Title: Deeper Kummer theory

Speakers: Theo Johnson-Freyd

Series: Mathematical Physics

Date: September 21, 2023 - 3:30 PM

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Abstract:

A tower is an infinite sequence of deloopings of symmetric monoidal ever-higher categories. Towers are places where extended functorial field theories take values. Towers are a "deeper" version of commutative rings (as opposed to "higher rings" aka E?-spectra). Notably, towers have their own opinions about Galois theory, and think that usual Galois groups are merely shallow approximations of deeper homotopical objects. In this talk, I will describe some steps in the construction and calculation of the deeper Galois group of a characteristic-zero field. In particular, I'll explain a homotopical version of the Kummer description of abelian extensions. This is joint work in progress with David Reutter.

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Zoom link: https://pitp.zoom.us/j/97950701035?pwd=Wk9FRSt2MkN3eWptTVltRVJncnFHdz09

Theorem [JF-Reutter]:

Suppose $T^\bullet \supseteq \mathbb{Q}$ is a tower.

Then TFAE:

(A) $T^\bullet$ is Galois-closed.

(B) (1) $\mathcal{M}(T) \cong I_{\mathbb{Q}/\mathbb{Z}}$

(2) for every finite gp G, there is a unique iso class of fibre functors $\mathrm{Rep}\, G \to T^1$.

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jt in progress w/ David Reutter

Deeper Galois Theory

today: say something about Kummer Theory

David Reutter

heory

bout

Theory

Towers: $\equiv$ targets for TQFT.

$n+1$ D TQFT Z

$Z(M^{n+1}) \in$ numbers $\quad \partial M = \emptyset$

David Reutter

theory

about

Towers $\equiv$ targets for TQFT.

n+1 D TQFT Z

$Z(M^{n+1}) \in A^0$ $\quad \partial M = \emptyset$ $\quad A^0$ a set

$Z(\emptyset) = 1$ $\quad\quad 1 \in A^0$.

$Z(M^n) \in A^1$ $\quad \partial M = \emptyset$ $\quad A^1$ a cat.

bordisms $\longrightarrow$ morphisms.

David Reutter

Theory

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Theory.

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bordisms $\longrightarrow$ morphisms.

$\hookrightarrow Z((\ )^{n+1})$ is multiplicative.

There is a unique iso class of fibre functors $\mathrm{Rep}\, G \to T$!

a k-times extended TQFT T is a functor

$$\mathrm{Bord}_{n-k,\dots,n+1} \xrightarrow{Z} \mathcal{A}^k$$

$\mathcal{A}^k$ is a k-category.

$$\emptyset \longmapsto 1$$

a fully extended TQFT is ∞-extended.

is multiplicative.

A <u>tower</u> is a sequence [Scheimbauer]

$$A^0,\ A^1,\ \ldots$$

s.t. $A^k \ni 1$ is a pointed k-cat,

together w/ equivs

$$A^{k-1} \simeq \Omega A^k := \mathrm{End}_{A^k}(1).$$

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N.B.: A *spectrum* is the same but w/ ~~k-cats~~ spaces.

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axioms:

A^k should be Karoubian i.e.:

(i) idemp. complete

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<u>axioms:</u>

A^k should be <u>Karoubian</u> ie:

(i) idemp. complete

(ii) additive

↳ has $\oplus$'s and $-1 \in A^0$.

If $A^\bullet$ is a tower, get $\mathbb{G}_m(A^\bullet) \in Sp$.

$\mathbb{G}_m(A)^n$ = invertible objects in A^n
+ morphisms

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$I_{\mathbb{C}^\times} \in Sp$ is the spectrum s.t.

$$[-, I_{\mathbb{C}^\times}] = \hom_{AbGrp}(\pi_0(-), \mathbb{C}^\times)$$

$$\parallel$$

$$\pi_0 \hom_{Sp}(-, I_{\mathbb{C}^\times})$$

N.B: A ~~spectrum~~ spectrum is the same but w/ ~~∞-cats~~ spaces.

Observation [Freed–Hopkins]

If $G^{0}_{m}(A) = I_{\mathbb{C}^\times}$, then invertible TQFTs are determined by their partition fns.

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Question (Hopkins):

Find a good tower T s.t.

$T^1 \cong sVec_{\mathbb{C}}$, $\mathbb{G}_m(T) = I_{\mathbb{C}^\times}$

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$$T^1 \simeq s\mathrm{Vec}_{\mathbb{C}}, \quad \mathbb{G}_m(T) \cong I_{\mathbb{C}^\times}$$

and convince your choice is good.

$$\pi_K \mathcal{B} = [\delta[K], \mathcal{B}] \qquad \mathcal{B} \in S_p$$

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$$\pi_K I_{\mathbb{C}^\times} = \mathrm{hom}(\pi_{-K}\, \mathbb{S}, \mathbb{C}^\times)$$

$\pi_k \mathcal{B} = [\mathbb{S}[k], \mathcal{B}]$ $\qquad \mathcal{B} \in \mathrm{Sp}$

$\pi_k I_{\mathbb{C}^\times} = \hom(\pi_{-k}\mathbb{S}, \mathbb{C}^\times)$ $\qquad \mathbb{S} = M\mathrm{fr}$

Koszul sign rule.

$$\mathrm{Gm}(\mathrm{sVec}_{\mathbb{C}}) = \begin{matrix} \mathbb{C}^\times \\ \mathbb{Z}/2 \end{matrix} \circlearrowleft \boxed{(-1)^{\int \xi^2}}$$

π_0	$\mathbb{C}^\times$	
	$\circlearrowleft$	$(-1)^{\int \xi^2}$
π_{-1}	$\mathbb{Z}/2$	
	$\circlearrowleft$	$\int \xi^2$
π_{-2}	$\mathbb{Z}/2$	
	$\circlearrowleft$	$\int \xi^2 \pmod 2$
π_{-3}	$\mathbb{Z}/24$	

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Find a good tower T s.t. and co your ch good

$T^1 \cong sVec^{f.d.}$ $\quad \mathbb{G}_m(T) \cong I_{\mathbb{C}^\times}$

Guess: $T^2 \cong sAlg_{\mathbb{C}}^{f.d.}$.

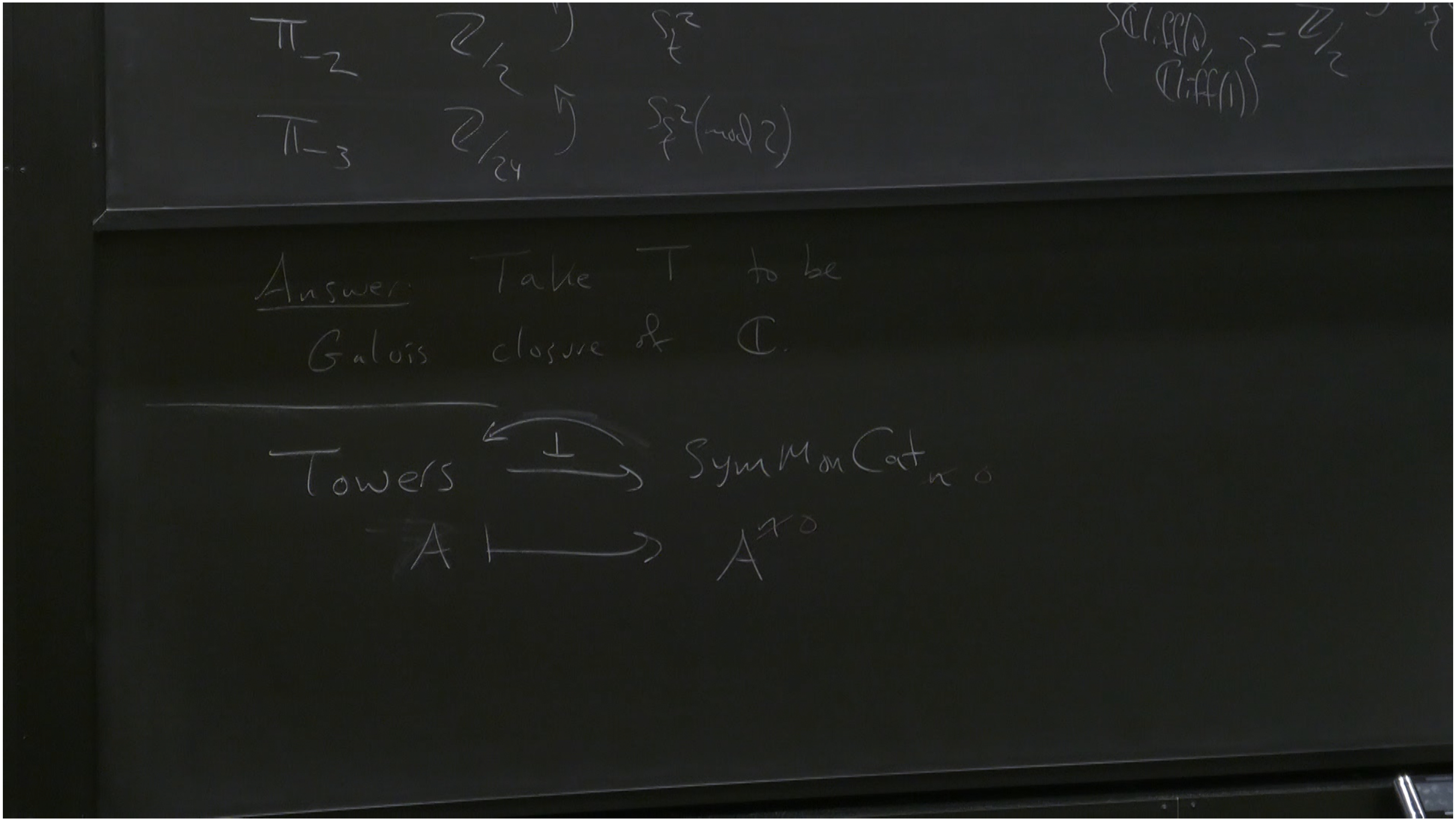
$\pi_k \mathbb{B} = [\mathbb{S}[k], \mathbb{B}] \qquad \mathbb{B} \in \mathrm{Sp}$

$\pi_k I_{\mathbb{C}^\times} = \hom(\pi_{-k} \mathbb{S}, \mathbb{C}^\times) \qquad \mathbb{S} = M\mathrm{fr}$

Koszul sign rule.

$$\begin{array}{ll}
\pi_0 & \mathbb{C}^\times \\
 & \quad \circlearrowleft \ (-1)^{Sq^2} \\
\pi_{-1} & \mathbb{Z}/2 \\
 & \quad \circlearrowleft \ Sq^2 \\
\pi_{-2} & \mathbb{Z}/2 \\
 & \quad \circlearrowleft \ Sq^2 (\mathrm{mod}\ 2) \\
\pi_{-3} & \mathbb{Z}/24
\end{array}$$

$$\mathbb{G}_m(\mathrm{sAlg}_{\mathbb{C}}) = \begin{array}{l} \mathbb{C}^\times \\ \quad \circlearrowleft \ \boxed{(-1)^{Sq^2}} \\ \{\mathbb{C}^{1|0}, \mathbb{C}^{0|1}\} = \mathbb{Z}/2 \\ \quad \circlearrowleft \ Sq^2 \\ \{\mathbb{C}\mathrm{liff}(0), \mathbb{C}\mathrm{liff}(1)\} = \mathbb{Z}/2 \end{array}$$


π_{-2} $\mathbb{Z}/2$ $S\xi^2$
$\{Cliff(0), Cliff(1)\} = \mathbb{Z}/2$
π_{-3} $\mathbb{Z}/24$ $S\xi^2 \pmod 2$
Answer: Take T to be Galois closure of $\mathbb{C}$.
$\mathrm{Towers} \rightleftarrows \mathrm{SymMonCat}_{\neq 0}$
$A \longmapsto A^{\neq 0}$

$\pi_{-2} \quad \mathbb{Z}/2 \quad Sq^2$

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<u>Answer:</u> Take T to be Galois closure of $\mathbb{C}$.

Turtle delooping + Karoubi completion

$$\begin{tikzcd} \text{Towers} \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathrm{SymMonCat}_{\geq 0} \arrow[l, bend right, shift left] \end{tikzcd}$$

$$A \longmapsto A^{\geq 0}$$

$\pi_{-2} \quad \mathbb{Z}/2 \quad Sq^2$

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Turtle delooping + Karoubi completion

$$\mathrm{Towers} \underset{\text{i.e.}}{\overset{\perp}{\rightleftarrows}} \mathrm{SymMonCat}_{\neq 0}$$

$$A \longmapsto A^{\neq 0}$$

Fact: Let K a field, G a finite gp

$$F \in \mathrm{Fun}_{\substack{\mathrm{sym}\otimes \\ K\text{-lin}}}\Big(\underset{\substack{\rotatebox{90}{$\in$} \\ \mathcal{O}(G)}}{\mathrm{Rep}_K G}, \mathrm{Vec}_K\Big).$$

$F(\mathcal{O}(G)) \in \mathrm{Vec}$ will be a com alg.

it will be a G-Galois extension of K

Fact: Let K a field, G a finite gp

$$F \in \mathrm{Fun}_{\substack{\mathrm{sym}\otimes \\ K\text{-lin}}}\left(\mathrm{Rep}_K G,\ \mathrm{Vec}_K\right) = \begin{matrix} G\text{-bundles} \\ \text{on} \\ \mathrm{Spec}\ K \end{matrix} = \mathrm{maps}\Big(\underbrace{\mathbb{B}\mathrm{Gal}^{\mathrm{abs}}(K)}_{\substack{\shortparallel \\ \mathrm{et}(K)}}, \mathbb{B}G\Big).$$

$$\begin{matrix} \mathrm{Rep}_K G \\ \Psi \\ \mathcal{O}(G) \end{matrix}$$

$F(\mathcal{O}(G)) \in \mathrm{Vec}$ will be a com alg.

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Given X a π-finite space, $T \in \mathrm{Tow}$

$$X \mapsto \mathrm{Hom}_{\mathrm{Tow}_{T/}}(T^X, T) \in \text{Spaces}$$

If this is proco representable, its corep'ing object is the étale space $et(T)$.

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If this is proco representable, its corep'ing object is the étale space $et(T)$.

<u>Prop:</u> If $T \supseteq \mathbb{Q}$, then $et(T)$ exists.

a functor

Theorem [JF–Reutter]:

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it in p

Deeper

today:

tests $\pi_{\leq 1}\, \mathrm{et}(T)$

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$\mathfrak{z}^t$ in ρ

Deeper

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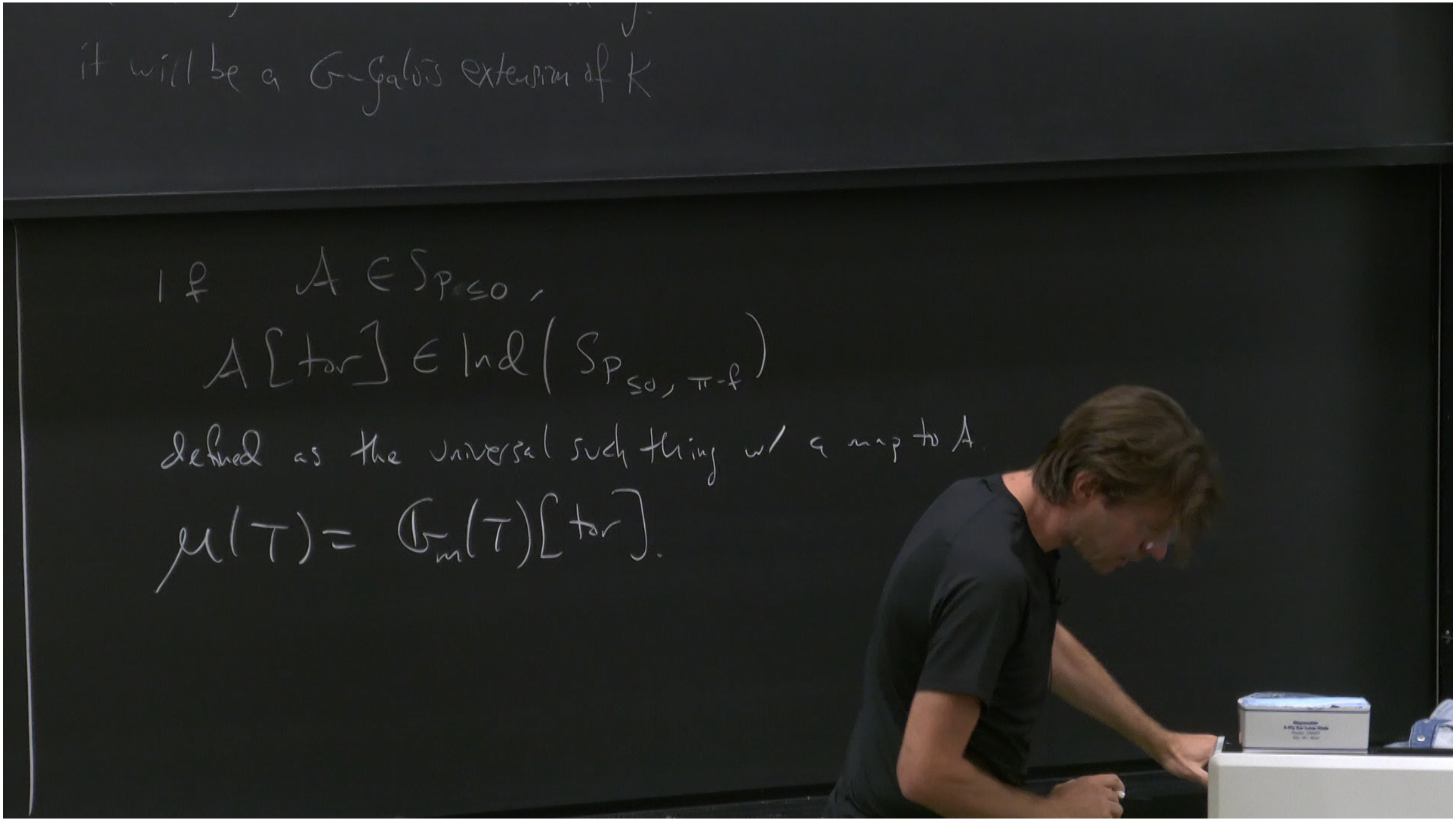
it will be a G-Galois extension of K
If $A \in Sp_{p,\leq 0}$,
$A[\mathrm{tor}] \in \mathrm{Ind}(Sp_{\leq 0,\pi\text{-}f})$
defined as the universal such thing w/ a map to A.
$\mu(T) = \mathbb{G}_m(T)[\mathrm{tor}]$.

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$\mu(T) = \mathbb{G}_m(T)[\mathrm{tor}]$.

$\pi_k(A[\mathrm{tor}]) \hookleftarrow (\pi_k A)[\mathrm{tor}]$

$\hookleftarrow \mathbb{Z}$'s in π_{k+1}

$\mathbb{Z}[\mathrm{tor}] = \Sigma^{-1} \mathbb{Q}/\mathbb{Z}$

$A^{k-1} \simeq \Omega A^k := \mathrm{End}_{A^k}(1)$.

N.B.: A spectrum is the same but w/ ~~tr-cats~~ spaces.

field $K \supseteq$ all the ℓ^{th} roots of 1.

ab gp A of exp. ℓ

$\mathrm{Kar}_{\oplus}(\mathrm{hom}(A, K^\times)) \xrightarrow{\sim} \mathrm{Rep}_A$

"A has good Fourier transform"

N.B.: A Spectrum is the same but w/ ~~∞-cats~~ spaces.

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$K^\times)) \xleftrightarrow{\sim} \mathrm{Rep}_A$

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Kummer Theory:

Then A-galois extensions of k

$\updownarrow$

$\mathrm{maps}(A^\vee, \Sigma H K^\times)$

$\|$

$H^1(A^\vee, K^\times)$

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(2) for every finite gp G, there is a unique iso class of fibre functors $\operatorname{Rep} G \to T^1$] tests $\pi_{\leq 1}\, et(T)$.

field

ab gp

$\mathrm{Kar}_{\oplus}$

Suppose you've built $T^n \in \mathrm{SymMonCat}_n$

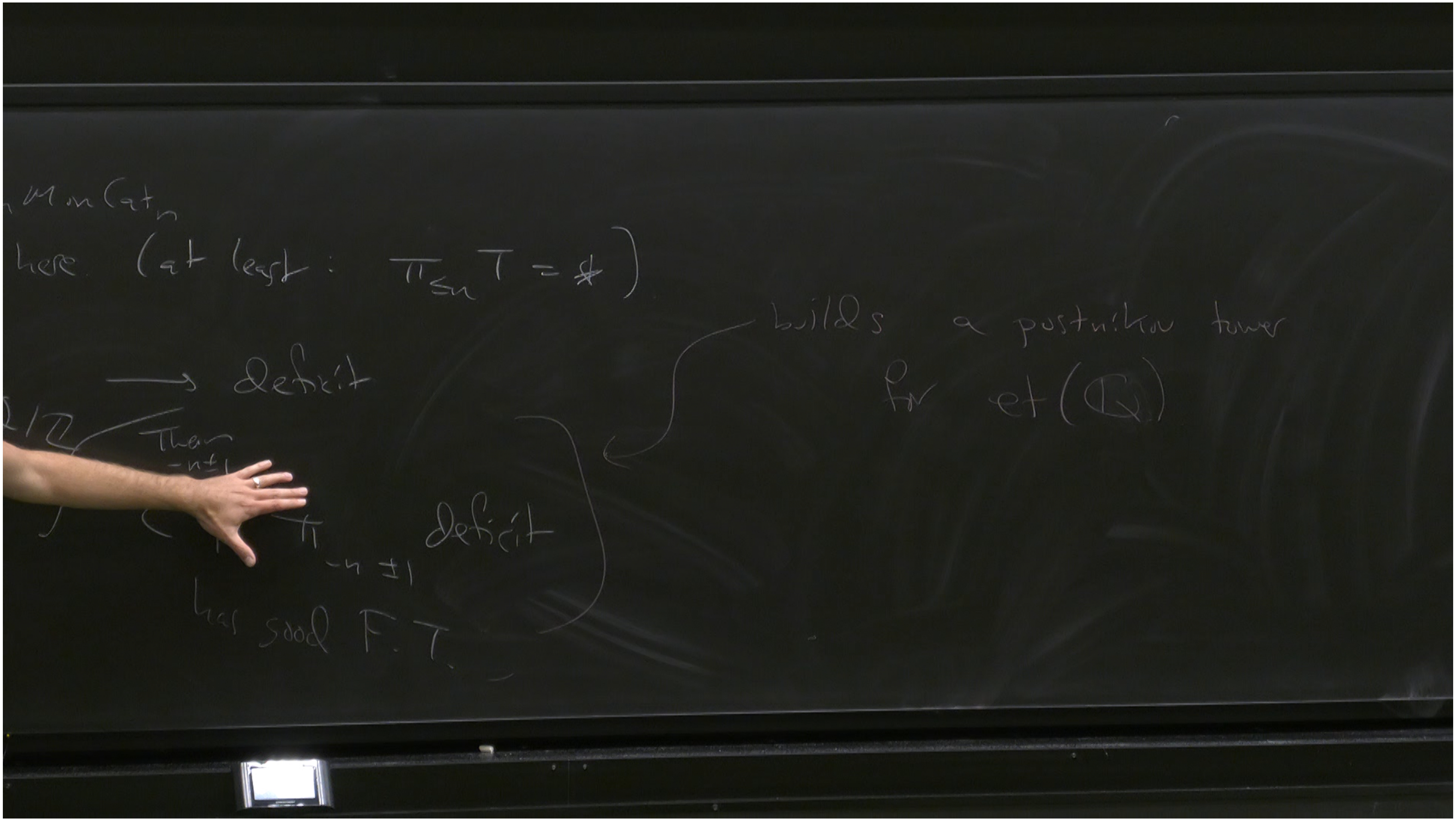
alg closed here (at least: $\pi_{\leq n} T = *$)

$$\mu(\text{turtle}(T^n)) \longrightarrow I_{\mathbb{Q}/\mathbb{Z}} \longrightarrow \text{deficit}$$

iso on $\pi_{\geq -n}$

Then $\Sigma^{-n\pm 1} H \pi_{-n\pm 1}$ deficit

has good F.T.


MonCat_n
here (at least: π_≤n T = *)
→ deficit
Then
deficit
-n ±1
has good F.T.
builds a postnikov tower
for et(Q)

π_0 et$(\mathbb{R})$

c.c. $= \mathbb{Z}/2 \longleftrightarrow \{1, i\}$ mod $\mathbb{R}^\times$

$(-1)^f = \mathbb{Z}/2 \longleftrightarrow \mathbb{C}^{1|0}, \mathbb{C}^{0|1}$

$*$

$\pi_{\leq n} T = *$)

builds a postnikov tower)

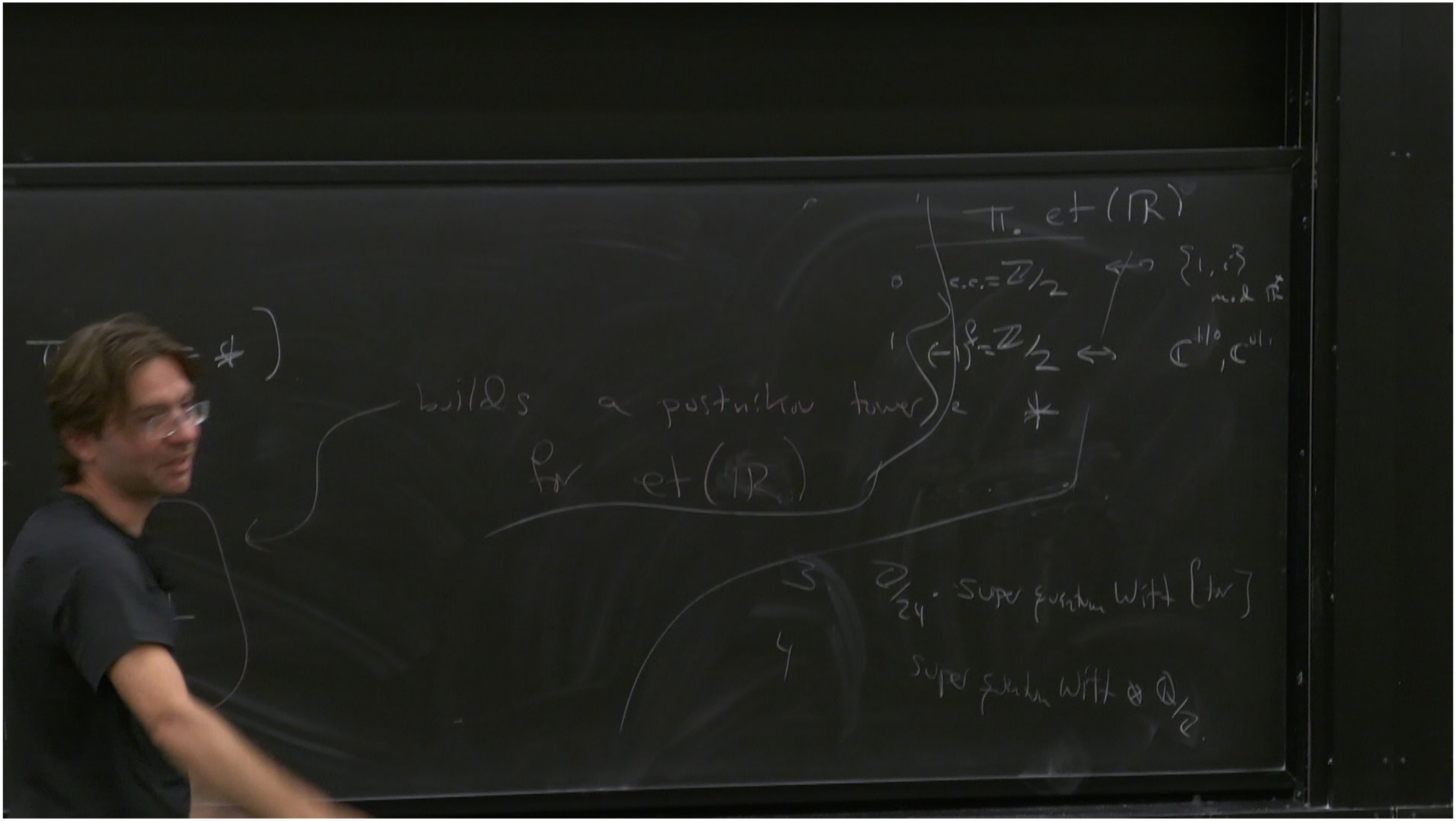
for et$(\mathbb{R})$

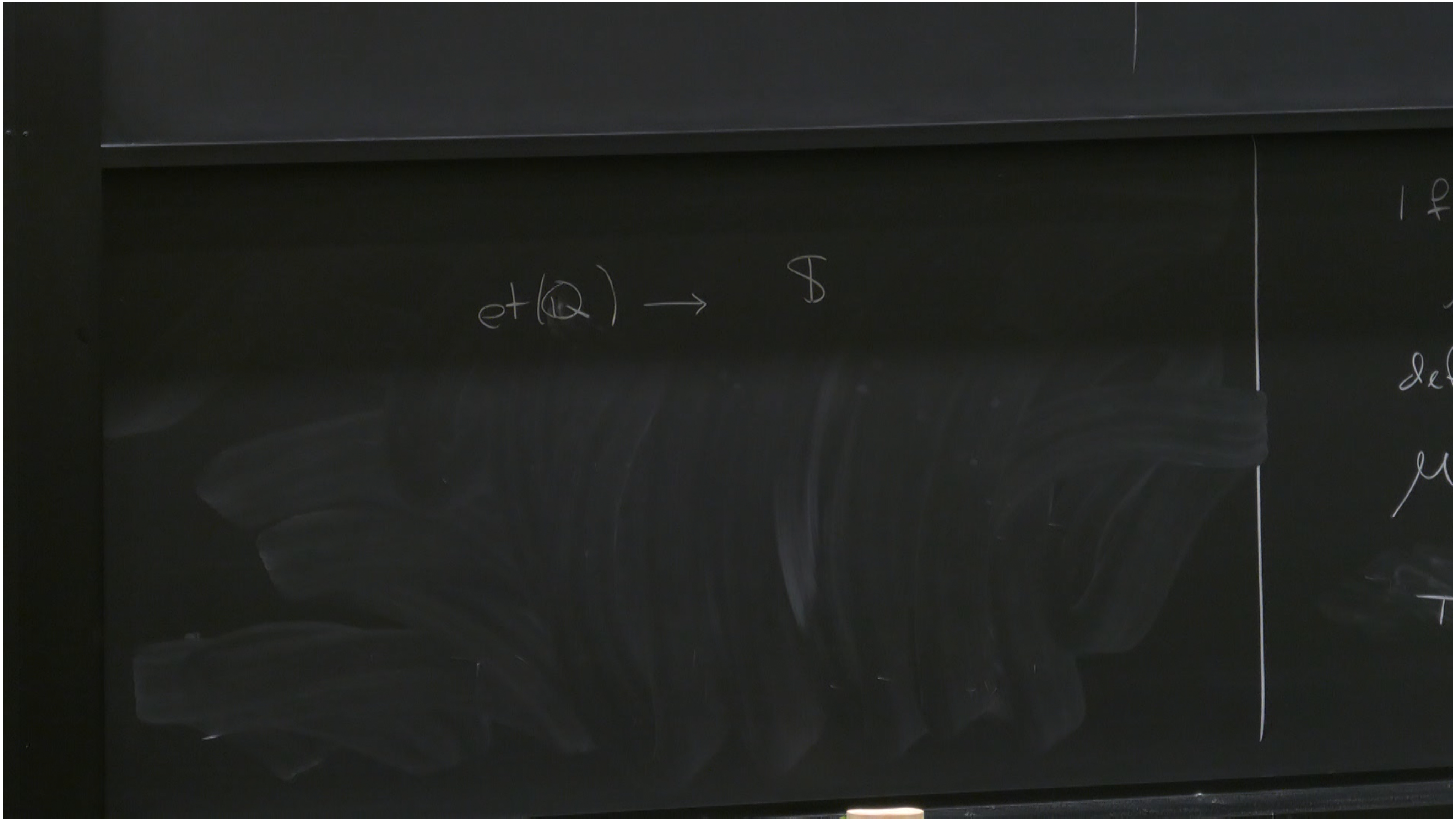
$\mathbb{Z}/24$ - super quantum Witt [thr]

deficit

± 1

F. T.


π₀ et(ℝ)
builds a postnikov tower
for et(ℝ)
super quantum Witt
super quantum Witt



$et(\mathbb{Q}) \longrightarrow \mathbb{S}$

$$et(\mathbb{Q}) \to B\mathbb{G}_m(\widehat{\mathbb{S}})$$

"cyclotomic character"

$$F \to \mathrm{et}(\mathbb{Q}) \to B\mathbb{G}_m(\widehat{\mathbb{S}})$$

"cyclotomic character"

$$F \longrightarrow \text{a certain finite } L\text{-thy}$$

"finite abelian CS thy?"

will be a G-galois extension of k

Thm: $F \to \ell$ is an iso in dim > 5

only failures: dim 0:

'ℓ'

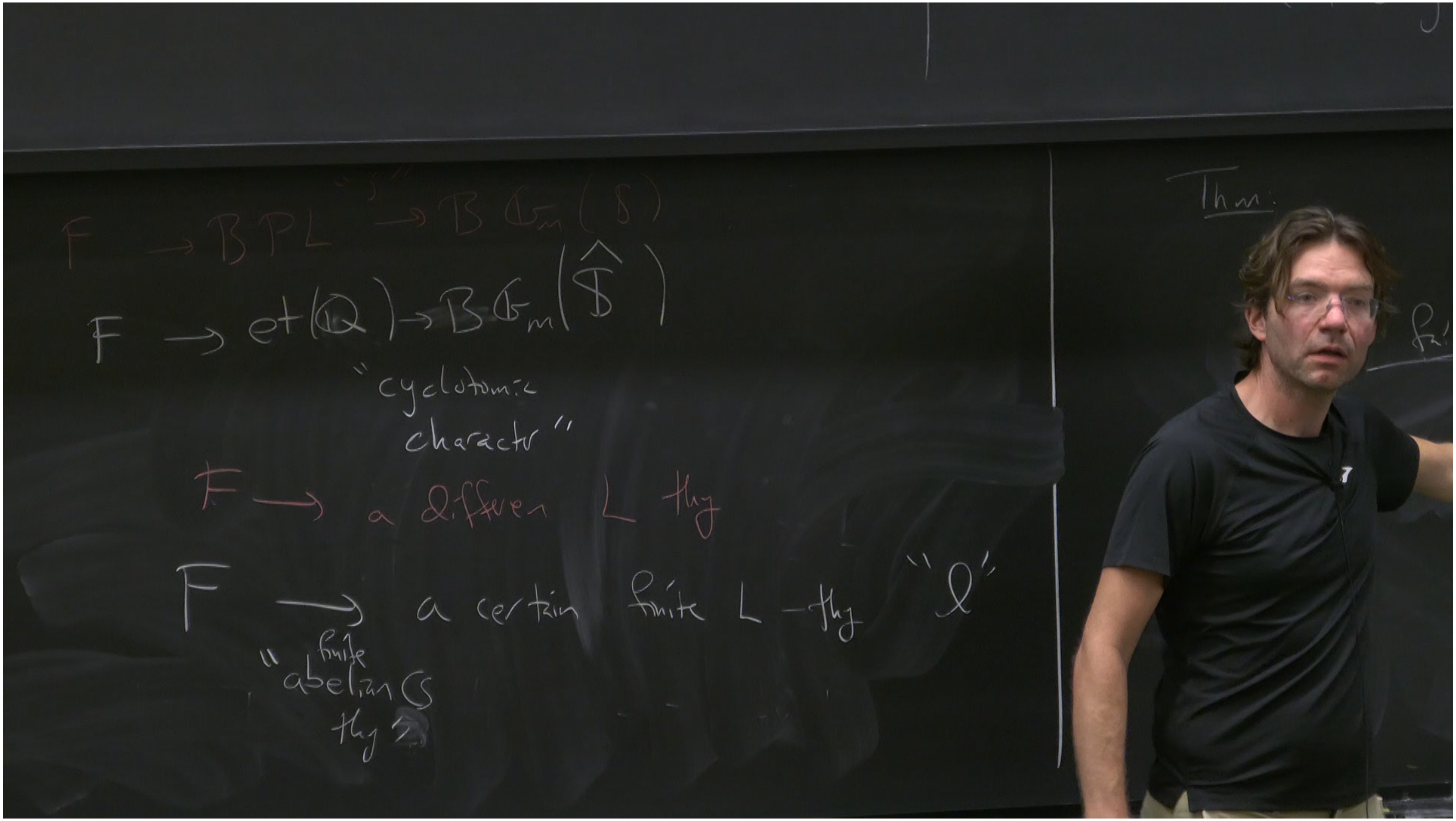
... of a G-galois extension of k

'ℓ'

Thm: $F \to \ell$ is an iso in dim > 5

only failures: dim 0: deficit of derived subgp of $\mathrm{Gal}^{abs}(\mathbb{Q})$

dim 4,5: deficit is truly quantum part of Witt.


F → BPL → BGm(S)
F → et(Q) → BGm(Ŝ)
"cyclotomic character"
F → a differen L thy
F → a certain finite L-thy "ℓ"
"finite abelian G thy"
Thm: