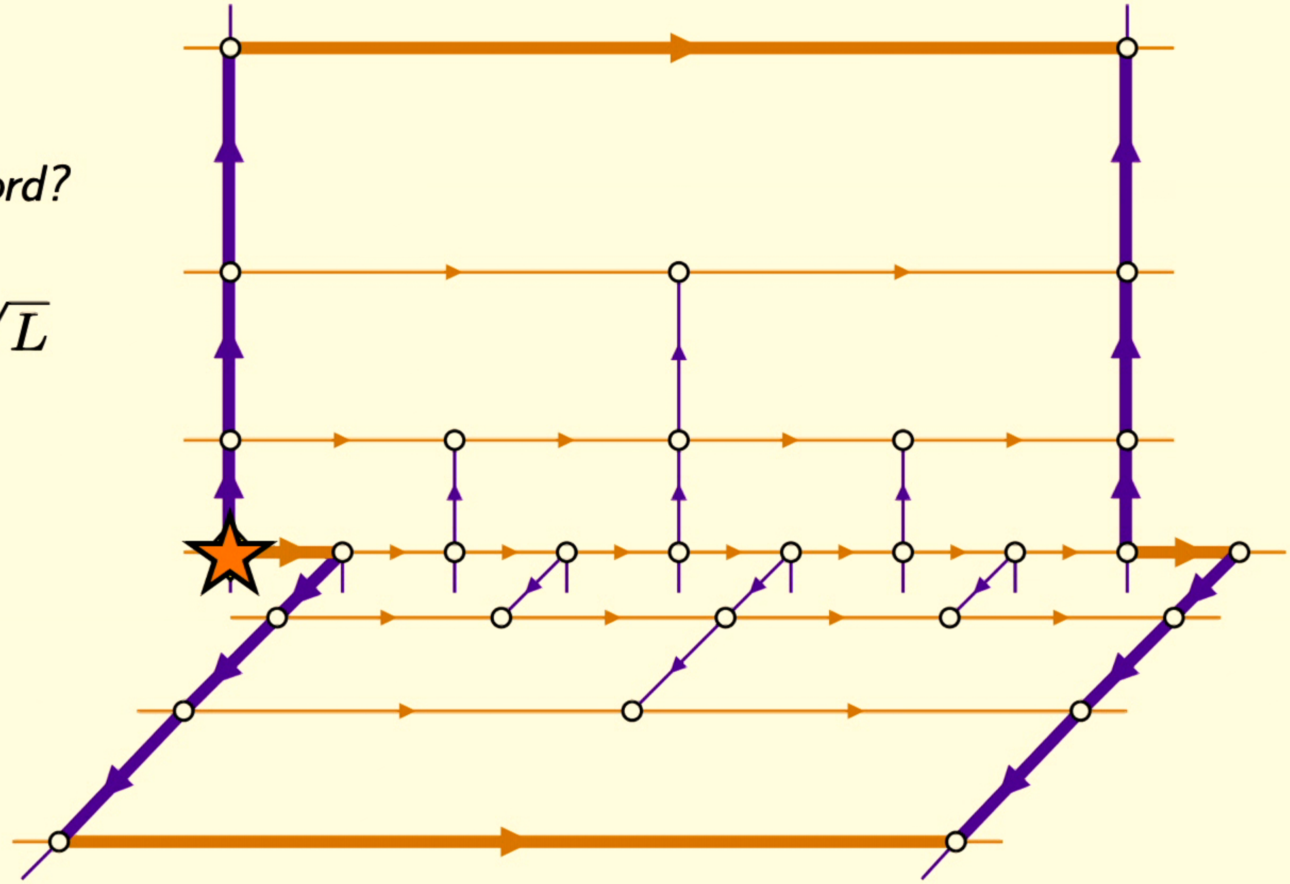


area of a typical length-L word?

