Title: Title: Braid group, Askey-Wilson algebra and centralizers of U_q(sl_2)

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Abstract: In this talk, I will consider the centralizer of the quantum group U_q(sl_2) in the tensor product of three identical spin representations. The case of spin 1/2 (fundamental representation) is understood within the framework of the Schur-Weyl duality for U_q(sl_N), and the centralizer is known to be isomorphic to a Temperley-Lieb algebra. The case of spin 1 has also been studied and corresponds to the Birman-Murakami-Wenzl algebra. For a general spin, I will explain how to describe explicitly the centralizer (by generators and relations) using a combination of the braid group algebra and the Askey-Wilson algebra, which has been introduced in the context of orthogonal polynomials.

Zoom link: https://pitp.zoom.us/j/98471794356?pwd=NjZFdjRFaDFON05HNkdTZS9hZTUvQT09

Braid group, Askey–Wilson algebra and centralizers of $U_q(\mathfrak{sl}_2)$

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based on joint work with
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$$U(\mathfrak{sl}_n) \xrightarrow{\pi} \mathrm{End}((\mathbb{C}^n)^{\otimes k}) \xleftarrow{\phi} \mathbb{C}S_k$$

Schur–Weyl duality

Vector space $\mathbb{C}^n$ of finite dimension n, take k-fold tensor product $(\mathbb{C}^n)^{\otimes k}$.

- Diagonal action of $X \in \mathfrak{sl}_n$ on $(\mathbb{C}^n)^{\otimes k}$

$$
\begin{aligned}
X \cdot (v_1 \otimes v_2 \otimes \cdots \otimes v_k) = \; & X \cdot v_1 \otimes v_2 \otimes \dots \otimes v_k \\
+ \; & v_1 \otimes X \cdot v_2 \otimes \dots \otimes v_k \\
+ \; & \dots \\
+ \; & v_1 \otimes v_2 \otimes \dots \otimes X \cdot v_k
\end{aligned}
$$

Extension of this action to universal enveloping algebra $U(\mathfrak{sl}_n)$.

- Action of symmetric group algebra $\mathbb{C}S_k$ on $(\mathbb{C}^n)^{\otimes k}$

$$\sigma_i \cdot (v_1 \otimes \cdots \otimes v_i \otimes v_{i+1} \otimes \cdots \otimes v_k) = v_1 \otimes \cdots \otimes v_{i+1} \otimes v_i \otimes \cdots \otimes v_k$$

$$U(\mathfrak{sl}_n) \xrightarrow{\pi} \mathrm{End}((\mathbb{C}^n)^{\otimes k}) \xleftarrow{\phi} \mathbb{C}S_k$$

The images of $U(\mathfrak{sl}_n)$ and $\mathbb{C}S_k$ in $\mathrm{End}((\mathbb{C}^n)^{\otimes k})$ are full mutual centralizers.

$$\phi(\mathbb{C}S_k) = \mathrm{End}_{U(\mathfrak{sl}_n)}((\mathbb{C}^n)^{\otimes k})$$
$$\pi(U(\mathfrak{sl}_n)) = \mathrm{End}_{\mathbb{C}S_k}((\mathbb{C}^n)^{\otimes k})$$

$$\mathrm{End}_{U(\mathfrak{sl}_n)}((\mathbb{C}^n)^{\otimes k}) = \{f \in \mathrm{End}((\mathbb{C}^n)^{\otimes k}) \mid f\pi(X) = \pi(X)f \quad \forall X \in U(\mathfrak{sl}_n)\}$$
$$\mathrm{End}_{\mathbb{C}S_k}((\mathbb{C}^n)^{\otimes k}) = \{f \in \mathrm{End}((\mathbb{C}^n)^{\otimes k}) \mid f\rho(\sigma) = \rho(\sigma)f \quad \forall \sigma \in \mathbb{C}S_k\}$$

$$(\mathbb{C}^n)^{\otimes k} \cong \bigoplus_{\substack{\lambda \vdash k \\ \ell(\lambda) \leq n}} \overbrace{M_\lambda}^{\mathbb{C}S_k \text{ irrep}} \otimes \overbrace{V_\lambda}^{U(\mathfrak{sl}_n) \text{ irrep}}$$

Sum over partitions λ of k with at most n rows.

