Title: Deeper Kummer theory

Speakers: Theo Johnson-Freyd

Collection/Series: Mathematical Physics

Subject: Mathematical physics

Date: September 21, 2023 - 3:30 PM

URL: https://pirsa.org/23090104

Abstract:

A tower is an infinite sequence of deloopings of symmetric monoidal ever-higher categories. Towers are places where extended functorial field theories take values. Towers are a "deeper" version of commutative rings (as opposed to "higher rings" aka E∞-spectra). Notably, towers have their own opinions about Galois theory, and think that usual Galois groups are merely shallow approximations of deeper homotopical objects. In this talk, I will describe some steps in the construction and calculation of the deeper Galois group of a characteristic-zero field. In particular, I'll explain a homotopical version of the Kummer description of abelian extensions. This is joint work in progress with David Reutter.

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Zoom link: https://pitp.zoom.us/j/97950701035?pwd=Wk9FRSt2MkN3eWptTVltRVJncnFHdz09

Theorem [JF - Reutter]:

Suppose $T^\otimes \supseteq \mathbb{Q}$ is a tower.

Then TFAE:

(A) $T^\otimes$ is Galois-closed.

(B) (1) $\mathcal{M}(T) \cong I_{\mathbb{Q}/\mathbb{Z}}$

(2) for every finite gp G, there is a unique iso class of fibre functors $\mathrm{Rep}\, G \to T'$.

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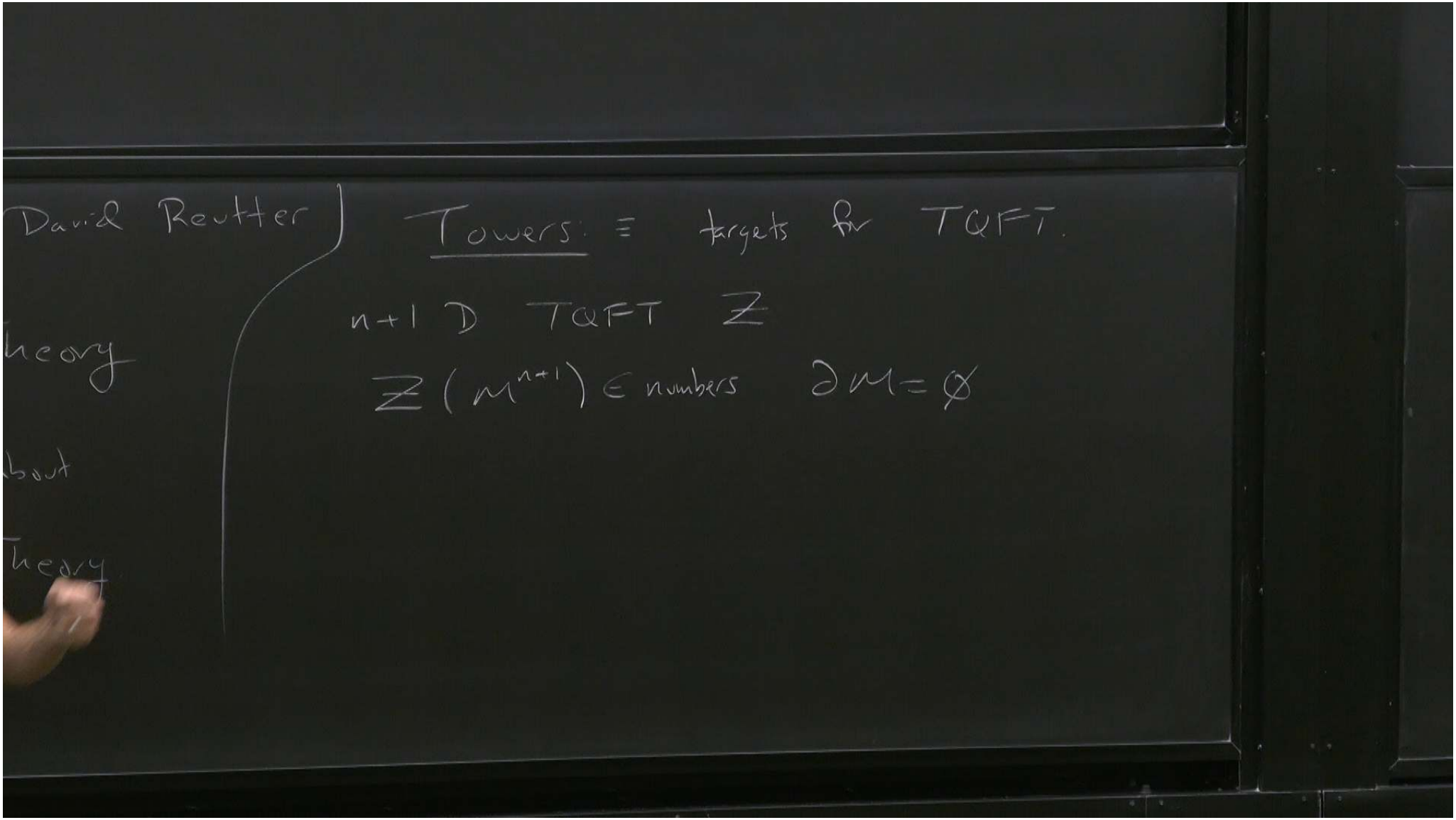
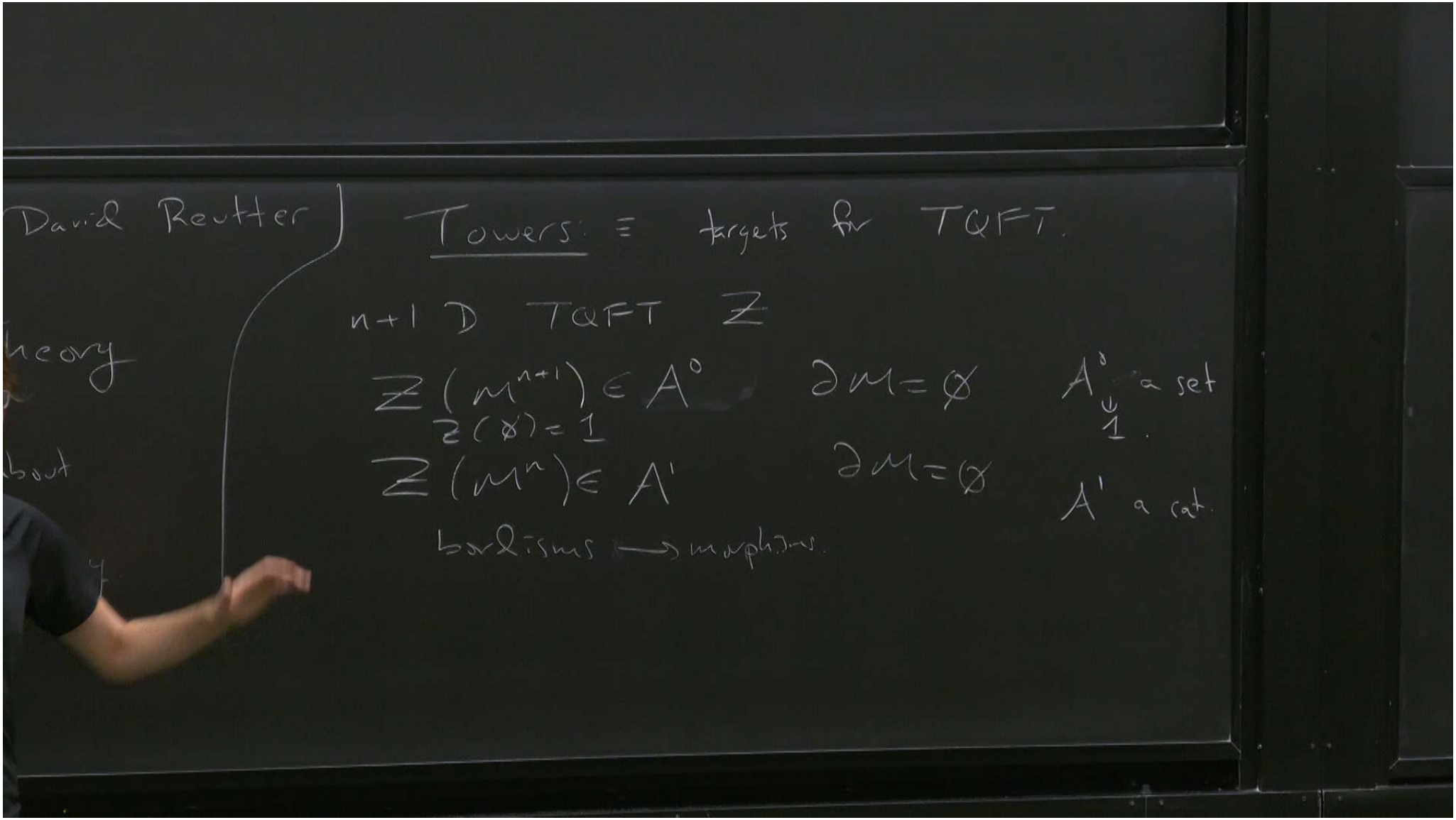
jt in progress w/ David Reutter

Deeper Galois Theory

today: say something about Kummer Theory


David Reutter
Towers: ≡ targets for TQFT.
n+1 D TQFT Z
Z(M^{n+1}) ∈ numbers ∂M = ∅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Towers: ≡ targets for TQFT.
n+1 D TQFT Z
$Z(M^{n+1}) \in A^0$ $\partial M = \emptyset$ A^0 a set
$Z(\emptyset) = 1$ $\downarrow$ 1.
$Z(M^n) \in A^1$ $\partial M = \emptyset$ A^1 a cat.
bordisms ⟶ morphisms.

David Reutter

Theory

about

Theory

Towers = targets for TQFT.

$n+1$ D TQFT Z

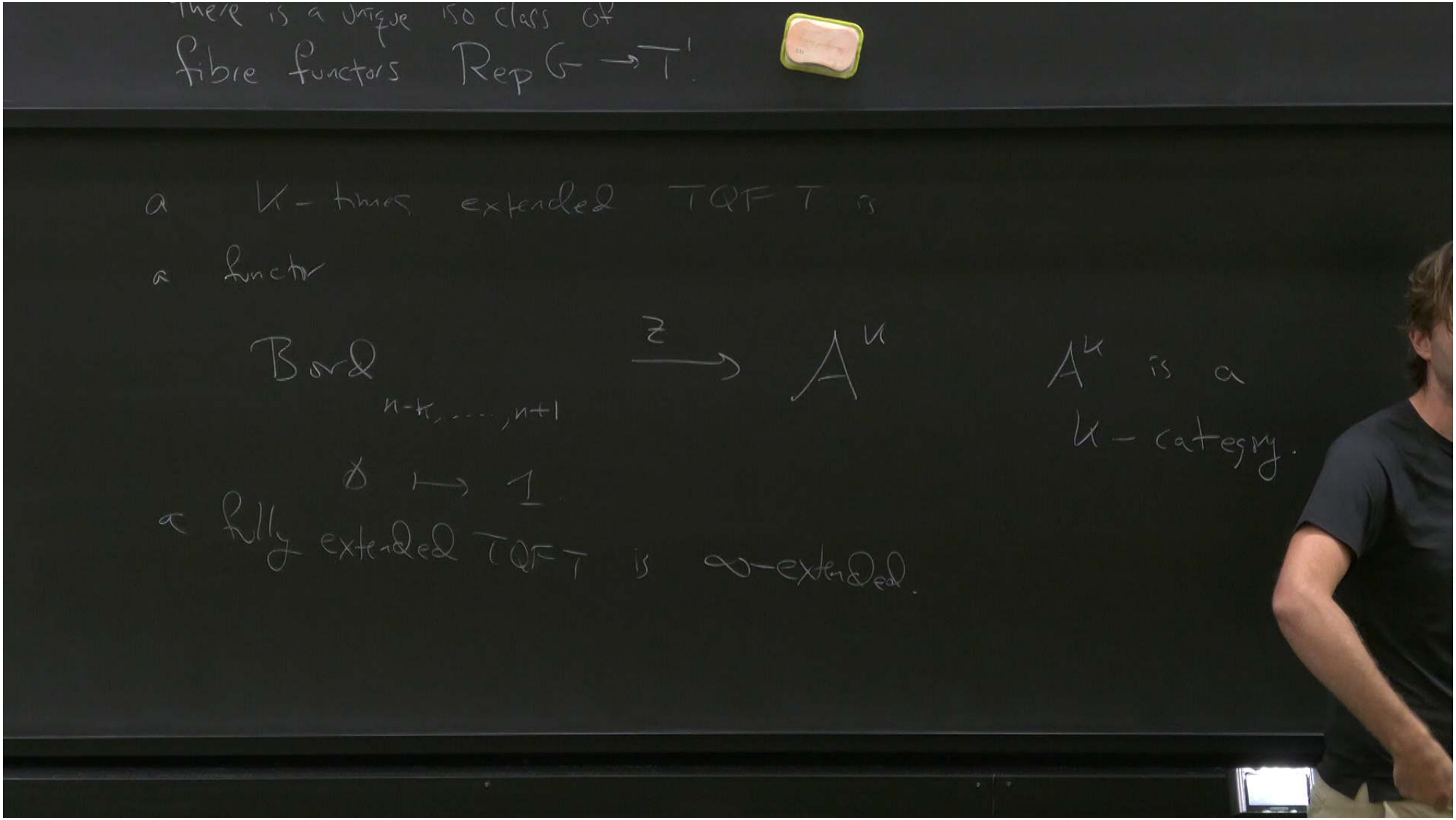
$Z(M^{n+1}) \in A^0$ $\quad \partial M = \emptyset$ $\quad A^0$ a set

$Z(\emptyset) = 1$ $\quad \downarrow 1$.