$$\text{Area}(w_{typ}) \sim 2^{\sqrt{L}}$$

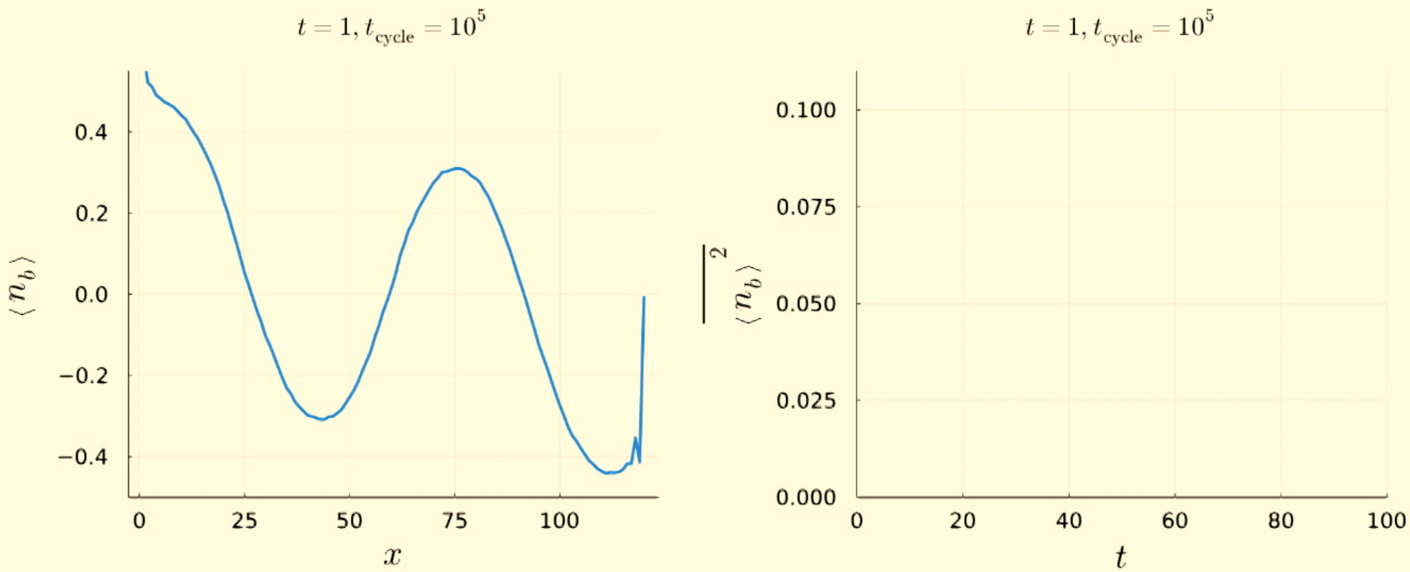
$$\Downarrow$$

$$\tau > 2^{\sqrt{L}}$$



The relation $ab = ba^2$ conserves the number of bs. Thus the number of bs is thus a conserved $U(1)$ charge.

As you might expect, the hydrodynamics of this charge is pretty crazy:



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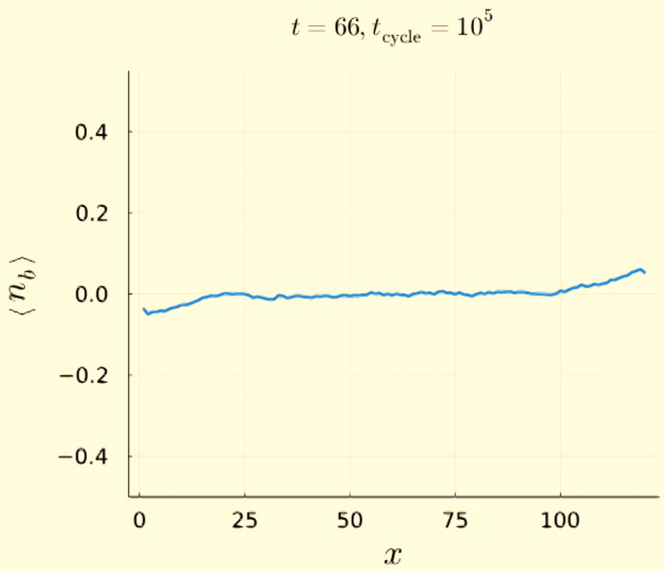

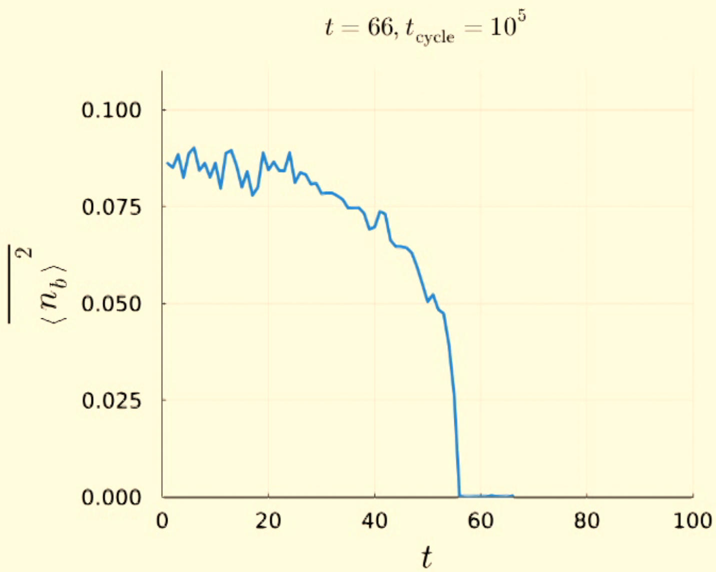

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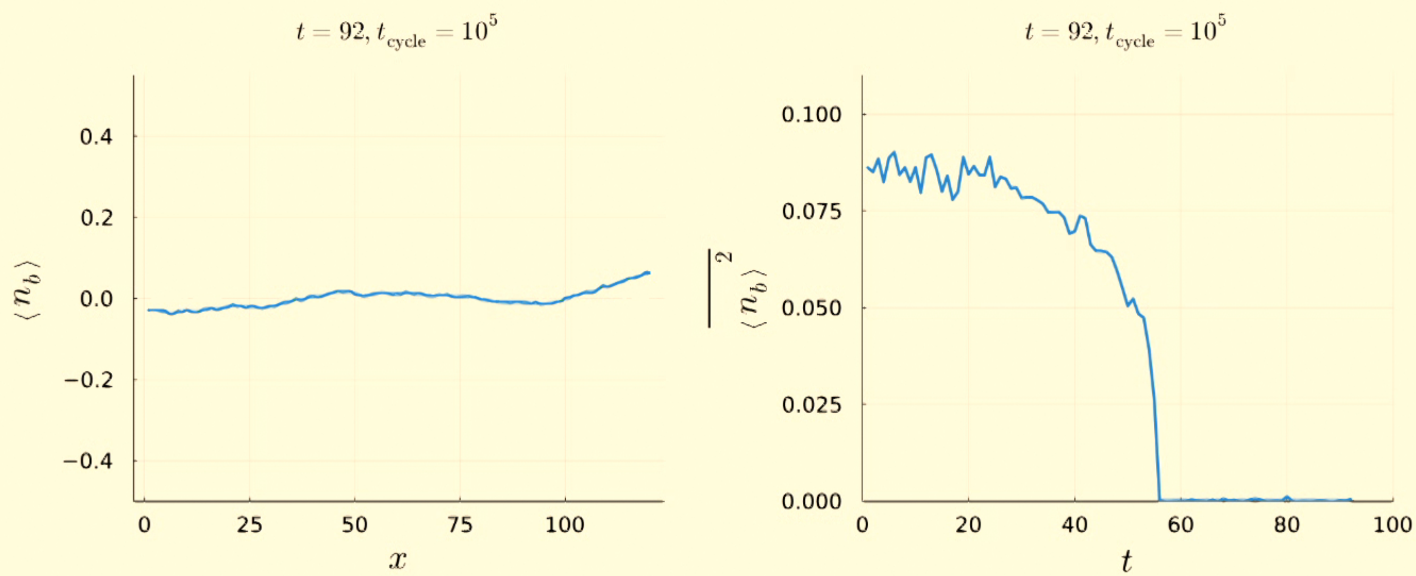
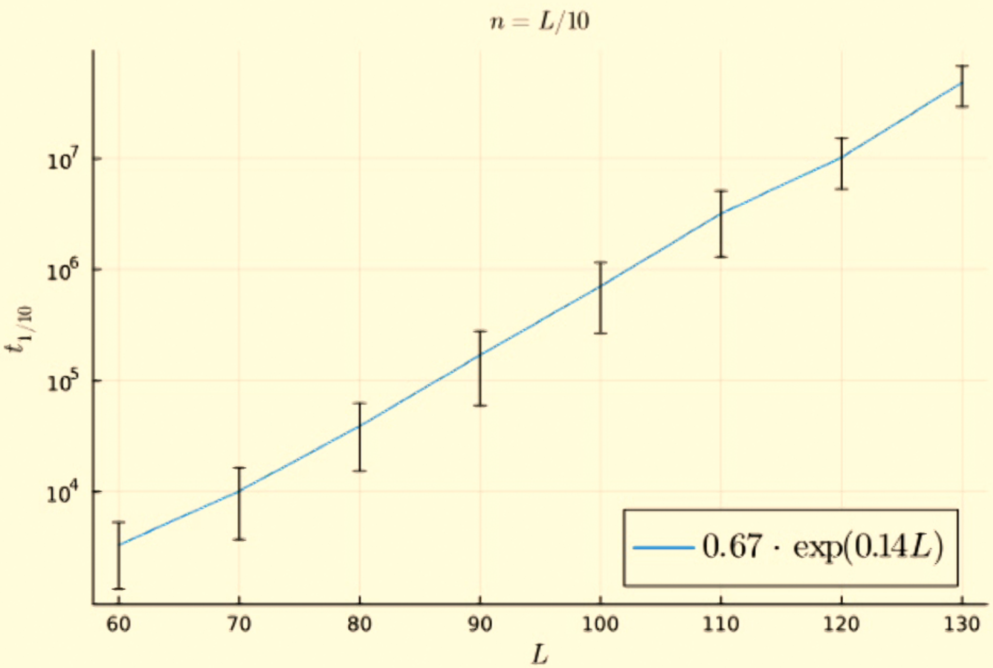


Can use this to numerically bound the thermalization time:



$$\tau \sim 2^{\alpha L}$$

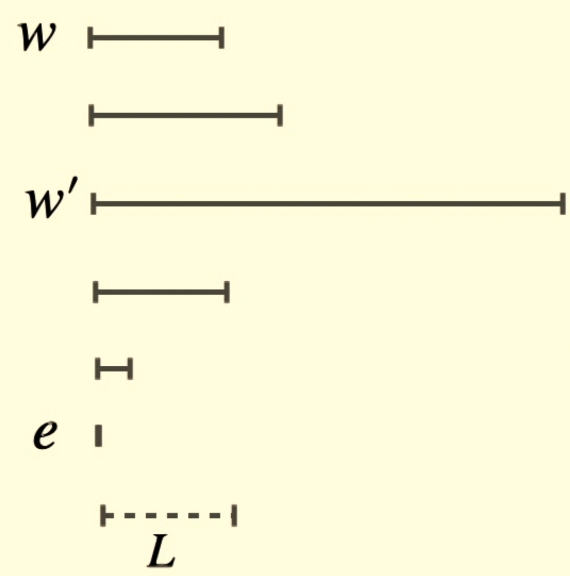
other things:

- *whole talk has been about time complexity; what about space complexity?*

  For some groups, words must get bigger before they can get smaller.

  Sometimes they must get much, much bigger! But physically, there is a cutoff at the system size L.

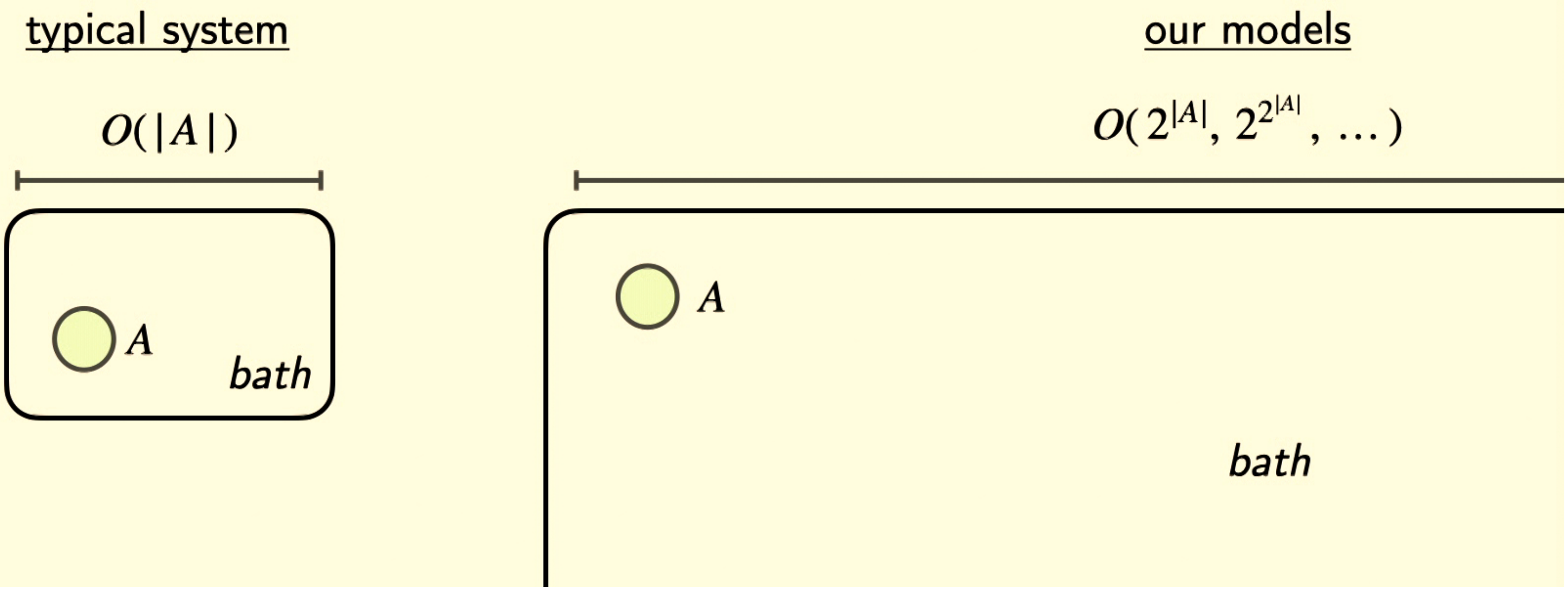
$$b^{-n} a b^{n} = \underbrace{aa \cdots aa}_{2^n}$$



can lead to an extreme form of “jamming" *other things:*

- *whole talk has been about time complexity; what about space complexity?*

  this has interesting implications for the size of a bath needed to thermalize a subsystem:



in the future:

- *robustness against group-breaking perturbations (in progress)*
- *different types of anomalous group hydrodynamics (in progress)*
- *more natural extensions to higher d*
- *ground states and entanglement*
- *better understanding of transport + connectivity of Hilbert space*
- *geometric understanding of thermalization (think about e.g. random graphs)*
- *...*

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