Different versions of Schur–Weyl duality

- $\mathfrak{so}_n, \mathfrak{sp}_n \leftrightarrow$ Brauer algebra $B_k(\pm n)$
- $U_q(\mathfrak{sl}_n) \leftrightarrow$ Hecke algebra $H_k(q)$
- $U_q(\mathfrak{so}_n), U_q(\mathfrak{sp}_n) \leftrightarrow$ Birman–Murakami–Wenzl algebra $BMW_k(q,\nu)$

$$U_q(\mathfrak{sl}_n) \xrightarrow{\pi} \mathrm{End}((\mathbb{C}^n)^{\otimes k}) \xleftarrow{\phi} H_k$$

Important remark

The map ϕ *surjects* on the centralizer but is *not injective in general.*

$\mathrm{End}_{U_q(\mathfrak{sl}_n)}((\mathbb{C}^n)^{\otimes k}) \cong$ Hecke algebra $H_k(q)$ iff $k \leq n$.
For $k \geq n+1$, quotient of $H_k(q)$ by q-antisymmetrizer on $n+1$ points.

$\mathrm{End}_{U_q(\mathfrak{sl}_2)}((\mathbb{C}^2)^{\otimes k}) \cong$ Temperley–Lieb algebra $TL_k(q)$.

Question: What if we chose different representations?

$U_q(\mathfrak{sl}_2)$ has irreps of dimension $2s+1$ for $s = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots$ ("spin").

Can consider rep on $\mathbb{C}^{2s_1+1} \otimes \mathbb{C}^{2s_2+1} \otimes \cdots \otimes \mathbb{C}^{2s_k+1}$.

- $\mathrm{End}_{U_q(\mathfrak{sl}_2)}((\mathbb{C}^2)^{\otimes k}) \cong TL_k$ (spins $\frac{1}{2}$)
- $\mathrm{End}_{U_q(\mathfrak{sl}_2)}((\mathbb{C}^3)^{\otimes k}) \cong$ quotient of BMW_k (spins 1)
  Non-trivial kernel for $k > 3$ generated by a quasi-idempotent in BMW_4.
  [Lehrer, Zhang]
- $\mathrm{End}_{U_q(\mathfrak{sl}_2)}((\mathbb{C}^{2s+1})^{\otimes k}) \cong$ quotient of braid group algebra $\mathcal{B}_k$.
  In general, kernel not known.
- $\mathrm{End}_{U_q(\mathfrak{sl}_2)}((\mathbb{C}^{2s_1+1}) \otimes (\mathbb{C}^{2s_2+1}) \otimes (\mathbb{C}^{2s_3+1})) \cong$ quotient of Askey–Wilson algebra $AW(3)$.
  Conjecture for the kernel. [Crampé, Vinet, Z.]

In particular,

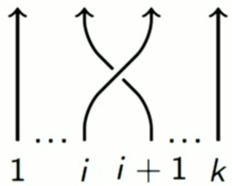
$$Z_s := \mathrm{End}_{U_q(\mathfrak{sl}_2)}((\mathbb{C}^{2s+1})^{\otimes 3})$$

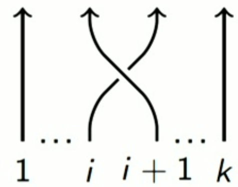
is a quotient of both $\mathcal{B}_3$ and $AW(3)$.

Braid group algebra and its quotients

$\mathcal{B}_k$ has invertible generators σ_i for $i = 1, 2, \ldots, k-1$ with relations

$$\sigma_i \sigma_j = \sigma_j \sigma_i, \qquad |i-j| > 1,$$
$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \qquad 1 \leq i \leq k-2.$$

$\sigma_i =$ 

, $\sigma_i^{-1} =$ 

- Knot theory, link invariants, TQFT;
- Yang–Baxter operator representations.
  Braid generators represented by R-matrices $R_{i,i+1}$.
- Integrable systems, statistical mechanics.

Surjective map $\mathcal{B}_k \to \mathrm{End}_{U_q(\mathfrak{sl}_2)}((\mathbb{C}^{2s+1})^{\otimes k})$.

$$\mathcal{B}_k$$

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$$

$$H_k(q)$$

$$(\sigma_i - 1)(\sigma_i + q^2) = 0$$

$$TL_k(q)$$

$$e_i e_{i+1} e_i = \delta^{-2} e_i$$

$$e_i := \frac{\sigma_i + q^2}{1 + q^2}$$

$$(\sigma_i - 1)(\sigma_i + q^2)(\sigma_i - q^6) = 0$$

$$BMW_k(q)$$

$$e_i \sigma_{i+1}^{\pm 1} e_i = \nu^{\pm 1} e_i$$

$$e_i := \frac{(\sigma_i + q^2)(\sigma_i - q^6)}{(1 + q^2)(1 - q^6)}$$

Askey-Wilson algebra

- AW polynomials: q-hypergeometric orthogonal polynomials.
  Discrete version: q-Racah polynomials.
- Top of q-Askey scheme. (classical OPs)
- Recurrence and q-difference operators satisfy $AW(3)$ algebra. [Zhedanov]

$$\mu(x)p_n(\mu(x)) = A_n p_{n+1}(\mu(x)) + B_n p_n(\mu(x)) + C_n p_{n-1}(\mu(x))$$
$$\lambda_n p_n(\mu(x)) = A(x)p_n(\mu(x+1)) + B(x)p_n(\mu(x)) + C(x)p_n(\mu(x-1))$$

- Racah problem for $U_q(\mathfrak{sl}_2)$:

$$(j_1 \otimes j_2) \otimes j_3 = j_1 \otimes (j_2 \otimes j_3)$$

  Change of basis coefficients are q-Racah polynomials.
  Racah coefficients, $6j$-symbols.

- $AW(3)$ is realized as the diagonal centralizer of $U_q(\mathfrak{sl}_2)$ in $U_q(\mathfrak{sl}_2)^{\otimes 3}$.
  Generators of $AW(3)$ mapped to intermediate Casimir elements for $U_q(\mathfrak{sl}_2)$: C_{12}, C_{23}, C_{13} and central elements C_1, C_2, C_3, C_{123}.

Surjective map $AW(3) \to \mathrm{End}_{U_q(\mathfrak{sl}_2)}(\mathbb{C}^{2j_1+1} \otimes \mathbb{C}^{2j_2+1} \otimes \mathbb{C}^{2j_3+1})$.

We will consider a specialization of $AW(3)$ corresponding to $j_1 = j_2 = j_3$.

It is generated by three elements A, B, C and central element κ

$$\frac{[A,B]_q}{q^2 - q^{-2}} + C = \frac{[C,A]_q}{q^2 - q^{-2}} + B = \frac{[B,C]_q}{q^2 - q^{-2}} + A = \frac{\chi_s}{\chi_0}(\kappa + \chi_s)$$

$[X, Y]_q = qXY - q^{-1}YX$

$\chi_s := q^{2s+1} + q^{-2s-1}$

There is a central element $\Omega \in AW(3)$, polynomial in A, B, C, κ.

$$\Omega = \chi_s(\kappa + \chi_s)(qA + q^{-1}B + qC) - q^2A^2 - q^{-2}B^2 - q^2C^2 - qABC.$$

Special Askey–Wilson algebra:

$$\Omega = \kappa(\kappa + \chi_s^3) + 3\chi_s^2 - \chi_0.$$

Objective

$$\begin{array}{ccccccccc}
AW(3) & & & & & & & & \mathcal{B}_3 \\
\big\downarrow & \nearrow & \overline{AW}(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}) & \cong & Z_{\frac{1}{2}} & \cong & TL_3(q) & \nwarrow & \big\downarrow \\
\overline{AW}(s,s,s) & \longrightarrow & \overline{AW}(1,1,1) & \cong & Z_1 & \cong & BMW_3(q) & \longleftarrow & ? \\
 & \searrow & \overline{AW}(\tfrac{3}{2},\tfrac{3}{2},\tfrac{3}{2}) & \cong & Z_{\frac{3}{2}} & \cong & ? & \swarrow &
\end{array}$$

Objective:
Find an explicit algebraic description of centralizer Z_s, for any spin s, by combining braid group and Askey–Wilson algebra relations.

Centralizers of $U_q(\mathfrak{sl}_2)$

Suppose q not root of unity.

Spin irreps V_s of dimension $2s+1$ ($V_s \cong \mathbb{C}^{2s+1}$).
Representation $U_q(\mathfrak{sl}_2) \to \text{End}(V_s) \sim (2s+1) \times (2s+1)$ matrices.

Can act on tensor products using coproduct,
$\Delta : U_q(\mathfrak{sl}_2) \to U_q(\mathfrak{sl}_2) \otimes U_q(\mathfrak{sl}_2)$

$$U_q(\mathfrak{sl}_2) \to \text{End}(V_s \otimes V_s \otimes V_s)$$

Question: What elements in $\text{End}(V_s^{\otimes 3})$ commute with action of $U_q(\mathfrak{sl}_2)$?

The set of all such elements is the centralizer Z_s.

Twofold tensor product

Tensor product decomposition of $U_q(\mathfrak{sl}_2)$ irreps:

$$V_s \otimes V_s \cong \bigoplus_{r=0}^{2s} V_r.$$

Example

For $s = \frac{1}{2}$,

$$V_{\frac{1}{2}} \otimes V_{\frac{1}{2}} \cong V_0 \oplus V_1.$$

If $X \in U_q(\mathfrak{sl}_2)$, this means that there is a basis such that

$$X \mapsto \left(\begin{array}{c|c} X_0 & 0 \\ \hline 0 & X_1 \end{array}\right) = \left(\begin{array}{c|ccc} * & & 0 & \\ \hline & * & * & * \\ 0 & * & * & * \\ & * & * & * \end{array}\right).$$

$E^{(r)} :=$ projector on the irrep V_r for $r = 0, 1, \ldots, 2s$.

Threefold tensor product

$$V_s \otimes V_s \otimes V_s \cong \bigoplus_{j=j_{\min}}^{3s} V_j^{\oplus d_j}, \qquad j_{\min} = 0 \text{ or } \frac{1}{2}.$$

d_j : degeneracies

Example

$$V_{\frac{1}{2}} \otimes V_{\frac{1}{2}} \otimes V_{\frac{1}{2}} \cong V_{\frac{1}{2}}^{\oplus 2} \oplus V_{\frac{3}{2}} = V_{\frac{1}{2}} \oplus V_{\frac{1}{2}} \oplus V_{\frac{3}{2}}.$$

$$X \mapsto \left(\begin{array}{cc|c} X_{\frac{1}{2}} & 0 & \\ 0 & X_{\frac{1}{2}} & 0 \\ \hline & & \\ \multicolumn{2}{c|}{0} & X_{\frac{3}{2}} \end{array} \right)$$

$$\dim(Z_s) = \sum_{j=j_{\min}}^{3s} d_j^2 = \frac{1}{2}(2s+1)((2s+1)^2+1)$$

R-matrix

There is a universal R-matrix $\mathcal{R} \in U_q(\mathfrak{sl}_2) \otimes U_q(\mathfrak{sl}_2)$.

From this, one can construct the braided R-matrix $\check{R} \in \mathrm{End}(V_s \otimes V_s)$.

$\check{R}$ is a matrix that commutes with the action of $U_q(\mathfrak{sl}_2)$ on $V_s \otimes V_s$.

$$\check{R} = \bigoplus_{r=0}^{2s} q_r \mathrm{Id}_{V_r} = \sum_{r=0}^{2s} q_r E^{(r)}, \qquad q_r = (-1)^r q^{r(r+1)}.$$

Define $\check{R}_{12} := \check{R} \otimes \mathrm{Id}_{V_s}$ and $\check{R}_{23} := \mathrm{Id}_{V_s} \otimes \check{R}$.

These clearly belong to Z_s (commute with action of $U_q(\mathfrak{sl}_2)$ on $V_s^{\otimes 3}$).

In fact, $\check{R}_{12}$ and $\check{R}_{23}$ generate Z_s.

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In fact, $\check{R}_{12}$ and $\check{R}_{23}$ generate Z_s.

They satisfy the **braided Yang–Baxter equation**

$$\check{R}_{12}\check{R}_{23}\check{R}_{12} = \check{R}_{23}\check{R}_{12}\check{R}_{23}.$$

Starting point for the algebraic description of Z_S

1. $\mathcal{B}_3$: invertible generators σ_1, σ_2 with braid relation

$$\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2.$$

2. $\mathcal{B}_3^s(q)$: add characteristic equations

$$\prod_{p=0}^{2s}(\sigma_i - q_p) = 0, \quad \text{for } i = 1, 2.$$

Define idempotents for $r = 0, 1, \ldots, 2s$ and $i = 1, 2$

$$e_i^{(r)} := \prod_{\substack{p=0 \\ p\neq r}}^{2s} \frac{\sigma_i - q_p}{q_r - q_p}.$$

$$\sigma_i = \sum_{r=0}^{2s} q_r e_i^{(r)}.$$

Intermediate Casimir elements

There is a Casimir element in $U_q(\mathfrak{sl}_2)$. Acts as $\chi_s \mathrm{Id}_{V_s}$ on V_s.

Its action on $V_s \otimes V_s$ is some matrix $C \in \mathrm{End}(V_s \otimes V_s)$.

C is a matrix that commutes with the action of $U_q(\mathfrak{sl}_2)$ on $V_s \otimes V_s$.

$$C = \bigoplus_{r=0}^{2s} \chi_r \mathrm{Id}_{V_r} = \sum_{r=0}^{2s} \chi_r E^{(r)}.$$

Define $C_{12} := C \otimes \mathrm{Id}_{V_s}$ and $C_{23} := \mathrm{Id}_{V_s} \otimes C$. These generate Z_s.

Define $C_{13} := \check{R}_{12} C_{23} \check{R}_{12}^{-1} = \check{R}_{23}^{-1} C_{12} \check{R}_{23}$,
and $C_{123} :=$ the Casimir element in $\mathrm{End}(V_s^{\otimes 3})$.

The triplet (C_{12}, C_{23}, C_{13}) together with the central element C_{123} satisfy the special AW relations.

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$$\frac{[C_{12}, C_{23}]_q}{q^2 - q^{-2}} + \check{R}_{12} C_{23} \check{R}_{12}^{-1} = \frac{\chi_s}{\chi_0}(C_{123} + \chi_s \mathrm{Id}_{V_s^{\otimes 3}}).$$

Askey–Wilson braid algebra

3. $AWB_3^s(q)$: Define elements

$$g_i = \sum_{r=0}^{2s} \chi_r e_i^{(r)}, \quad \text{for } i = 1, 2.$$

Impose AW relations for the triplet $(g_1, g_2, \sigma_1 g_2 \sigma_1^{-1})$.

$$\frac{[g_1, g_2]_q}{q^2 - q^{-2}} + \sigma_1 g_2 \sigma_1^{-1} = \frac{[\sigma_1 g_2 \sigma_1^{-1}, g_1]_q}{q^2 - q^{-2}} + g_2 = \frac{[g_2, \sigma_1 g_2 \sigma_1^{-1}]_q}{q^2 - q^{-2}} + g_1$$
$$=: \frac{\chi_s}{\chi_0}(\kappa + \chi_s).$$

4. Special $AWB_3^s(q)$: Impose special relation for central element Ω.

$$\Omega = \kappa(\kappa + \chi_s^3) + 3\chi_s^2 - \chi_0.$$

Final quotient

5. $\mathcal{A}_s$: add relation

$$e_1^{(0)} g_2 e_1^{(0)} = \frac{\chi_s^2}{\chi_0} e_1^{(0)}.$$

Claim: $\mathcal{A}_s \cong Z_s$.

Some remarks:

- $\mathcal{A}_s$ generated by σ_1, σ_2 or g_1, g_2 or $e_1^{(r)}, e_2^{(r)}$ for $r = 0, 1, \ldots, 2s$.
- Characteristic equations for g_i

$$\prod_{p=0}^{2s} (g_i - \chi_p) = 0, \quad \text{for } i = 1, 2.$$

- Step 5: equivalent to adding

$$\kappa e_1^{(0)} = \chi_s e_1^{(0)}.$$

Isomorphism

$\mathcal{A}_s$ has been constructed such that we have a **homomorphism** with the centralizer Z_s (mapping $\mathcal{A}_s \to Z_s$ that preserves relations).

$$\sigma_1 \mapsto \check{R}_{12}, \quad \sigma_2 \mapsto \check{R}_{23},$$

$$g_1 \mapsto C_{12}, \quad g_2 \mapsto C_{23}, \quad \sigma_1 g_2 \sigma_1^{-1} \mapsto C_{13}, \quad \kappa \mapsto C_{123},$$
$$e_1^{(r)} \mapsto E_{12}^{(r)}, \quad e_2^{(r)} \mapsto E_{23}^{(r)}.$$

It is **surjective** because the centralizer Z_s is generated by $\check{R}_{12}, \check{R}_{23}$ (or C_{12}, C_{23} or projectors $E_{12}^{(r)}, E_{23}^{(r)}$).

This means that $\dim(\mathcal{A}_s) \geq \dim(Z_s)$.

The complicated part is to show that it is **injective** (no other relations are required).

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This means that $\dim(\mathcal{A}_s) \geq \dim(Z_s)$.

The complicated part is to show that it is **injective** (no other relations are required).

Suffice to show that $\dim(\mathcal{A}_s) \leq \dim(Z_s)$.

Injectivity

The idea is to find a spanning set for $\mathcal{A}_s$ with cardinality $= \dim(Z_s)$.

This spanning set will therefore be a basis for $\mathcal{A}_s \cong Z_s$.

Outline

- PBW basis for $AW(3)$, and $\sigma_1 g_2 \sigma_1^{-1} = \sigma_2^{-1} g_1 \sigma_2$

$$\{g_1^a g_2^b (\sigma_2^{-1} g_1 \sigma_2)^c \kappa^p \mid a, b, c, p \in \mathbb{N}\}$$

- Characteristic polynomials for g_1 and g_2

$$\{g_1^a g_2^b \sigma_2^{-1} g_1^c \sigma_2 \kappa^p \mid 0 \leq a, b, c \leq 2s,\ \ p \in \mathbb{N}\}$$

- $g_1^a \to e_1^{(a)}$, $g_2^b \sigma_2^{-1} \to g_2^b$, $g_1^c \to e_1^{(c)}$, σ_2 invertible, κ central

$$\{e_1^{(a)} g_2^b e_1^{(c)} \kappa^p \mid 0 \leq a, b, c \leq 2s,\ \ p \in \mathbb{N}\}$$

- $AW(3)$ relations imply

$$e_1^{(a)} g_2^b e_1^{(c)} = 0 \quad \text{if } |a-c| > b.$$

$$\{e_1^{(a)} g_2^b e_1^{(c)} \kappa^p \mid 0 \leq a, c \leq 2s,\ |a-c| \leq b \leq 2s,\ p \in \mathbb{N}\}$$

- Special $AW(3)$ relations imply

$$e_1^{(a)} g_2^{|a-c|+n} e_1^{(c)} = P_n^{(a,c)}(\kappa) e_1^{(a)} g_2^{|a-c|} e_1^{(c)}$$

$$\{e_1^{(a)} g_2^{|a-c|} e_1^{(c)} \kappa^p \mid 0 \leq a, c \leq 2s,\ p \in \mathbb{N}\}$$

- braid + special $AW(3)$ + final relation imply

$$\prod_{r=|a-s|}^{a+s} (\kappa - \chi_r) e_1^{(a)} = 0.$$

$$\{e_1^{(a)} g_2^{|a-c|} e_1^{(c)} \kappa^p \mid 0 \leq a, c \leq 2s,\ 0 \leq p \leq n(a,c)\}$$

This set has cardinality $= \dim(Z_s) \Rightarrow \mathcal{A}_s \cong Z_s$.

Conclusion

Summary

- Algebraic description of the centralizer of $U_q(\mathfrak{sl}_2)$ in the tensor product of 3 identical copies of spin-s irrep, for any s.
- Combine braid group relations on 3 strands with special $AW(3)$.

Some perspectives

- n copies: $\mathcal{B}_n$ and higher rank $AW(n)$.
- q root of unity?