$Z(M^n) \in A^1$ $\quad \partial M = \emptyset$ $\quad A^1$ a cat.

bordisms $\longrightarrow$ morphisms.

$\hookrightarrow Z((\ )^{n+1})$ is multiplicative.


There is a unique iso class of
fibre functors $\mathrm{Rep}\, G \to T'$.
a k-times extended TQFT is
a functor
$\mathrm{Bord}^{n-k,\dots,n+1} \xrightarrow{Z} A^k$
$\emptyset \longmapsto \underline{1}$
A^k is a k-category.
a fully extended TQFT is ∞-extended.

A tower is a sequence [Scheimbauer]

$$A^0, A^1, \dots$$

s.t. $A^{k} \ni 1$ is a pointed k-cat,

together w/ equivs

$$A^{k-1} \simeq \Omega A^k := \mathrm{End}_{A^k}(1).$$

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N.B. A spectrum is the same but w/ ~~k-categories~~ spaces.

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$A^0, A^1, \dots$

s.t. $A^k \ni \mathbb{L}$ is a pointed k-cat,

together w/ equivs

$$A^{k-1} \simeq \Omega A^k := \mathrm{End}_{A^k}(\mathbb{1}).$$

axioms:

A^k should be Karoubian ie:

(i) idemp. complete

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A tower is a sequence [Scheimbauer]

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together w/ equivs

$$A^{k-1} \simeq \Omega A^k := \mathrm{End}_{A^k}(1)$$

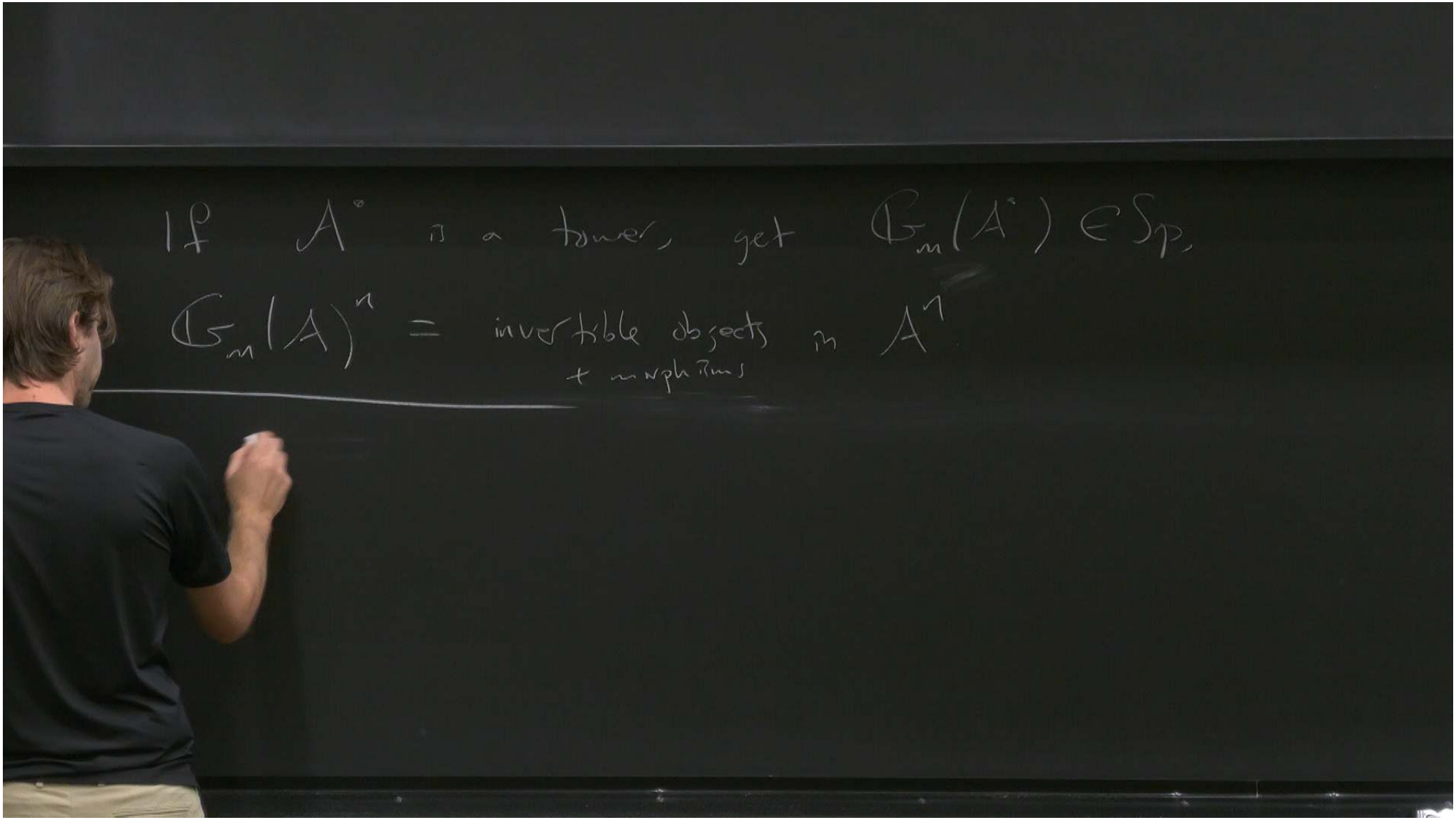
axioms:

A^k should be Karoubian ie:

(i) idemp. complete

(ii) additive
↳ has $\oplus$-s and $1 \in A^0$.

N.B. A spectrum is the same but w/ ~~k-cats~~ spaces.


If $A^\bullet$ is a tower, get $\mathbb{G}_m(A^\bullet) \in Sp$.
$\mathbb{G}_m(A)^n$ = invertible objects in A^n
+ morphisms

If $\mathcal{A}^\otimes$ is a tower, get $\mathbb{G}_m(\mathcal{A}^\otimes) \in Sp$.

$\mathbb{G}_m(\mathcal{A})^n$ = invertible objects in $\mathcal{A}^n$ + morphisms

---

$I_{\mathbb{C}^\times} \in Sp$ is the spectrum s.t.

$$[-, I_{\mathbb{C}^\times}] = \hom_{AbGp}(\pi_0(-), \mathbb{C}^\times)$$

$$\| $$

$$\pi_0 \hom_{Sp}(-, I_{\mathbb{C}^\times})$$

N.B. A spectrum is the same but w/ ~~the~~ ~~gs~~ spaces.

Observation [Freed-Hopkins]

If $\mathcal{I}^{\cdot}_{m}(A) = I_{\mathbb{C}^\times}$, then invertible TQFTs are determined by their partition fns.

N.B. A spectrum is the same but w/ ~~the~~ gps spaces.

Observation / {Freed-Hopkins}

If $\mathcal{G}_m^i(A) = I_{\mathbb{C}^\times}$, then invertible TQFTs are determined by their partition fns.

Question (Hopkins):

Find a good tower T s.t.

$$T^1 \simeq sVec_{\mathbb{C}^\times} \qquad \mathcal{G}_m(T) = I_{\mathbb{C}^\times}$$

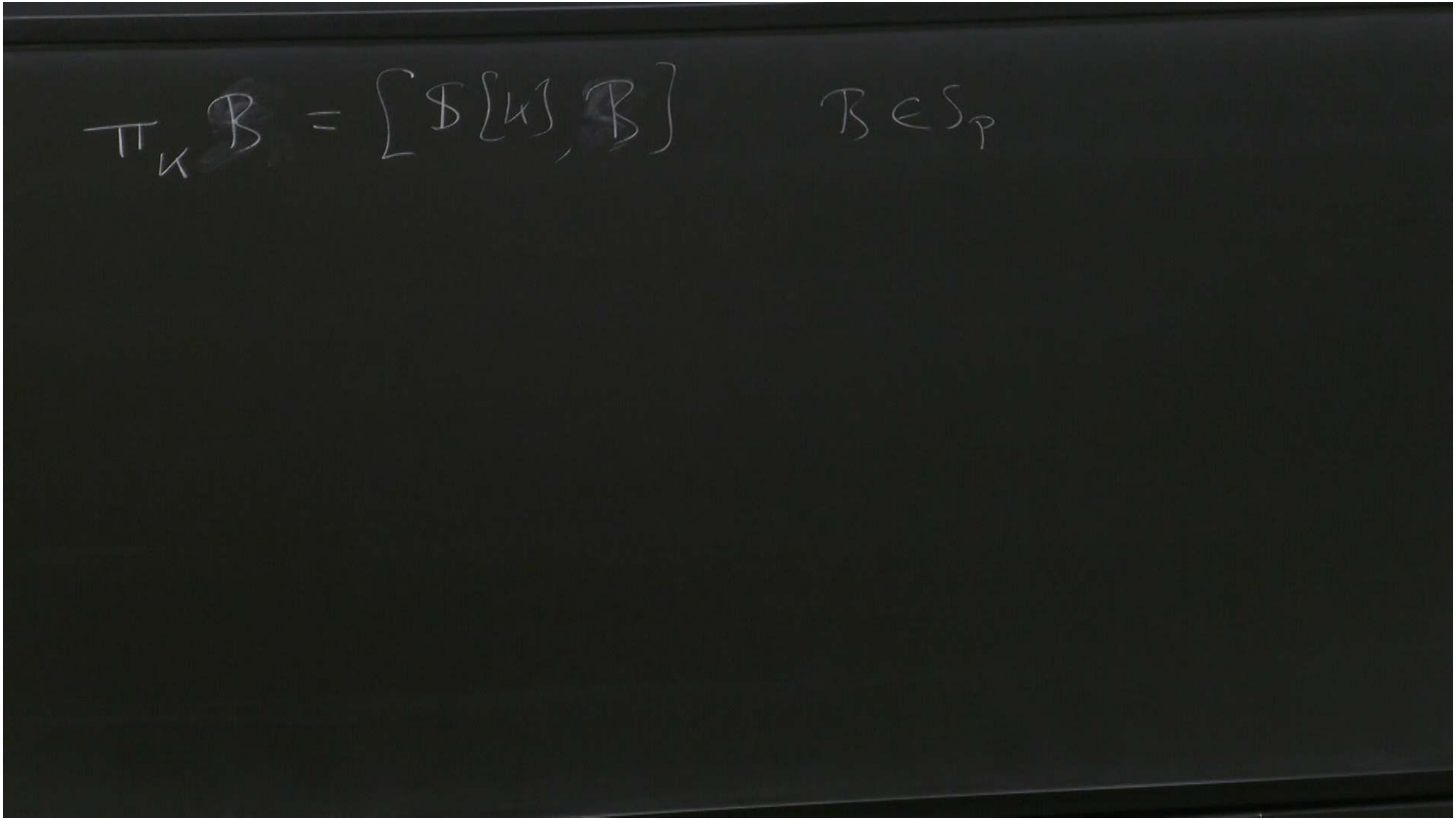
N.B. A spectrum is the same but w/ ~~the~~ spaces.

Observation (Freed-Hopkins):

If $\mathcal{G}_m^i(A) = I_{\mathbb{C}^\times}$, then invertible TQFTs are determined by their partition fns.

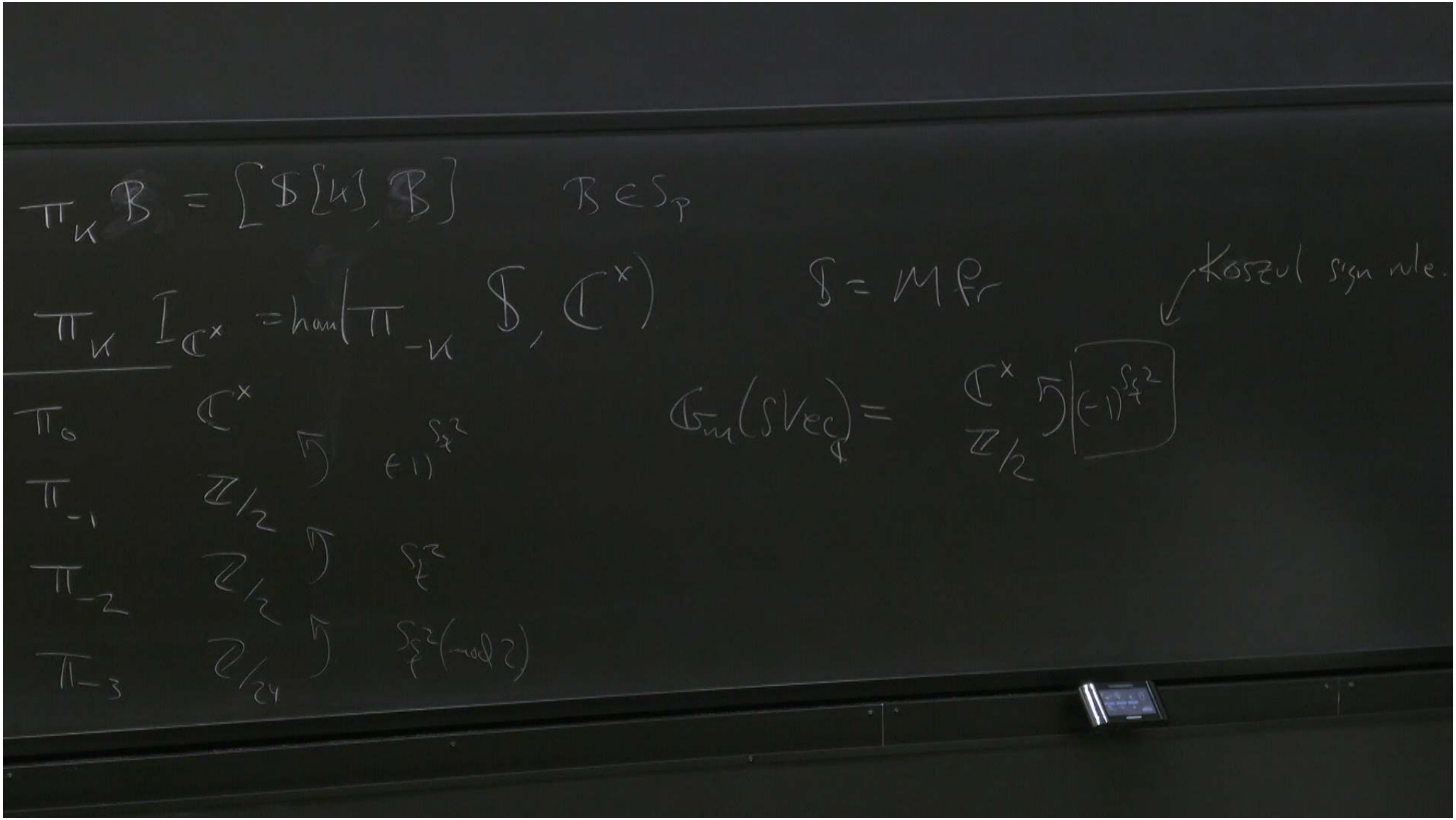
Question (Hopkins):

Find a good tower T s.t. $T^1 \cong sVec_{\mathbb{C}^\times}$, $\mathcal{G}_m(T) \cong I_{\mathbb{C}^\times}$ and convince your choice is good.


$\pi_4 B = [\$[4], B]$
$B \in S_p$

$$\pi_\mu B = [\$[\mu], B] \qquad B \in S_p$$

$$\pi_\mu I_{\mathbb{C}^\times} = \hom(\pi_{-\mu} \$, \mathbb{C}^\times)$$


$\pi_\mu \mathcal{B} = [\mathfrak{d}[\mu], \mathcal{B}]$ $\mathcal{B} \in S_p$
$\pi_\mu I_{\mathbb{C}^\times} = \hom(\pi_{-\mu}, \mathfrak{g}, \mathbb{C}^\times)$ $\mathfrak{g} = M_{fr}$
Koszul sign rule.
π_0 $\mathbb{C}^\times$ $\circlearrowleft (-1)^{S_f^2}$
π_{-1} $\mathbb{Z}/2$ $\circlearrowleft$ S_f^2
π_{-2} $\mathbb{Z}/2$ $\circlearrowleft$ $S_f^2 \pmod 2$
π_{-3} $\mathbb{Z}/24$
$G_m(SVect) = \mathbb{C}^\times \wr \mathbb{Z}/2$ $(-1)^{S_f^2}$

Observation [Freed-Hopkins]

If $G_m^i(A) = I_{\mathbb{C}^\times}$, then invertible TQFT

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partition fns.

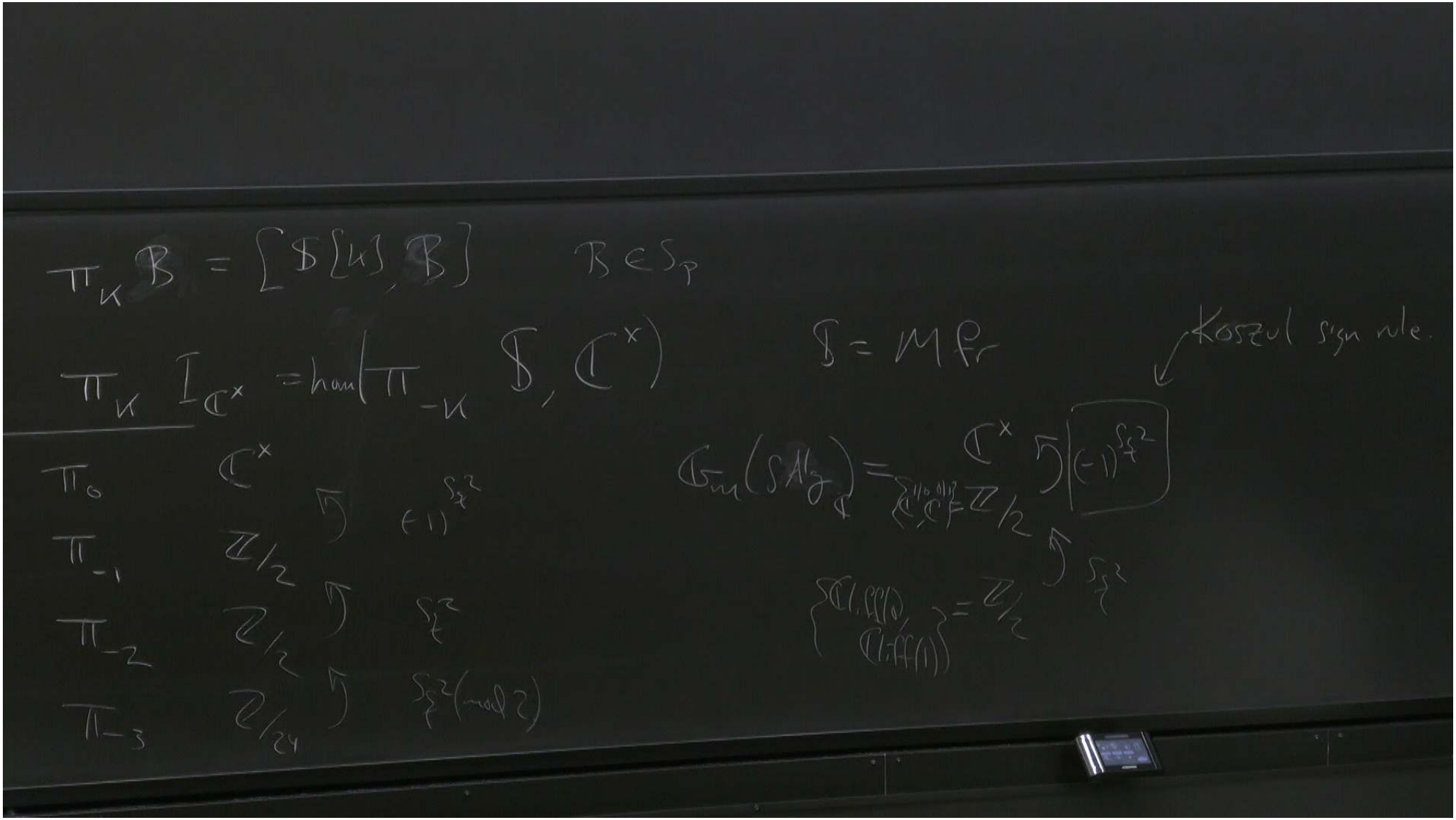
Question (Hopkins):

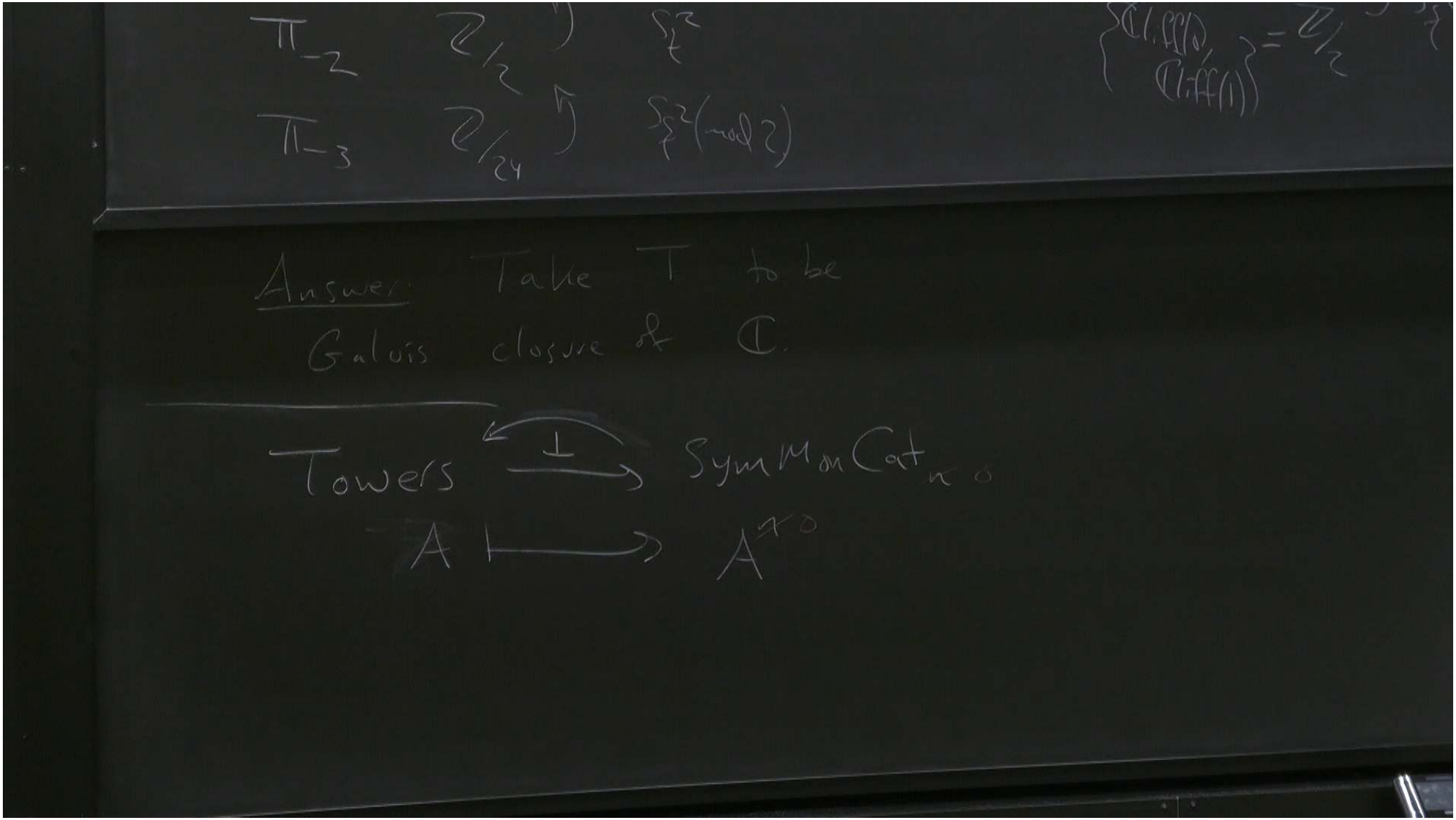
Find a good tower T s.t. and co

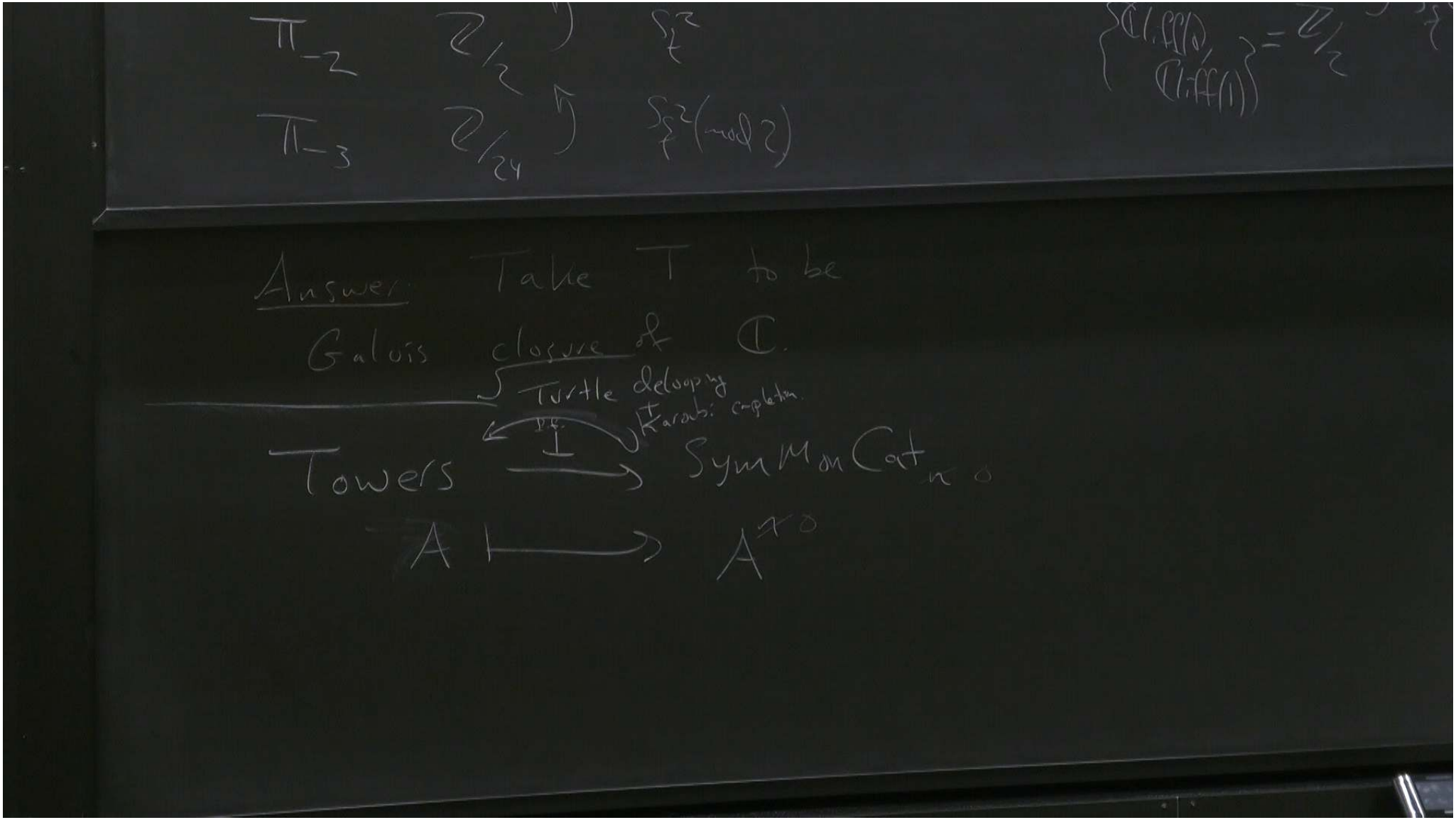
$T^1 \cong sVec^{f.d.}$ $G_m(T) \cong I_{\mathbb{C}^\times}$ your ch

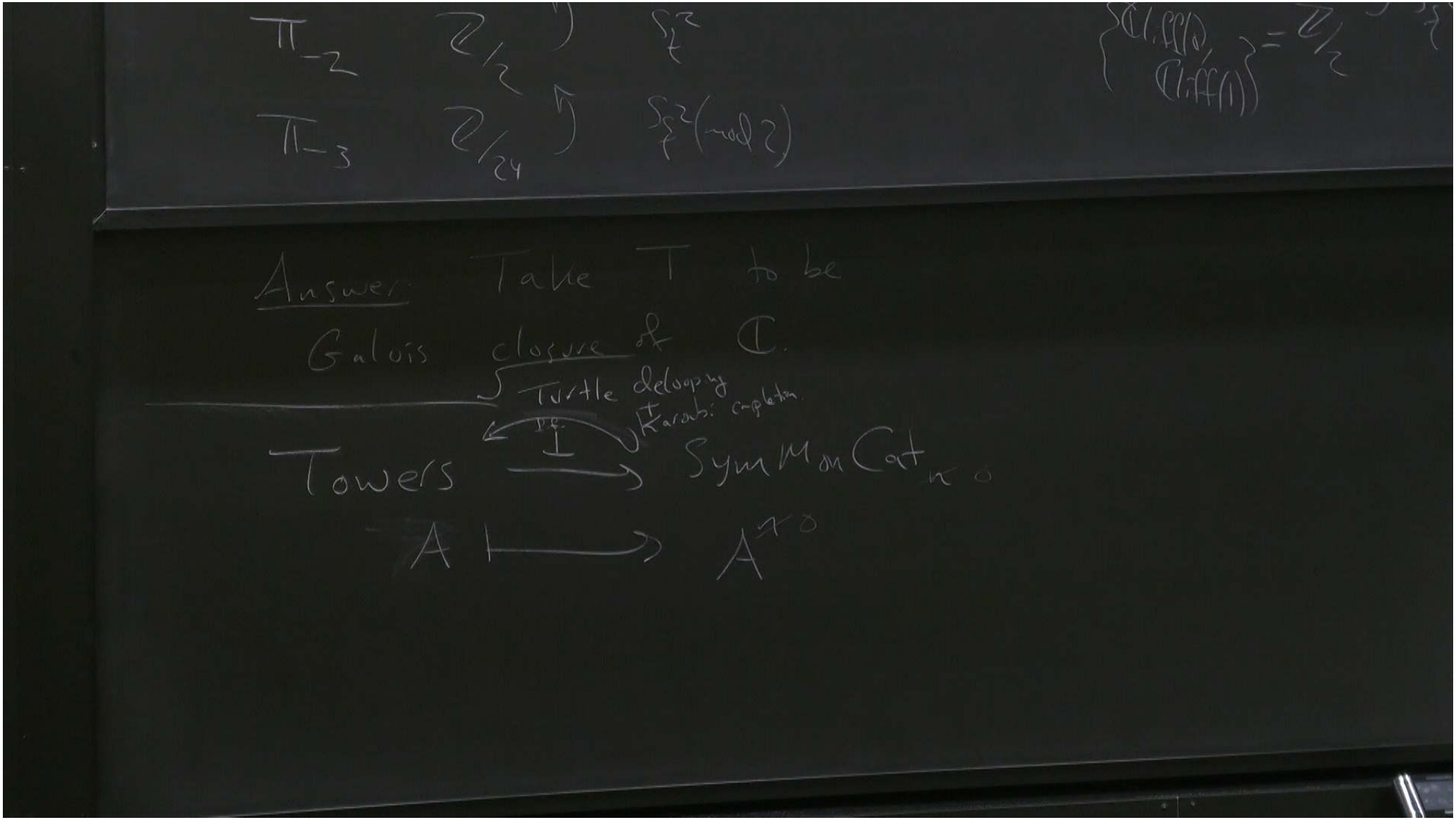
goo

Guess: $T^2 \cong sAlg_{\mathbb{C}}^{f.d.}$


$\pi_k \mathbb{B} = [\mathbb{S}[4], \mathbb{B}]$ $\mathbb{B} \in S_p$
$\pi_k I_{\mathbb{C}^\times} = hom(\pi_{-k} \mathbb{S}, \mathbb{C}^\times)$ $\mathbb{S} = M\mathbb{F}_r$
Koszul sign rule.
π_0 $\mathbb{C}^\times$ $(-1)^{S_\xi^2}$
π_{-1} $\mathbb{Z}/2$ S_ξ^2
π_{-2} $\mathbb{Z}/2$ $S_\xi^2 \pmod 2$
π_{-3} $\mathbb{Z}/24$
$G_m(SA)_\xi = \mathbb{C}^\times$ $(-1)^{S_\xi^2}$
$\mathbb{Z}/2$ S_ξ^2
$= \mathbb{Z}/2$


π_{-2} $\mathbb{Z}/2$ S^2
π_{-3} $\mathbb{Z}/24$ $S^2(\mathrm{mod}\ 2)$
$\{\mathbb{C}, \mathrm{fld}, \mathbb{C}\mathrm{liff}(1)\} = \mathbb{Z}/2$
Answer: Take T to be
Galois closure of $\mathbb{C}$.
Towers $\perp$ SymMonCat$_{n<0}$
$A \longmapsto A^{\otimes 0}$


Answer: Take T to be
Galois closure of ℂ.
Turtle delooping
Farab: completion.
Towers
SymMonCat
A
A^TO


Answer: Take T to be
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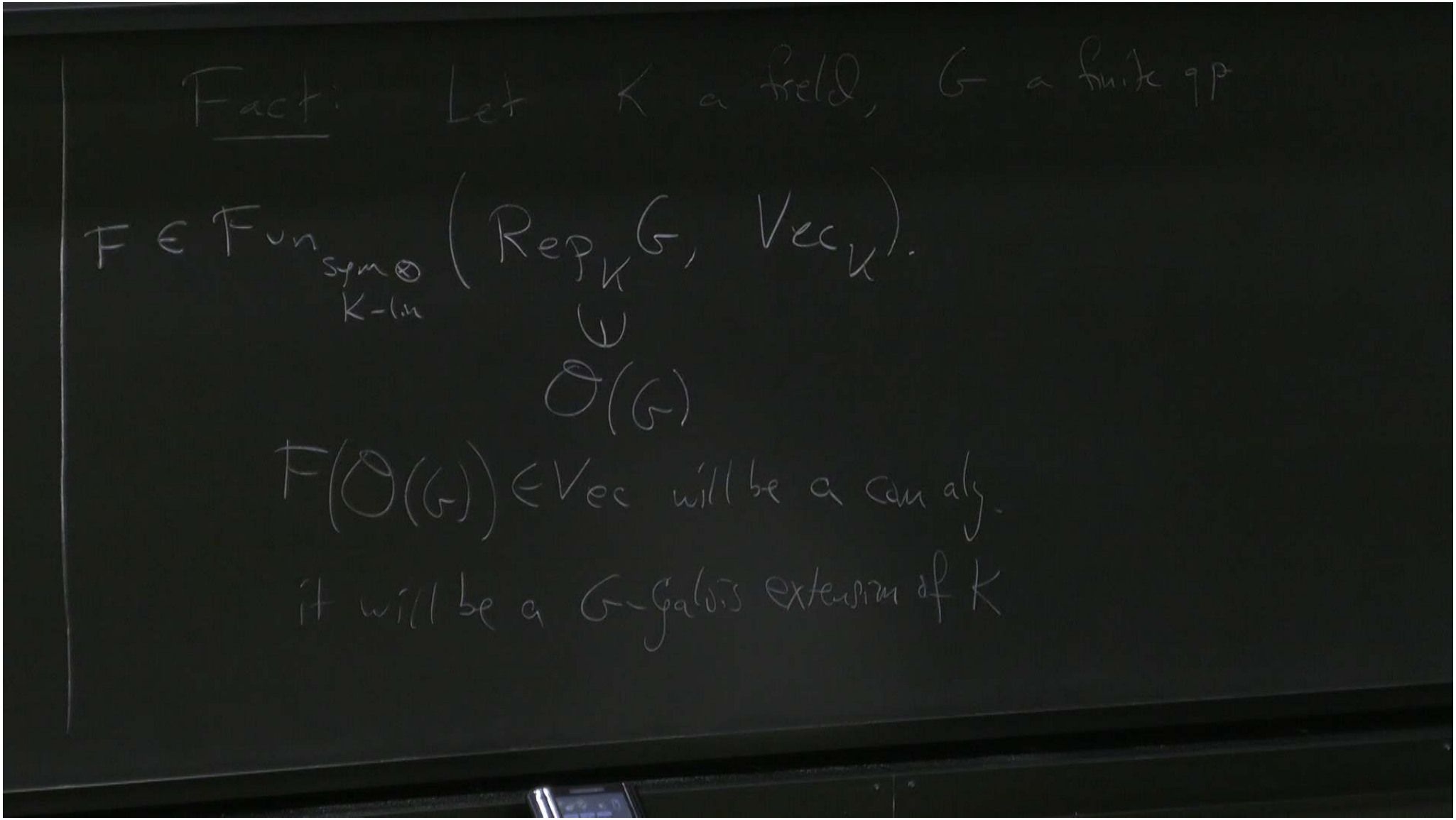
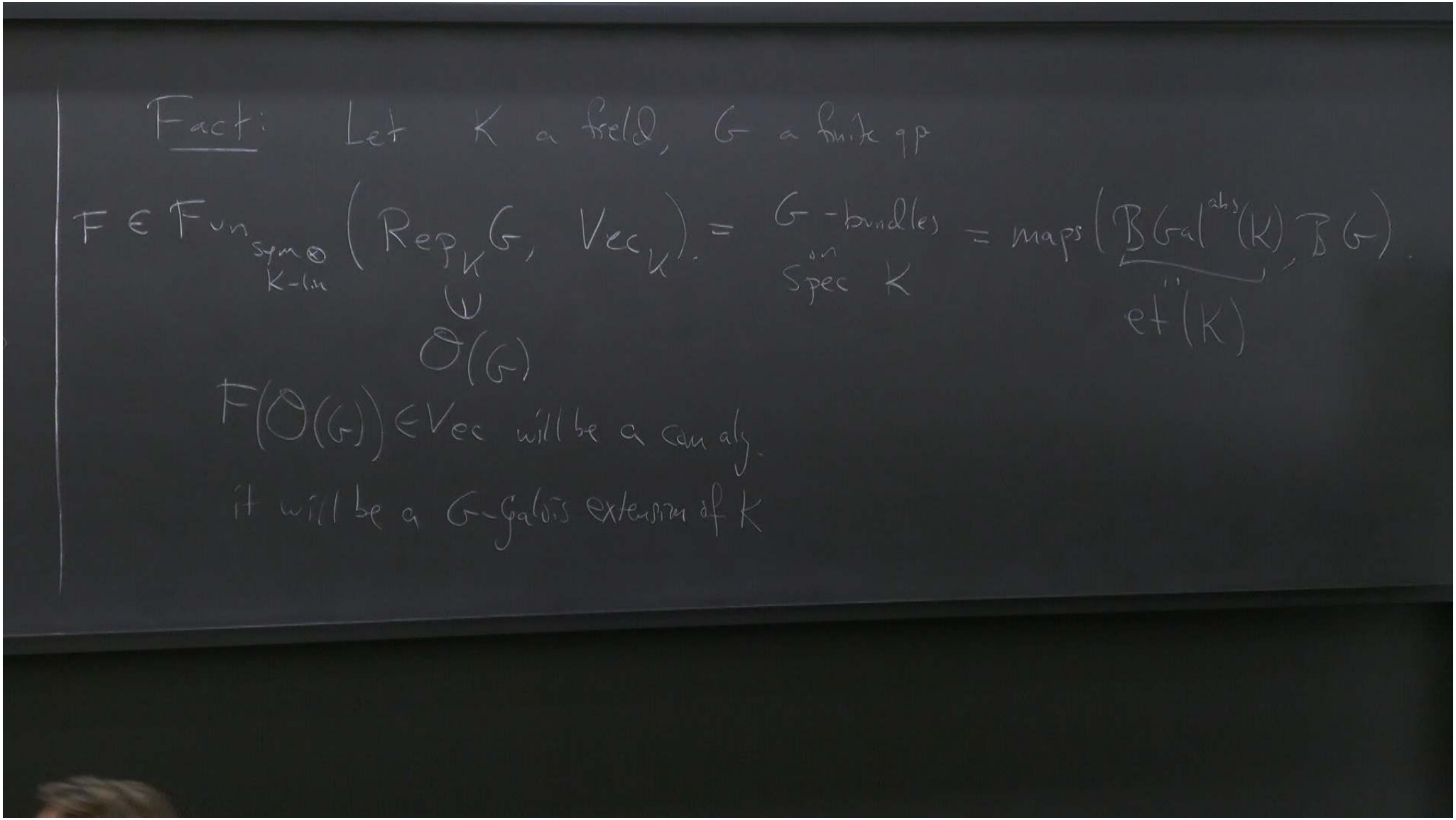
Turtle deloop by Karoubi completion
Towers
SymMonCat


Fact: Let K a field, G a finite gp
$F \in \mathrm{Fun}^{\mathrm{sym}\otimes}_{K\text{-lin}}(\mathrm{Rep}_K G, \mathrm{Vec}_K)$.
$\cup$
$\mathcal{O}(G)$
$F(\mathcal{O}(G)) \in \mathrm{Vec}$ will be a com alg.
it will be a G-Galois extension of K


Fact: Let K a field, G a finite gp
$F \in Fun^{sym\otimes}_{K\text{-lin}}(Rep_K G, Vec_K) = G\text{-bundles on } Spec\ K = maps(\underbrace{BGal^{ab}(K)}_{et}, BG)$
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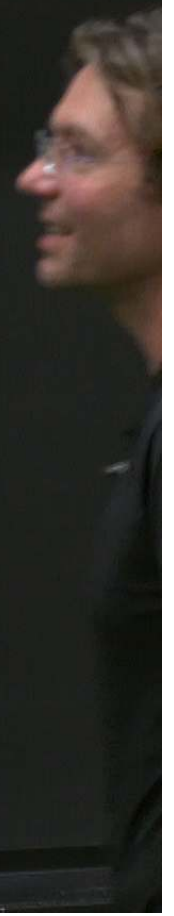
$F(\mathcal{O}(G)) \in Vec$ will be a com alg.
it will be a G-Galois extension of K

Given X a π-finite space, $T \in \mathrm{Tow}$

$$X \longmapsto \mathrm{Hom}_{\mathrm{Tow}_{T/}}(T^X, T) \in \mathrm{Spaces}$$

If this is proco representable, its corep'ng object is the étale space $\mathrm{et}(T)$.



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If this is proco representable, its corep'ng object is the étale space $\mathrm{et}(T)$.

Prop: If $T \geq \mathbb{Q}$, then $\mathrm{et}(T)$ exists.

a functor

Theorem [JF - Reutter]:

Suppose $T^\bullet \supseteq \mathbb{Q}$ is a tower.

Then TFAE:

(A) $T^\bullet$ is Galois-closed.

(B) (1) $\mathcal{M}(T) \cong I_{\mathbb{Q}/\mathbb{Z}}$

(2) for every finite gp G, there is a unique iso class of fibre functors $\mathrm{Rep}\, G \to T^1$

] tests $\pi_{\leq 1}\, \mathrm{et}(T)$

yt in p

Deeper

today:

a functor

Theorem [JF-Reutter]:

Suppose $T^\bullet \supseteq \mathbb{Q}$ is a tower.

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tests $\pi_{\leq 1}\, et(T)$

gt in ...

Deeper

today:

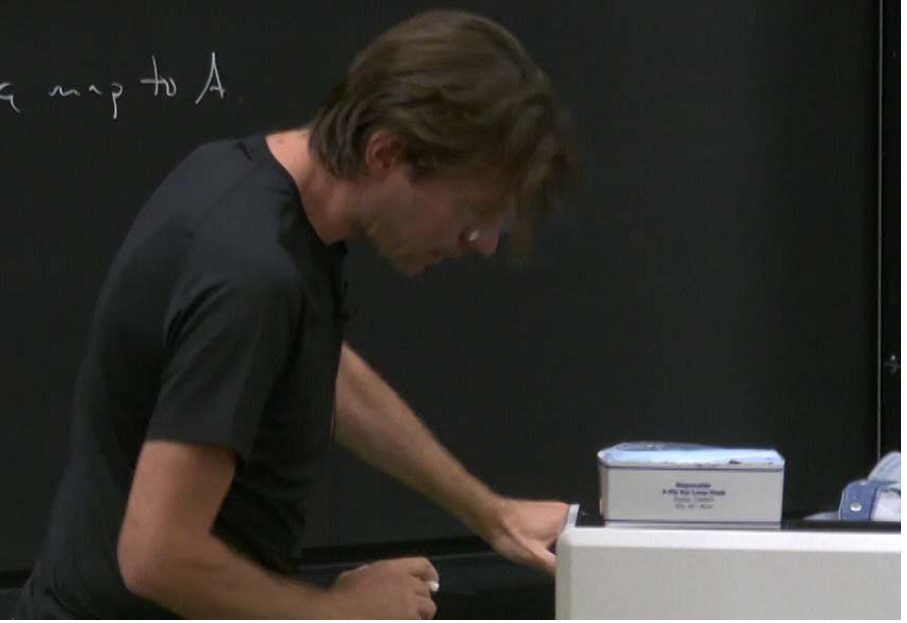
it will be a G-Galois extension of K

If $A \in \mathrm{Sp}_{\leq 0}$,

$A[\mathrm{tor}] \in \mathrm{Ind}(\mathrm{Sp}_{\leq 0, \pi\text{-f}})$

defined as the universal such thing w/ a map to A.

$\mathcal{M}(T) = \mathbb{G}_m(T)[\mathrm{tor}]$.



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If $A \in \mathrm{Sp}_{\leq 0}$,

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defined as the universal such thing w/ a map to A.

$\mathcal{M}(T) = \mathbb{G}_m(T)[\mathrm{tor}]$.

$\pi_{\ast}(A[\mathrm{tor}]) \xleftarrow{\sim} (\pi_{\ast} A)[\mathrm{tor}]$

from $\mathbb{Z}$'s in $\pi_{\ast+1}$

$\mathbb{Z}[\mathrm{tor}] = \Sigma^{-1}\mathbb{Q}/\mathbb{Z}$

$A^{k-1} \simeq \Omega A^k := \mathrm{End}_{A^4}(1)$

N.B. A spectrum is the same but w/ ~~t.v.s~~ spaces.

field $K \supseteq$ all the ℓ^{th} roots of 1

ab gp A of exp. ℓ

$\mathrm{Rep}_A \xrightarrow{\sim} \mathrm{Kar}_\oplus(\mathrm{Hom}(A, K^\times))$

"A kind of Fourier transform"

(T)

N.B. A spectrum is the same but w/ ~~ab gps~~ spaces.

Kummer Theory:

Then A-galois extensions of K

$\updownarrow$

maps$(A^\vee, \Sigma H K^\times)$

$=$

$H^1(A^\vee, K^\times)$

...ll the ℓth roots of 1

...exp. ℓ

...$K^\times)$) $\simeq$ Rep_A

$\uparrow$

"A has good Fourier transform"

Theorem [JF - Reutter]:

Suppose $T^\otimes \supseteq \mathbb{Q}$ is a tower.

Then TFAE:

(A) $T^\otimes$ is Galois-closed.] $et(T) = *$.

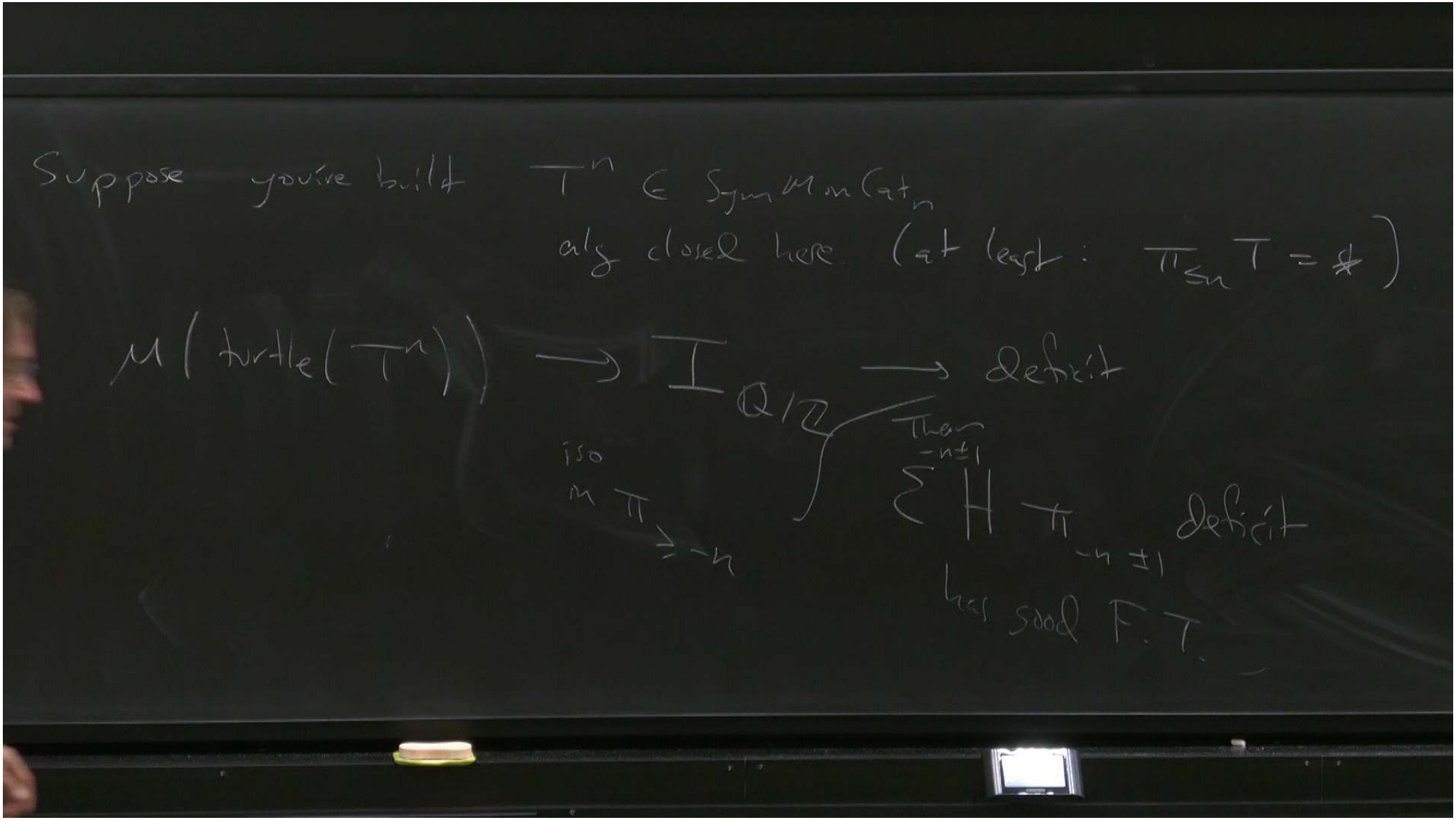
(B) (1) $\mu(T) \cong I_{\mathbb{Q}/\mathbb{Z}}$] tests ~~tests~~ abelian extensions

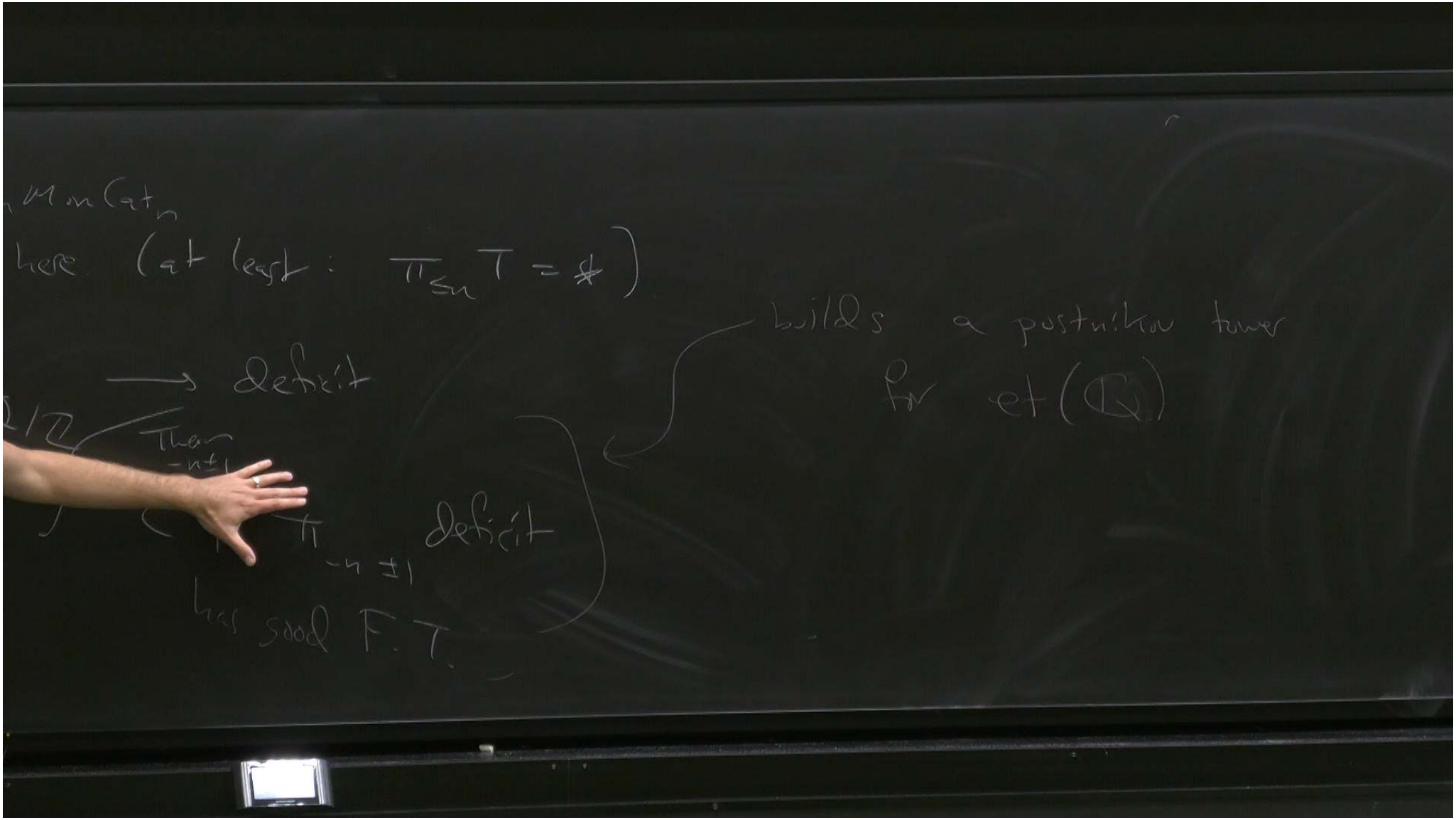
(2) for every finite gp G, there is a unique iso class of fibre functors $\mathrm{Rep}\, G \to T$] tests $\pi_{\leq 1} et(T)$

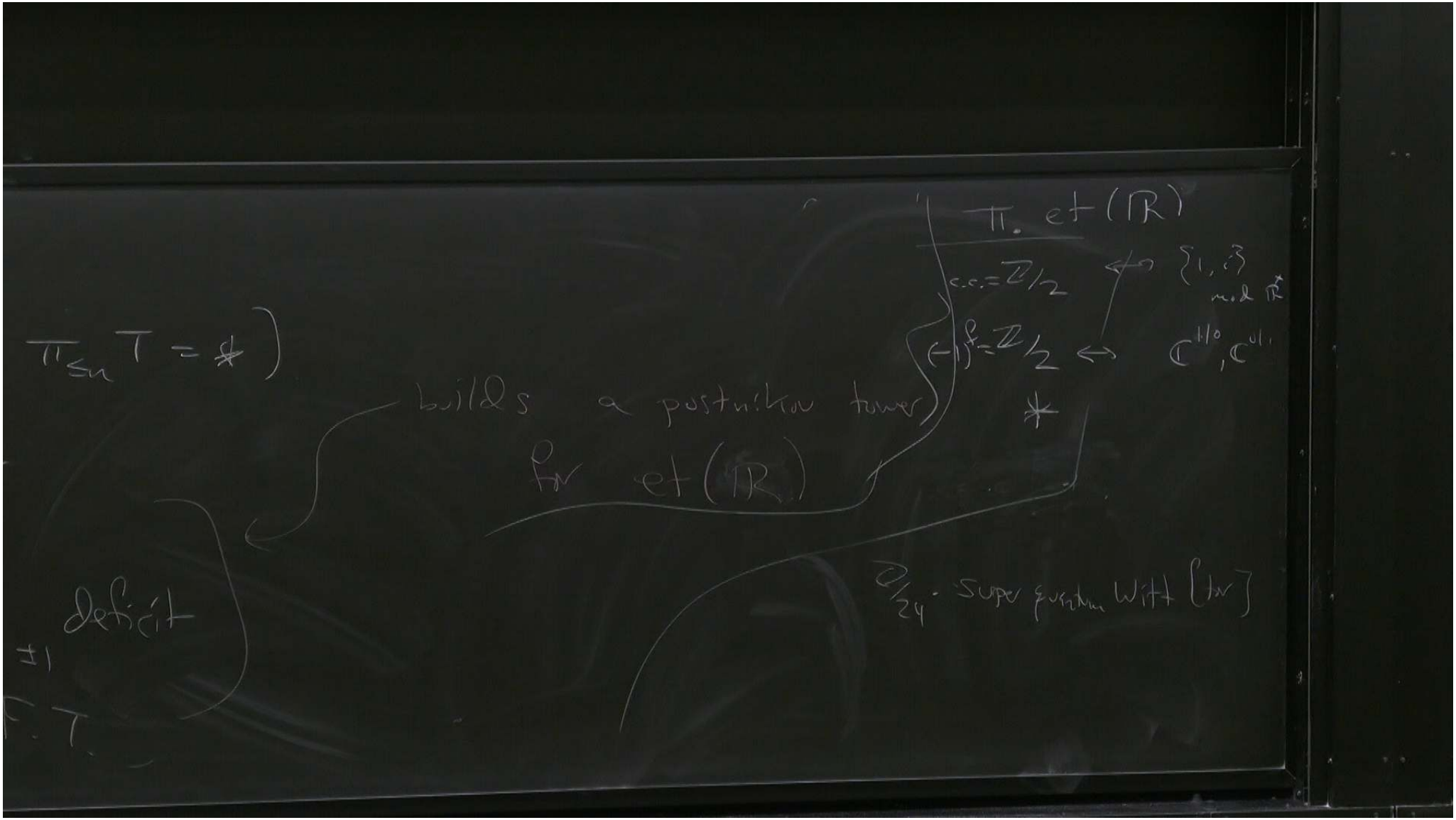
field

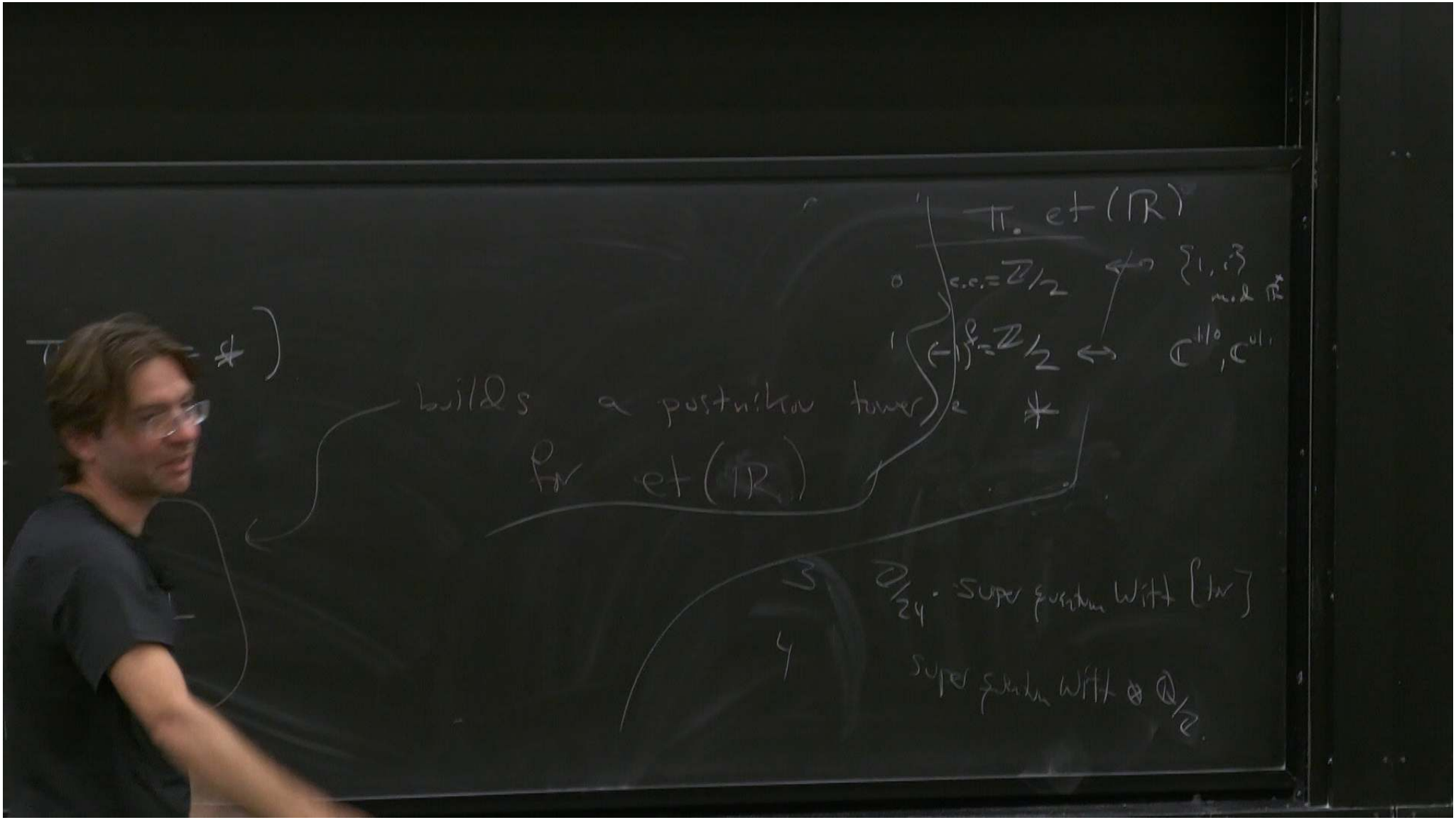
ab gp

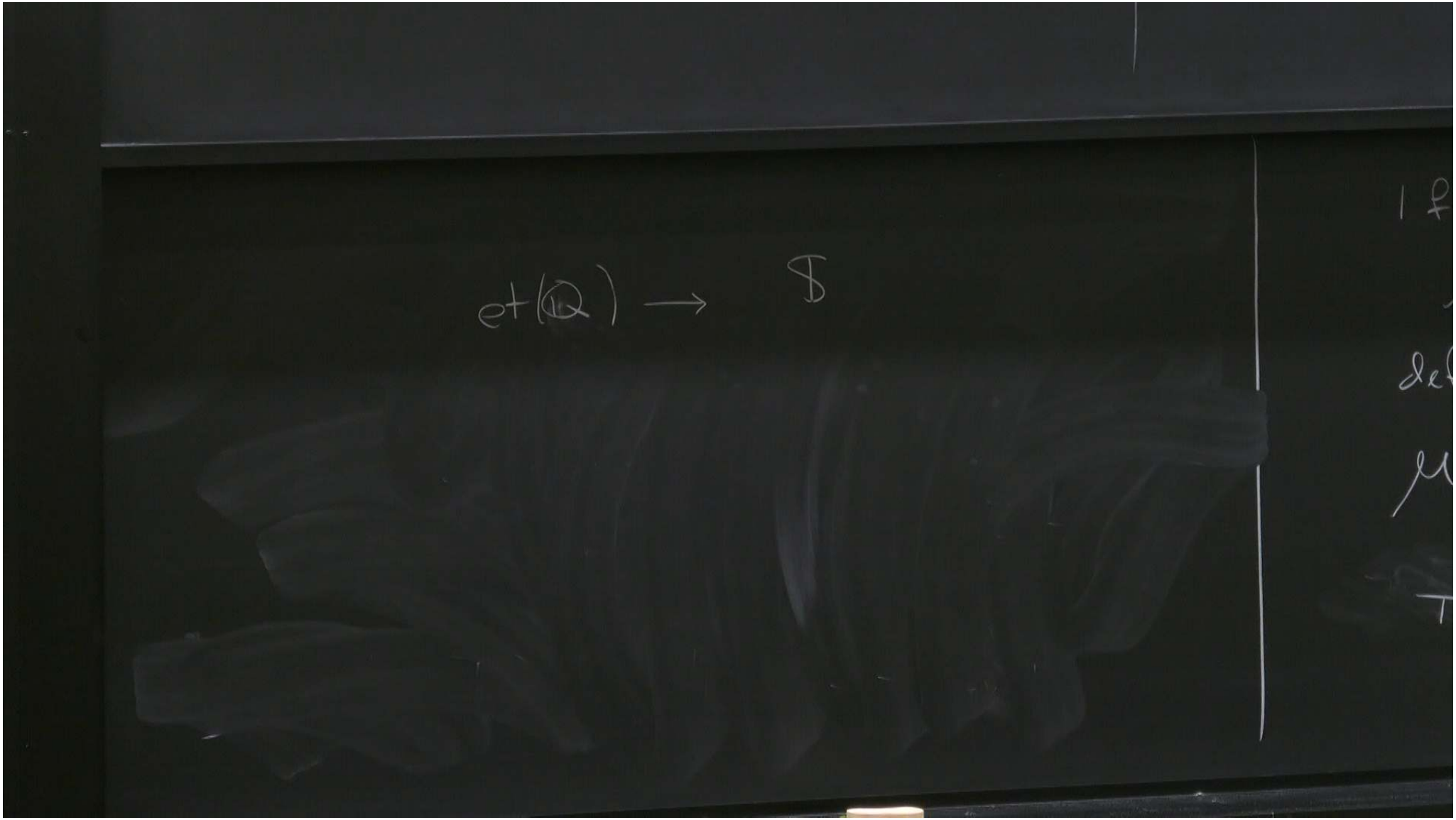
Kar

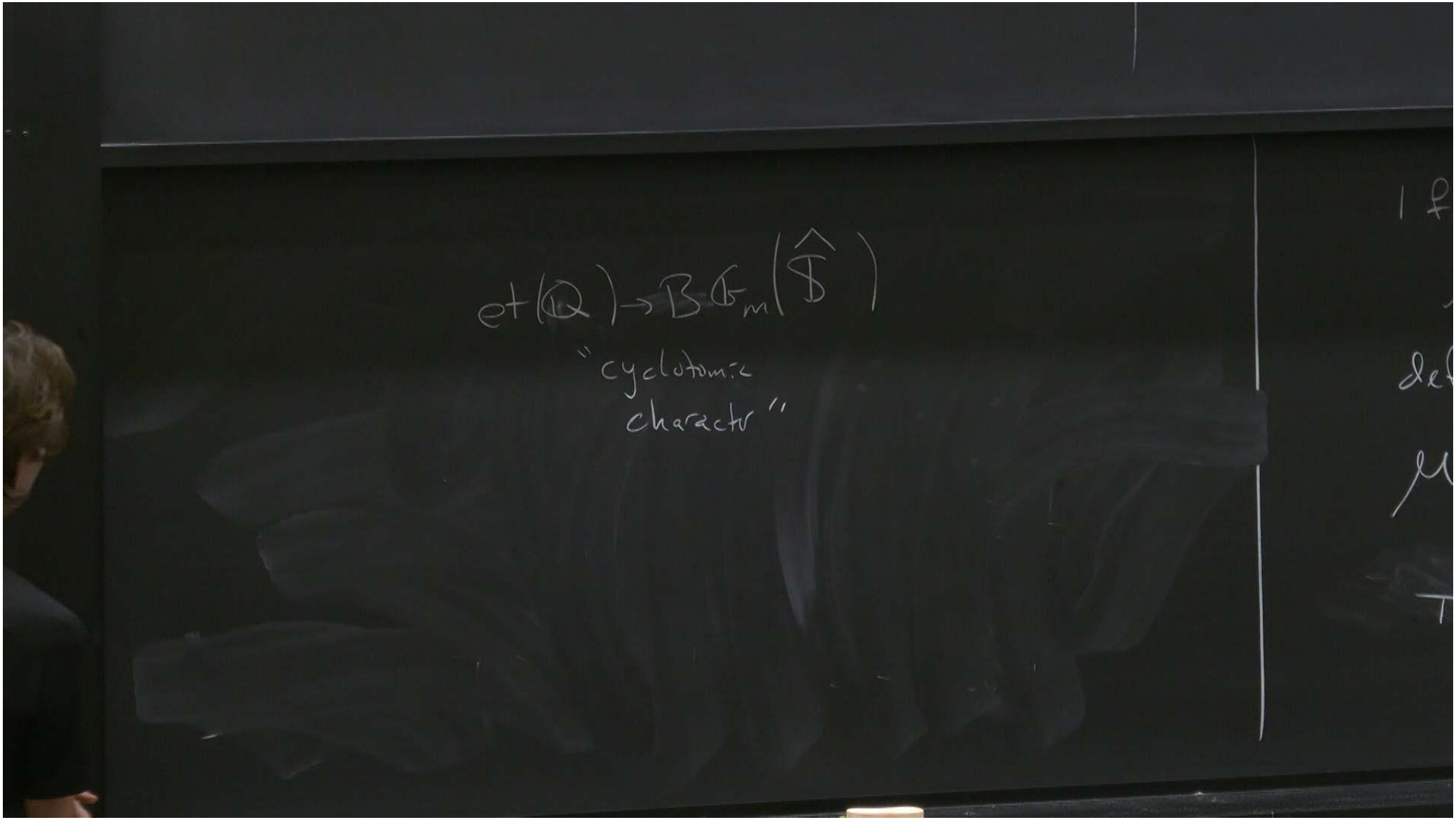

Suppose you've built $T^n \in \mathrm{SymMonCat}_n$
alg closed here (at least: $\pi_{\le n} T = *$)
$M(\mathrm{turtle}(T^n)) \longrightarrow I_{Q/Z} \longrightarrow$ deficit
iso
$M \pi_{\ge -n}$
Then
$\Sigma^{-n\pm 1} H \pi_{-n\pm 1}$ deficit
has good F.T.

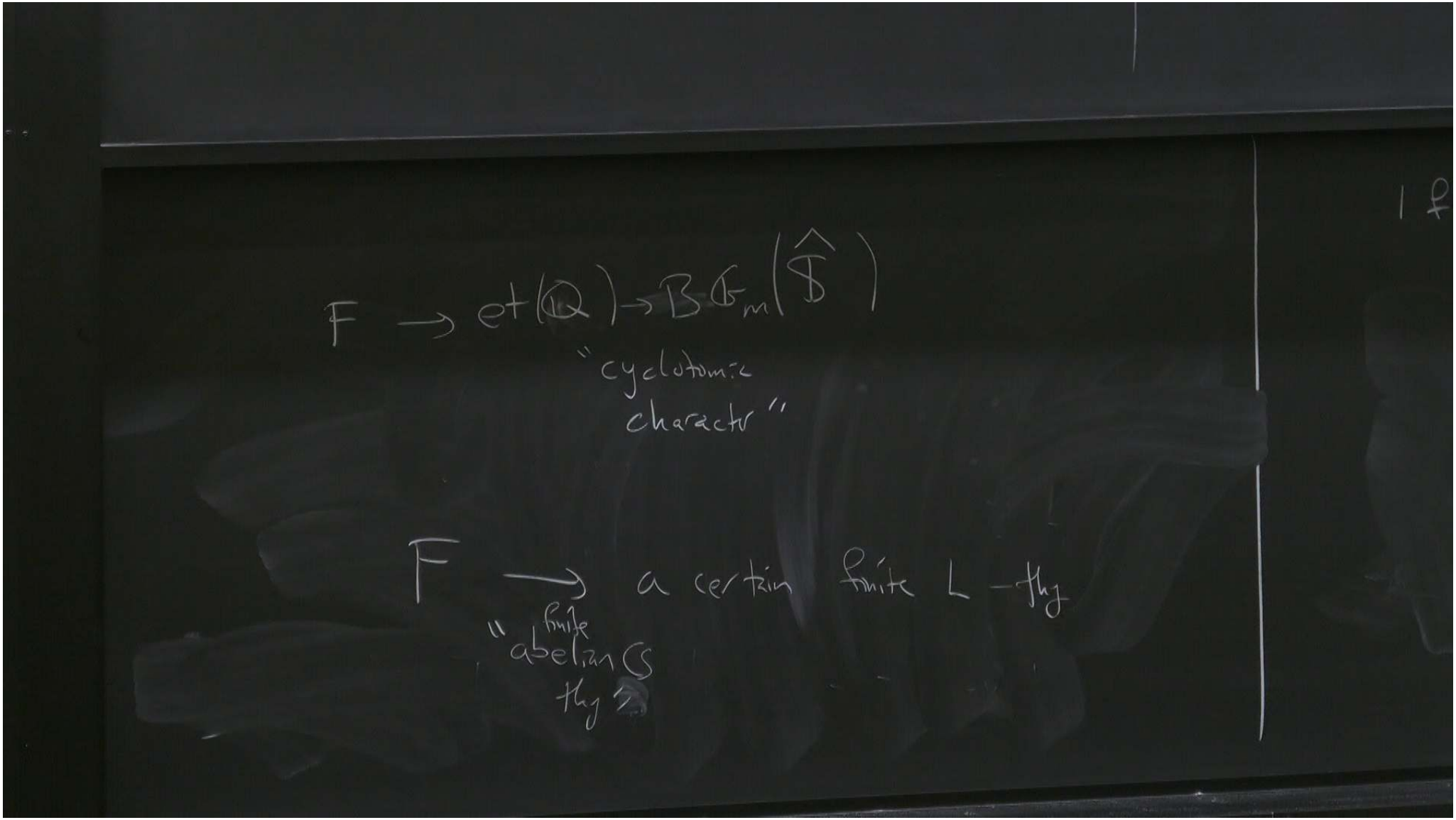

builds a postnikov tower
for et(Q)
deficit
deficit

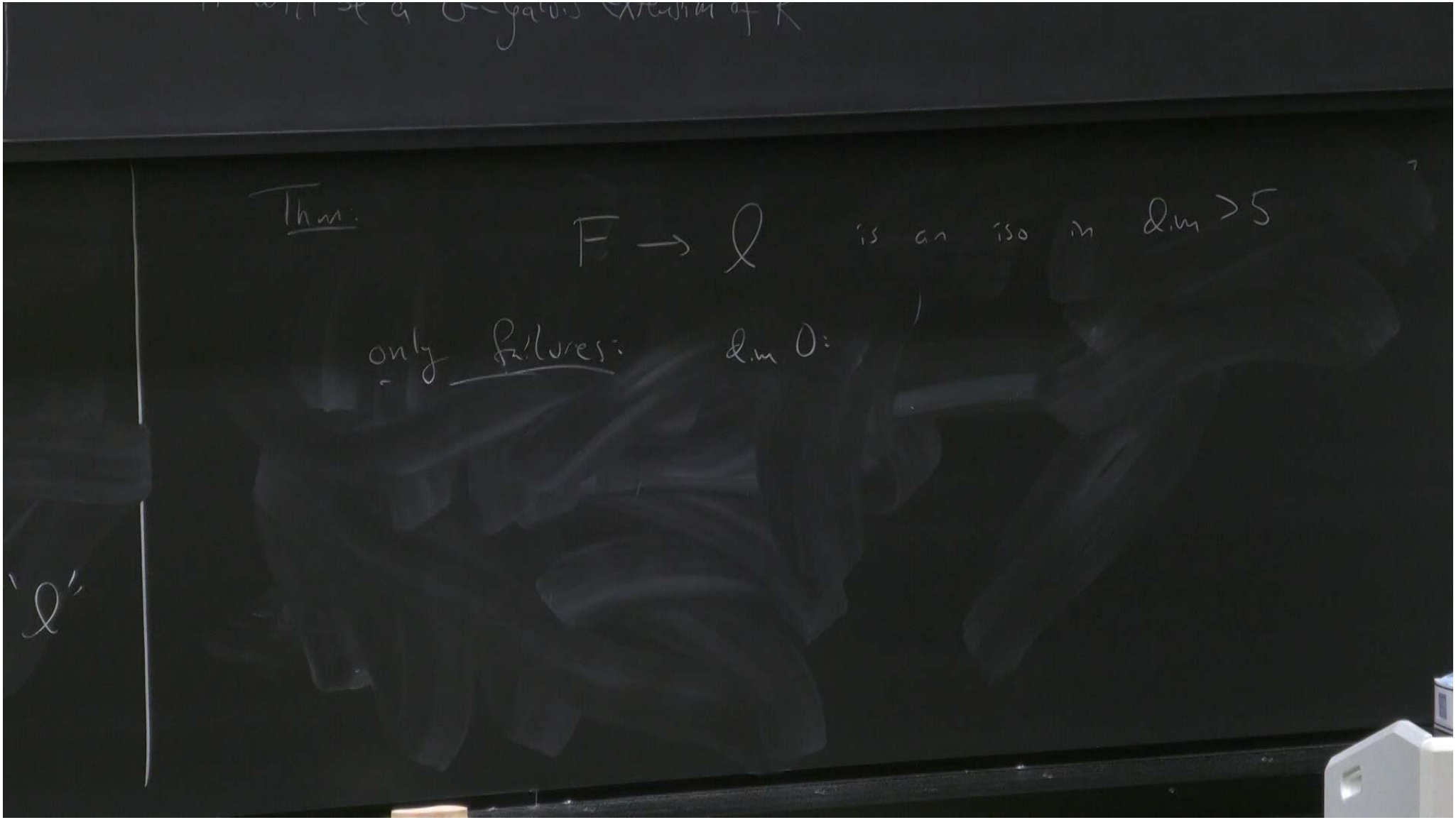

builds a postnikov tower
for et(ℝ)
deficit
$\mathbb{Z}/24$ - super question with [Ur]

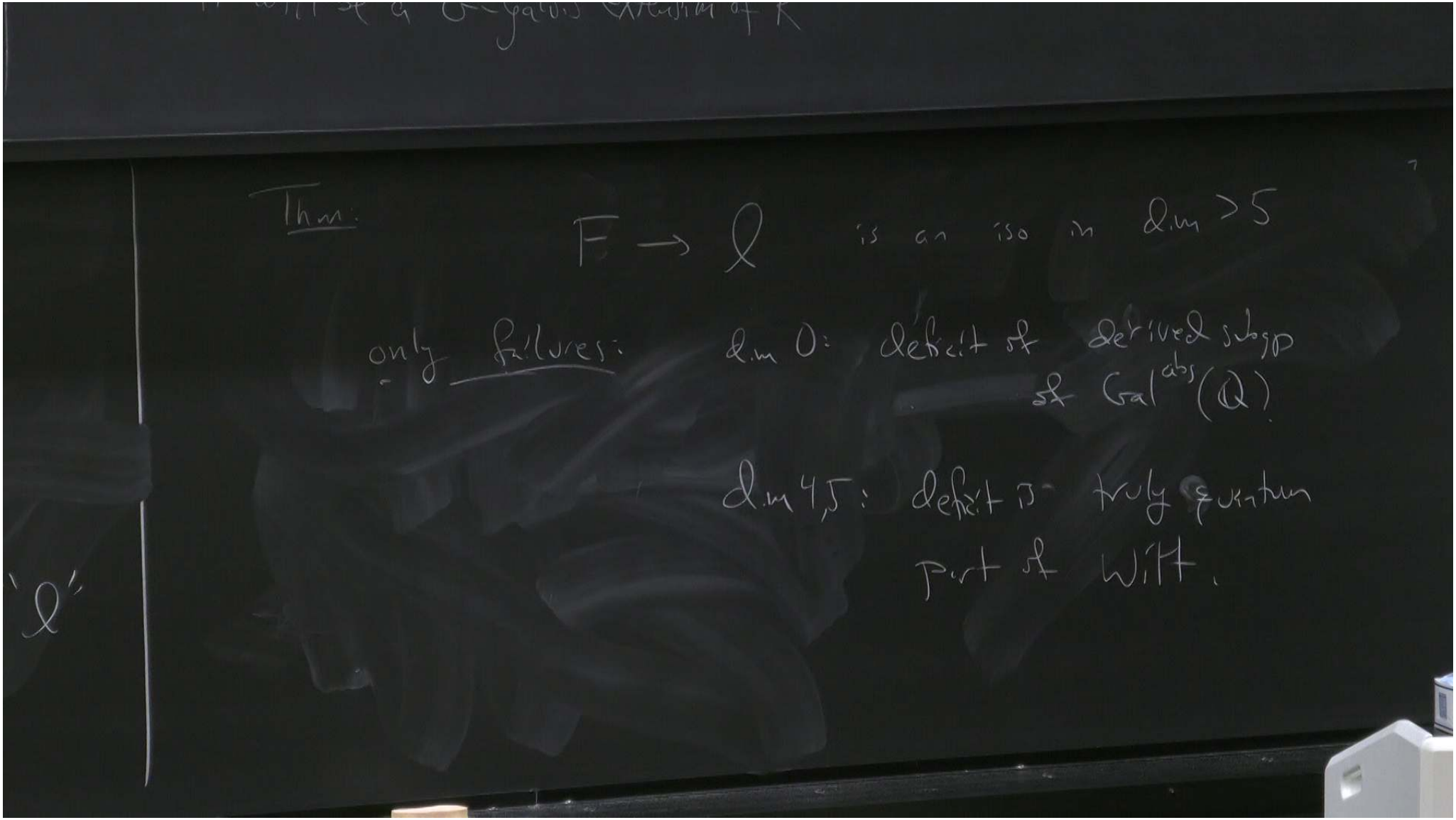

builds a postnikov tower
for et(ℝ)
π. et(ℝ)
c.c. = ℤ/2
(-1)^f = ℤ/2
ℂ^{1/0}, ℂ^{0/1}
mod 8
ℤ/24 · super quantum Witt [M]
super quantum Witt ⊗ ℚ/ℤ

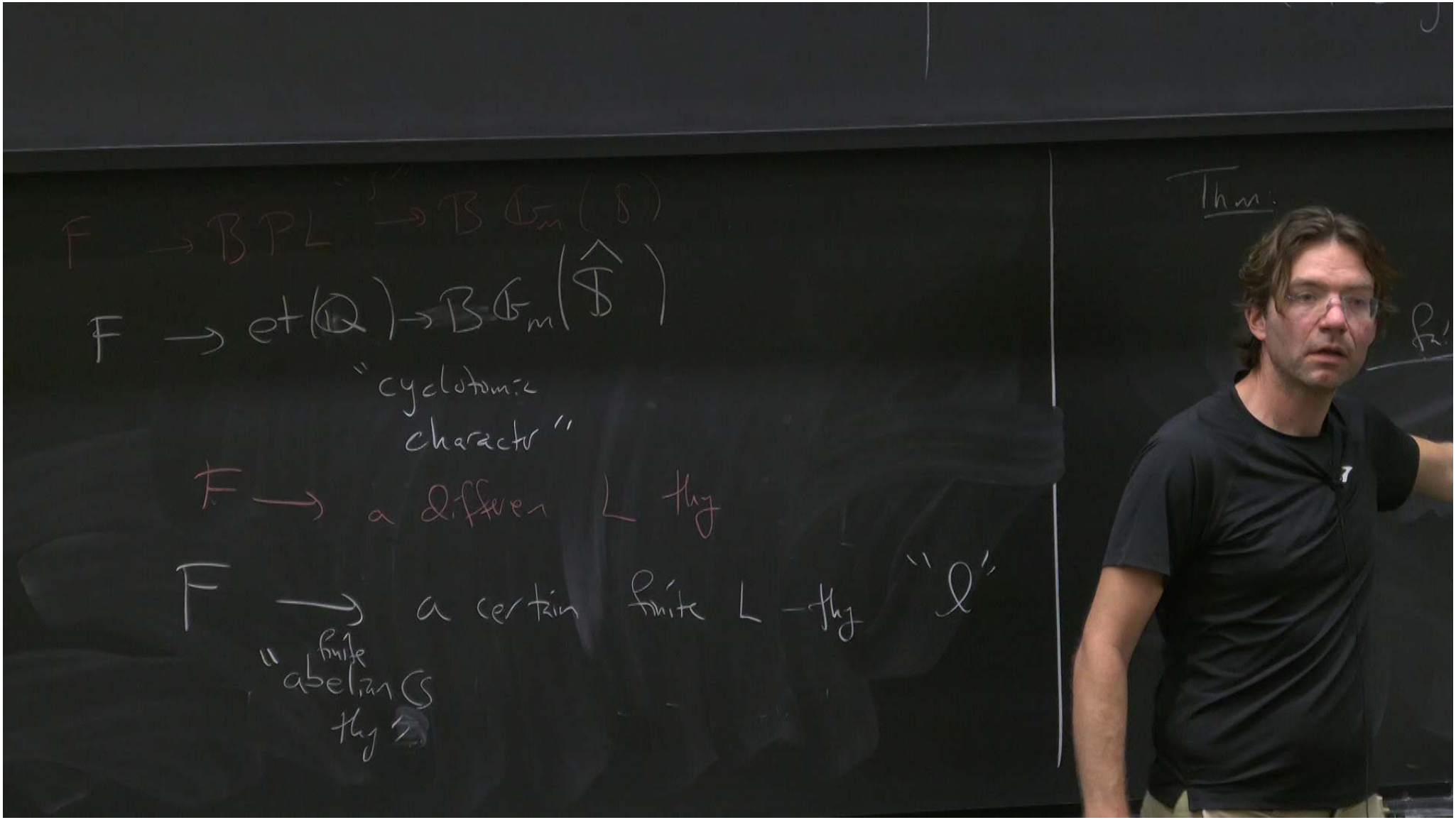

et(Q) → $


$et(\mathbb{Q}) \to B\mathbb{G}_m(\hat{\mathbb{S}})$
"cyclotomic character"


$F \to et(\mathbb{Q}) \to B\mathbb{G}_m(\hat{\mathbb{S}})$
"cyclotomic character"
$F \to$ a certain finite L-thy
"finite abelian $\mathbb{S}$ thy"


Thm:
$F \to \ell$ is an iso in $\dim > 5$
only failures: $\dim 0$:
ℓ'


Thm:
$F \to \ell$ is an iso in dim > 5
only failures:
dim 0: deficit of derived subgp of $Gal^{abs}(\mathbb{Q})$
dim 4,5: deficit is truly quantum part of Witt.
ℓ


$F \to BPL \to B\mathbb{G}_m(\mathbb{S})$
$F \to et(\mathbb{Q}) \to B\mathbb{G}_m(\hat{\mathbb{S}})$
"cyclotomic character"
$F \to$ a diffuse L-thy
$F \to$ a certain finite L-thy "ℓ"
"finite abelian $\mathbb{S}$-thy"
Thm: