Title: Classical Physics Lecture - 092623

Speakers:

Collection: Classical Physics 2023/24

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Neumann-type Green's function

problem: $\begin{cases} \Delta\varphi = -4\pi\rho & \text{in } R \\ \vec{s}\cdot\vec{\nabla}_i\varphi = -f & \text{at } \partial R \end{cases}$

Gauss's law: $4\pi\int_R \rho = \oint_{\partial R} f$

$\left[\int_R \Delta\varphi = \int_R \vec{\nabla}(\vec{\nabla}\varphi) = \oint_{\partial R}\vec{s}\cdot\vec{\nabla}\varphi\right]$

Green's identity:

$$\int_R \Phi(y)\,\Delta\Psi(y) = \int_R \Psi(y)\Delta\Phi(y) + \oint_{\partial R}\vec{s}\cdot\left(\Phi(y)\vec{\nabla}\Psi(y) - \Psi(y)\vec{\nabla}\Phi(y)\right)$$

$\begin{cases} \Delta_y G_N(y,x) = \delta^3(y-x) \\ \vec{s}\cdot\vec{\nabla}_y G_N(y,x) = \dfrac{1}{\text{Area}(\partial R)} \end{cases}$

$\Phi(y) = \varphi(y)$ & $\Psi($

Green's identity:

$\varphi(x) \neq -4\pi\int_R \rho(y)G_N(y$

Rmk: $\varphi(x)$

If

ion

∂R

$\oint_{\partial R} f$

$$\begin{cases} \Delta_y G_N(y,x) = \delta^3(y-x) & \text{in } R \\ \vec{s}\cdot\vec{D}_y G_N(y,x) = \dfrac{1}{\mathrm{Area}(\partial R)} & \text{at } \partial R \end{cases}$$

$\Phi(y) = \varphi(y)$ & $\Psi(y) = G_N(y,x)$

Green's identity:

$$\varphi(x) = -4\pi \int_R \rho(y) G_N(y,x) + \boxed{\oint_{\partial R} \frac{\varphi(y)}{\mathrm{Area}(\partial R)}} + \oint_{\partial R} f(y) G_N(y,x)$$

$$\left(\Phi(y)\vec{D}\Psi(y) - \Psi(y)\vec{D}\Phi(y)\right)$$

$\langle \varphi \rangle_{\partial R} = \text{const.}$

<u>Rmk</u>: $\varphi(x)$ is a sol

iff $\varphi(x)+c$ is a sol. $\Rightarrow \langle \varphi \rangle_{\partial R}$ doesn't really matter

$$\begin{cases} \Delta_y G_N(y,x) = \delta^3(y-x) & y \in R \\ \vec{\xi}\cdot\vec{D}_y G_N(y,x) = \dfrac{1}{\text{Area}(\partial R)} & y \in \partial R \end{cases} \qquad (x \in \dot{R})$$

$$\Phi(y) = \varphi(y) \quad \& \quad \Psi(y) = G_N(y,x)$$

Green's identity:

$$\varphi(x) = -4\pi \int_R \rho(y) G_N(y,x) + \boxed{\oint_{\partial R} \frac{\varphi(y)}{\text{Area}(\partial R)}} + \oint_{\partial R} f(y) G_N(y,x)$$

$$\langle \varphi \rangle_{\partial R} = \text{const.}$$

$$\Phi(y)\vec{D}\Psi(y) - \Psi(y)\vec{D}\Phi(y)$$

Rmk: $\varphi(x)$ is a sol iff $\varphi(x) + c$ is a sol. $\Rightarrow$ $\langle \varphi \rangle_{\partial R}$ doesn't really matter

(y,x)

doesn't really matter

Method of images

$$\begin{cases} \Delta_y G_D(y,x) = \delta^3(y-x) & y \in \mathring{R} \\ G_D(y,x) = 0 & y \in \partial R \end{cases} \qquad (x \in \mathring{R})$$

General recipe

- Find a "special" sol for part with sources:

$$\Delta_y G_0(y,x) = \delta^3(y-x) \qquad \left[\text{e.g. } G_0(y,x) = \frac{1}{4\pi}\frac{1}{|\vec{y}-\vec{x}|}\right]$$

- Look for a sol. of the homog. eq.

$$\Delta_y F(y,x) = 0$$

such that $G_D(y,x) = G_0(y,x) + F(y,x)$

satisfies the boundary condition

Method of images:

If we are lucky ($\equiv$ if the shape of R is simple enough) we can build $F(y,x) = \sum_\alpha q_\alpha G_0(y, x_\alpha)$, $x_\alpha \in \mathbb{R}^3 \setminus R$

image charges' position

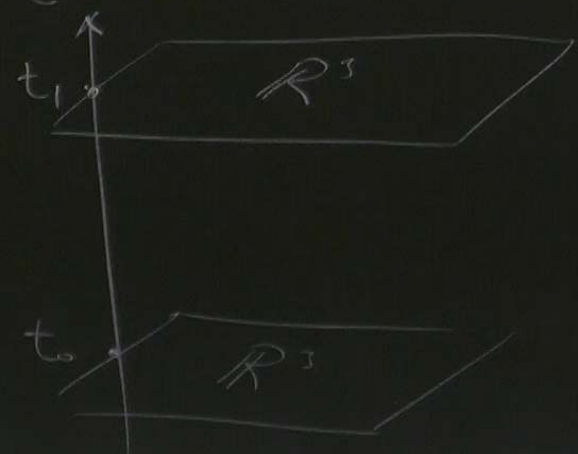
image charges's charge.

WAVE EQ

$$\begin{cases} \Box \varphi(t,\vec{x}) = -4\pi \rho(\vec{x},t) \\ \varphi(t_0,\vec{x}) = \phi_0(\vec{x}) \\ \partial_t \varphi(t_0,\vec{x}) = \pi_0(\vec{x}) \end{cases}$$

initial conditions

(or final cond if $t_0 \to t_1$)



$x_\alpha \in \mathbb{R}^3 \setminus \mathbb{R}$

...ge charges' position

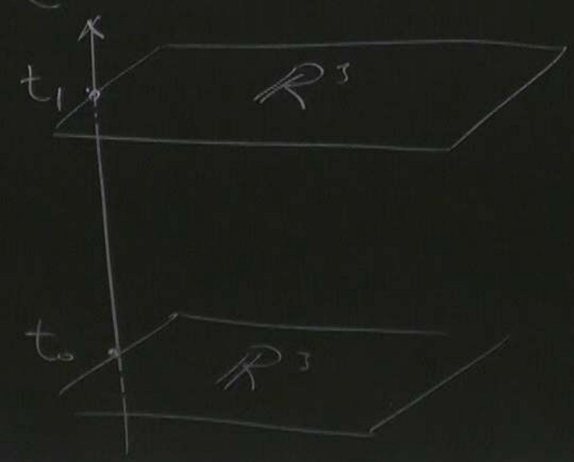
charge.

WAVE EQ

$$\begin{cases} \Box\varphi(t,\vec{x}) = -4\pi\rho(\vec{x},t) \\ \varphi(t_0,\vec{x}) = \phi_0(\vec{x}) \\ \partial_t\varphi(t_0,\vec{x}) = \pi_0(\vec{x}) \end{cases}$$

initial conditions

(or final cond if $t_0 \to t_1$)



$(\vec{x}, t)$

$\boxed{\text{initial conditions}}$

(or final cond if $t_0 \to t_1$)

Goal: use Green's functions to write down sol of $\varphi(t,\vec{x})$ as integrals of ρ, ϕ_0, π_0

$$\int_M \Phi(y)\,\Box\,\Psi(y) = \int_M \Psi(y)\,\Box\,\Phi(y) + \int_{\mathbb{R}^3} \Phi(\vec{y},t)\,\partial_t \Psi(\vec{y},t)\Big|_{t_0}^{t_1} - \Psi(\vec{y},t)\,\partial_t \Phi(\vec{y},t)\Big|_{t_0}^{t_1}$$

$\Phi = \varphi$

$\Psi = G = \begin{cases} 0 & \text{at } t_1 \\ \neq & \text{at } t_0 \end{cases}$

initial v.p.

initial bound value pb

$$\begin{cases} \Box_y G_R(y,x) = \delta^4(y-x) & , \ y,x \in M \\ G_R(y,x) = 0 & \text{if } y^0 > x^0 \end{cases}$$

"retarded"

time coord

Gr

$\longrightarrow \varphi($

final bvp:

$$\begin{cases} \Box_y G_A(y,x) = \delta^4(y-x) \\ G_A(y,X) = 0 & \text{if } y^0 < x^0 \end{cases}$$

$$4\pi \int_M \rho(y)\, G_R(y,x) + \int_{\mathbb{R}^3 \times \{t_0\}} \left(\pi_0(\vec{y})\, G_R((t_0,\vec{y}),x) - \phi_0(\vec{y}) \frac{\partial}{\partial y^0} G_R((t_0,\vec{y}),x) \right)$$

bulk source

initial conditions

but with $t_0 \to t_1$

& sign swapped

Retarded Green's f.

$G(y,x) \equiv D(r)$, $r^\mu = x^\mu - y^\mu$

fulk. $\Box_y G(y,x) = \delta^4(y-x)$

iff $\Box D(r) = \delta^4(r)$

Fourier: $D(r) = \int d^4k \; e^{-ik_\mu r^\mu} \hat{D}(k)$

$\hat{D}(k) = \frac{1}{(2\pi)^4} \int d^4r \; e^{+ik_\mu r^\mu} D(r)$

iff $\boxed{\hat{D}(k) = -\frac{1}{(2\pi)^4} \frac{1}{k_\mu k^\mu}}$

Pf

$\Box D(r) = \int d$

$\delta^4(r)$

for $\int d^4k \; e^{ik_\mu r}$

$D(r) = -$

$k^\mu = (\omega$

$\hat{D}(k)$

$e^{ik_\mu r^\mu} D(r)$

Pf

$$\Box D(r) = \int d^4k\, \underbrace{(-ik_\mu)(-ik^\mu)}\, e^{-ik_\mu r^\mu} \underbrace{\hat{D}(k)} = \frac{1}{(2\pi)^4}$$

$$\shortparallel$$

$$\delta^4(r)$$

for $\int d^4k\, e^{ik_\mu r^\mu} = (2\pi)^4\, \delta^4(r)$, $\nabla_\mu e^{-ik_\nu r^\nu} = -ik_\mu e^{ik_\nu r^\nu}$

$$D(r) = -\frac{1}{(2\pi)^4}\int d^4k\, \frac{e^{+ik_\mu r^\mu}}{k^\mu k_\mu} = \frac{1}{(2\pi)^4}\int d\vec{k}\, e^{i\vec{k}\vec{r}} \int d\omega\, \frac{e^{-i\omega t}}{\omega^2 - \vec{k}^2}$$

$\uparrow$ poles!

$k^\mu = (\omega, \vec{k})$

iff $\Box D(r) = \delta^4(r)$

Fourier: $D(r) = \int d^4k \; e^{-ik_\mu r^\mu} \hat{D}(k)$

$$\hat{D}(k) = \frac{1}{(2\pi)^4} \int d^4r \; e^{+ik_\mu r^\mu} D(r)$$

iff $\boxed{\hat{D}(k) = -\frac{1}{(2\pi)^4} \frac{1}{k_\mu k^\mu}}$ (naive)

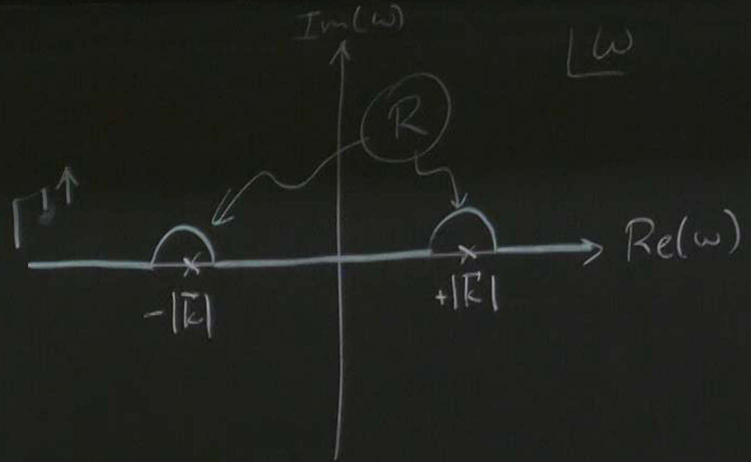
for $\int d^4k \; e^{ik_\mu r^\mu} = (2\pi)^4 \delta^4(r)$,

$$D(r) = -\frac{1}{(2\pi)^4} \int d^4k \; \frac{e^{+ik_\mu r^\mu}}{k^\mu k_\mu}$$

$k^\mu = (\omega, \vec{k})$

$$(2\pi)^4\, k_\mu k^\mu$$

ω

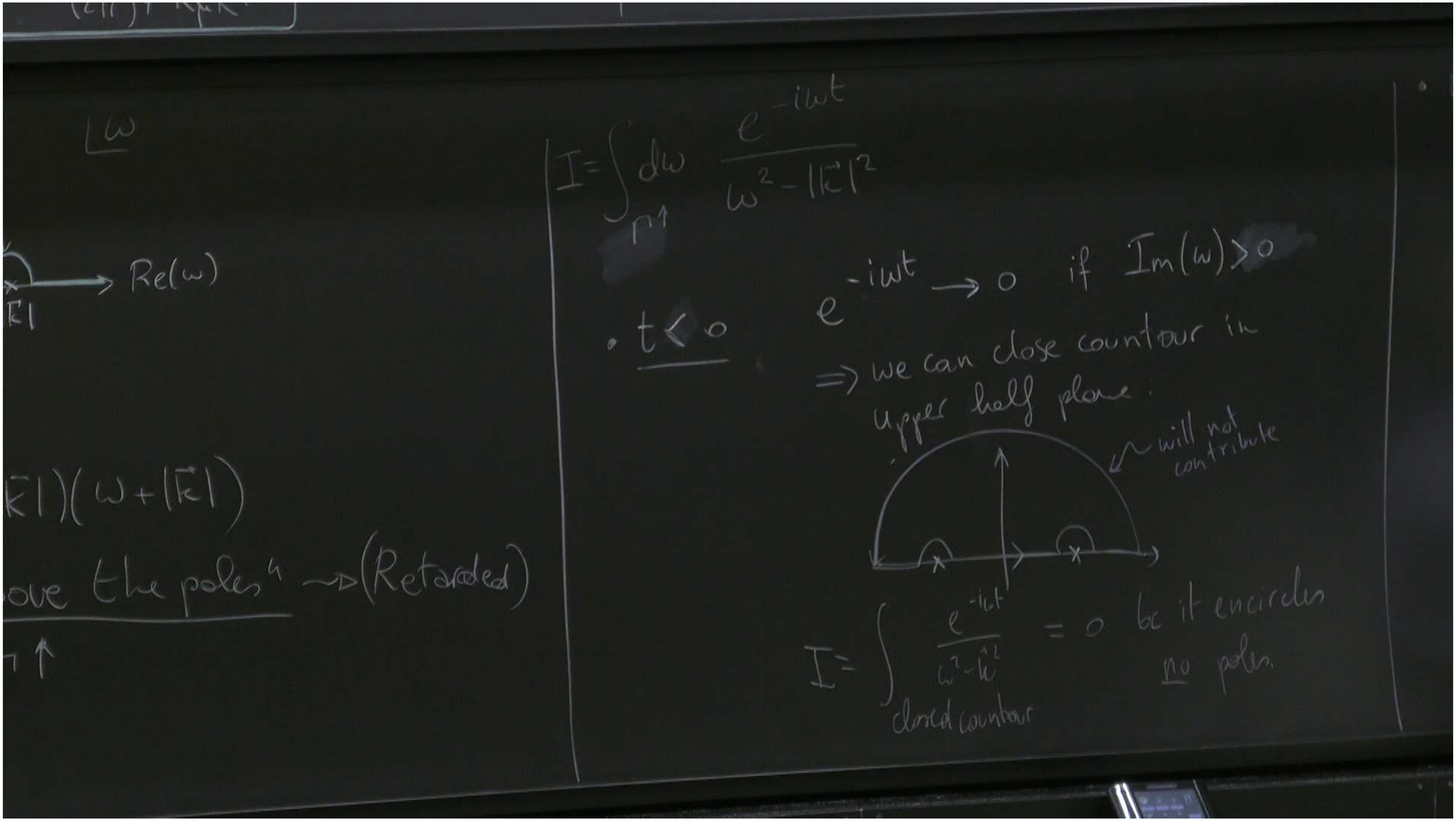


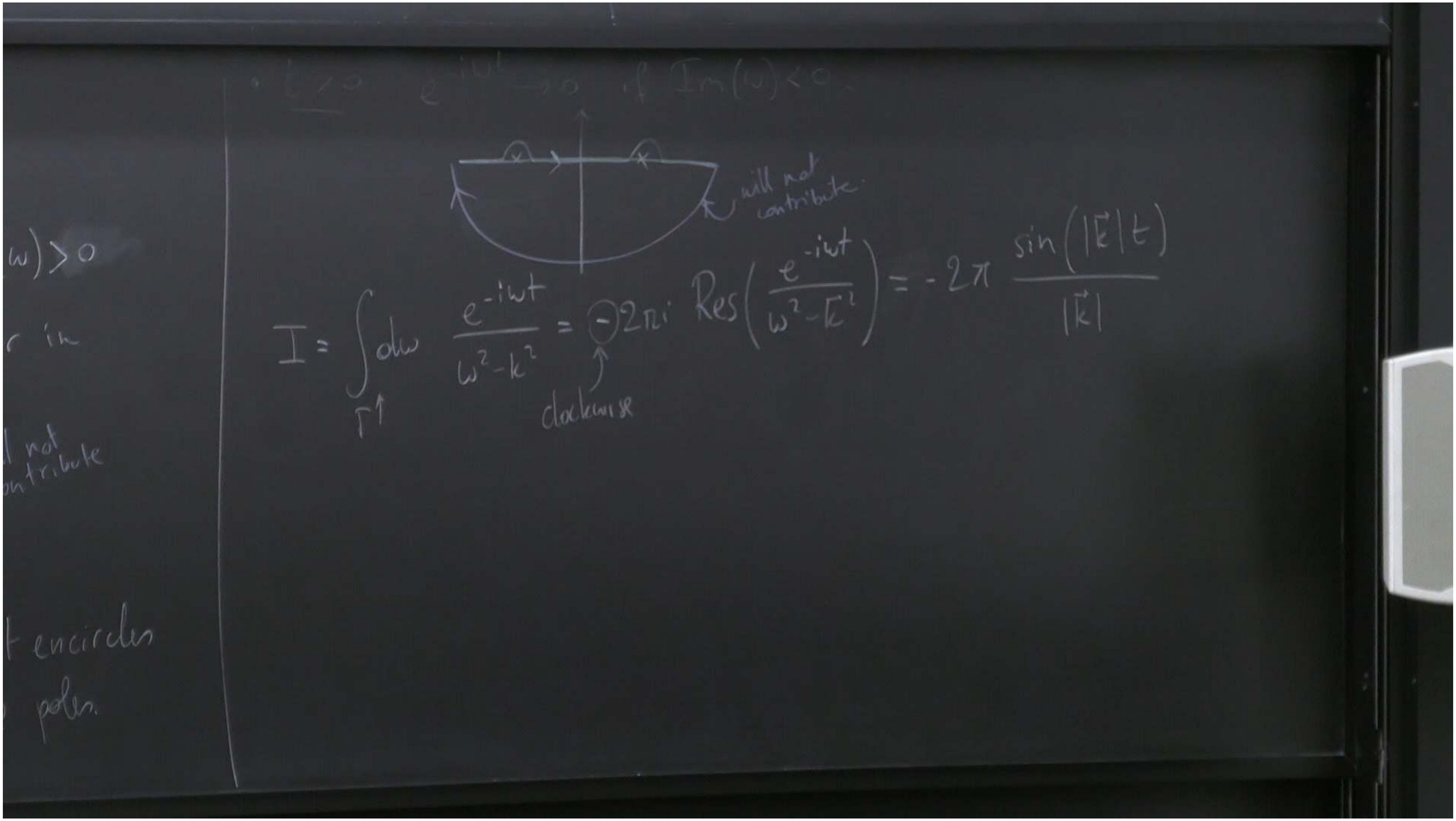
$$\omega^2 - \vec{k}^2 = (\omega - |\vec{k}|)(\omega + |\vec{k}|)$$

Prescription "above the poles" $\rightarrow$ (Retarded)

Countour $\Gamma^\uparrow$

$$\int_{\Gamma^\uparrow} d\omega \, \frac{e^{-i\omega t}}{\omega^2 - |\vec{k}|^2}$$


Re(ω)
the poles
(Retarded)
I = ∫ dω e^{-iωt} / (ω² - |k|²)
e^{-iωt} → 0 if Im(ω) > 0
t < 0
⇒ we can close countour in upper half plane.
will not contribute
I = ∫ closed countour e^{-iωt} / (ω² - k²) = 0 bc it encircles no poles.



t > 0 $e^{-i\omega t} \to 0$ if $\mathrm{Im}(\omega) < 0$
will not contribute.
$I = \int_{\Gamma} d\omega \, \frac{e^{-i\omega t}}{\omega^2 - k^2} = -2\pi i \, \mathrm{Res}\left(\frac{e^{-i\omega t}}{\omega^2 - \vec{k}^2}\right) = -2\pi \, \frac{\sin(|\vec{k}| t)}{|\vec{k}|}$
clockwise
$\omega) > 0$
r in
l not contribute
t encircles
poles.

closed countour

$$D_R(t,\vec{r}) \underset{\Gamma\uparrow}{=} -\frac{\Theta(t)}{(2\pi)^3}\int d^3\vec{k}\; e^{i\vec{k}\vec{r}}\;\frac{\sin(|\vec{k}|t)}{|\vec{k}|}\,, \qquad \Theta(t)=\begin{cases}1 & \text{if } t>0\\ 0 & \text{if } t\leq 0\end{cases}$$

Align our spherical coords for $\vec{k}$ along $\vec{r}$, $\vec{k}=(\underbrace{k}_{\text{positive}},\overset{[0,\pi]}{\theta},\overset{[0,2\pi]}{\varphi})$, $\vec{k}\cdot\vec{r}=k\overset{|\vec{r}|}{\rho}\cos\theta$.

$$= -\frac{\Theta(t)}{(2\pi)^{\cancel{3}2}}\int_0^\infty dk\int_0^\pi d\theta\,\cancel{\int_0^{2\pi} d\varphi}\; k^2\sin\theta\; e^{ik\rho\cos\theta}\;\frac{\sin(kt)}{k}$$

$$=$$

$$D_R(t,\vec r) = -\frac{\Theta(t)}{(2\pi)^3}\int d^3\vec k \; e^{i\vec k\vec r}\,\frac{\sin(|\vec k|t)}{|\vec k|} \;,\quad \Theta(t) = \begin{cases} 1 & \text{if} \\ 0 & \text{if} \end{cases}$$

$\Gamma \uparrow$

Align our spherical coords for $\vec k$ along $\vec r$, $\vec k = (k, \theta, \varphi)$, $\vec k$

k positive, $\theta \in [0,\pi)$, $\varphi \in [0,2\pi)$

$(1+d=4)$

$$= -\frac{\Theta(t)}{(2\pi)^{3/2}}\int_0^\infty dk\int_0^\pi d\theta\cancel{\int_0^{2\pi} d\varphi}\; k^2\sin\theta\, e^{ik\rho\cos\theta}\,\frac{\sin(kt)}{k}$$

$d\theta$

$$= +\frac{\Theta(t)}{4\pi^2}\int_0^\infty dk\; \cancel{k}\,\sin(kt)\left(\frac{e^{-ik\rho}-e^{ik\rho}}{i\cancel{k}\rho}\right) = -\frac{\Theta(t)}{8\pi^2 \ldots}$$

closed contour

$= \begin{cases} 1 & \text{if } t > 0 \\ 0 & \text{if } t \leq 0 \end{cases}$

$$\boxed{D_R(t,\vec{r}) = -\frac{1}{4\pi|\vec{r}|}\Theta(t)\,\delta(t-|\vec{r}|)}$$

$[0, 2\pi)$ $\quad |\vec{r}|$

$\theta, \varphi)\,, \quad \vec{k}\cdot\vec{r} = k\rho\cos\theta.$

positive

both $t, \rho > 0$

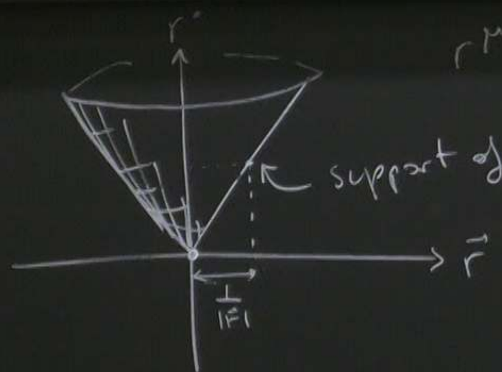
$$) = -\frac{\Theta(t)}{8\pi^2\rho}\int_{-\infty}^{\infty} dk\left(e^{ikt} - e^{-ikt}\right)e^{-ik\rho} = -\frac{\Theta(t)}{4\pi\rho}\left(\delta(t-\rho) + \cancel{\delta(t+\rho)}\right)$$

$$\boxed{D_R(t,\vec{r}) = -\frac{1}{4\pi|\vec{r}|}\Theta(t)\,\delta(t-|\vec{r}|) = -\frac{1}{2\pi}\Theta(t)\,\delta(r_\mu r^\mu)}$$

both $t, \rho > 0$

$$(t)\,e^{-ik\rho} = -\frac{\Theta(t)}{4\pi\rho}\left(\delta(t-\rho)+\cancel{\delta(t+\rho)}\right)$$

$= \ldots \frac{\ldots}{4\pi^2} \int_0 dk \; k \sin(kr) \left(\frac{\ldots}{ik\rho} \right) \ldots (8\pi^2 \rho) \ldots -\infty$



support of D_R is on future light cone

Lienard-Wiechert pot

$$\Box A^\mu = -4\pi j^\mu$$

where j^μ is given by a point charge:

$$j^\mu(x) = q \int d\lambda \, \frac{d\gamma^\mu}{d\lambda} \, \delta^4(x - \gamma(\lambda))$$

$$A_R^\mu(x) = -4\pi \int d^4y \;\; j^\mu(y)\, G_R(y,x)$$

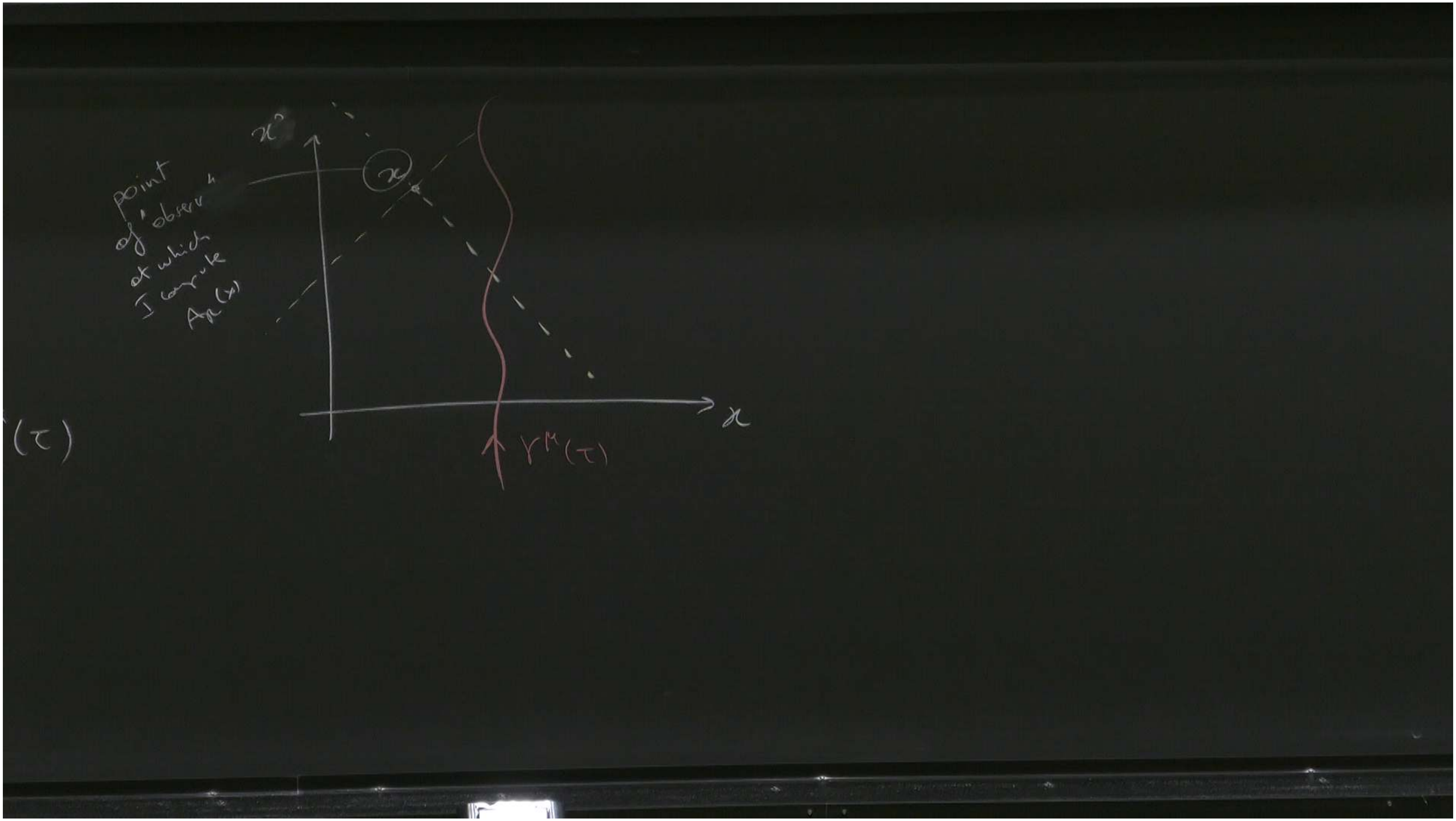
$$= -4\pi \int d^4y \;\; D_R(x-y)\, j^\mu(y)$$

cov. expr of D_R

$$= 2 \int d^4y \;\; \Theta(x^0-y^0)\, \delta\left((x-y)^2\right) j^\mu(y)$$

integrate $\int d^4y\, \delta^4(y-\gamma)$

$$= 2q \int d\tau \;\; \Theta(x^0-\gamma^0(\tau))\, \delta\left((x-\gamma(\tau))^2\right) u^\mu(\tau)$$


point of "observ'n" at which I compute $A_\mu(x)$
x^0
x
x
$y^\mu(\tau)$
(τ)

$$A_R^{\mu}(x) = -4\pi \int d^4y \; j^{\mu}(y)\, G_R(y,x)$$

$$= -4\pi \int d^4y \; D_R(x-y)\, j^{\mu}(y)$$

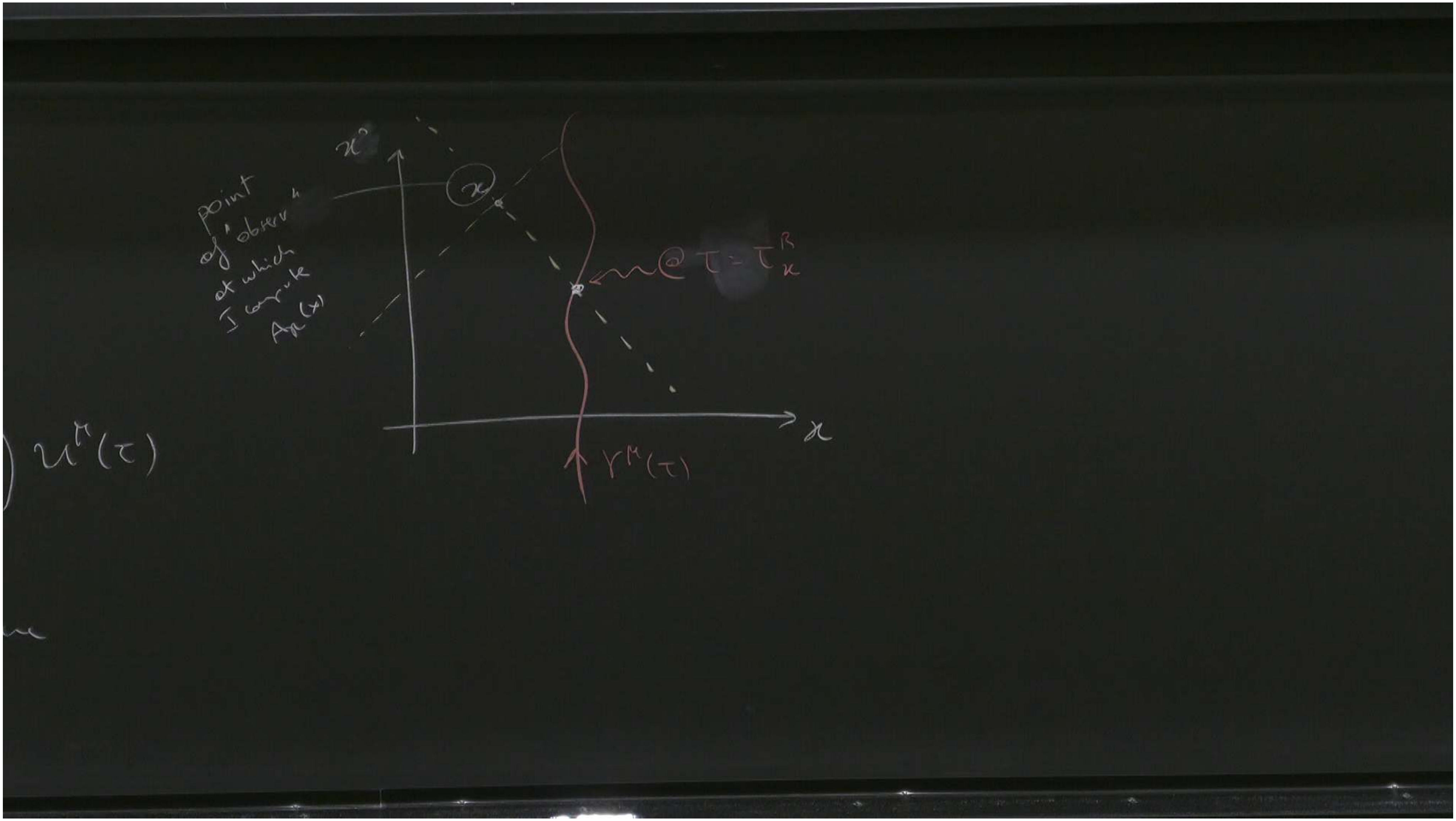
cov. expr of D_R

$$= 2 \int d^4y \; \Theta(x^0-y^0)\, \delta\big((x-y)^2\big)\, j^{\mu}(y)$$

integrate $\int d^4y\, \delta^4(y-\gamma)$

$$= 2q \int d\tau \; \underbrace{\Theta(x^0-\gamma^0(\tau))\, \delta\big((x-\gamma(\tau))^2\big)}_{\text{selects the only point at which the past lightcone of } x \text{ intercepts } \gamma^{\mu}(\tau)}\, u^{\mu}(\tau)$$

poin of at J



x^0
point of "observ'n" at which I compute A_μ(x)
x
@ τ = τ_x^R
x
r^μ(τ)
u^μ(τ)

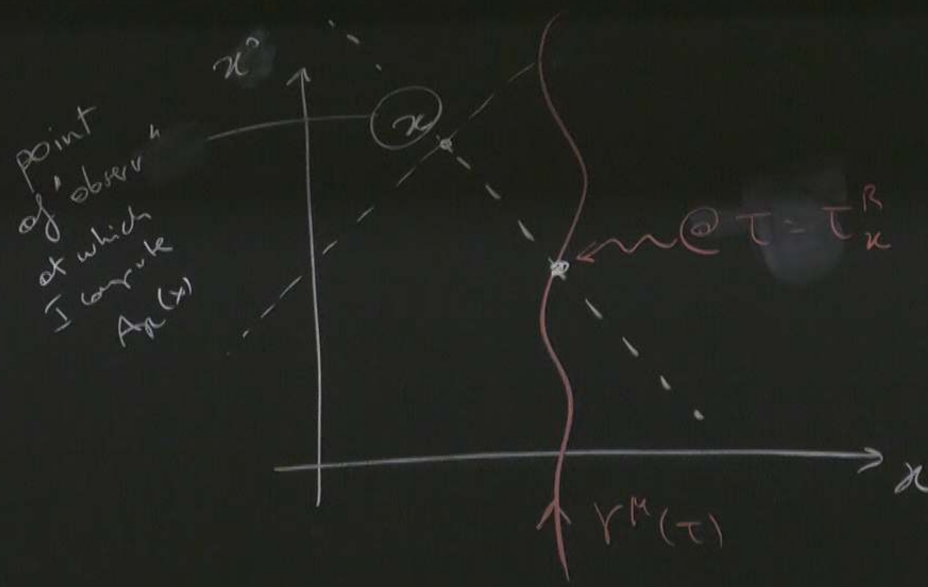
$j^\mu(y)\, G_R(y,x)$

$D_R(x-y)\, j^\mu(y)$

$\Theta(x^0-y^0)\; \delta\left((x-y)^2\right) j^\mu(y)$

$$\underbrace{\Theta(x^0-\gamma^0(\tau))\; \delta\left((x-\gamma(\tau))^2\right)}_{\text{selects the only point at which the past lightcone of } x \text{ intercepts } \gamma^\mu(\tau)\text{.}}\; u^\mu(\tau)$$

selects the only point at which the past lightcone of x intercepts $\gamma^\mu(\tau)$.
This happens at $\gamma^\mu(\tau_x^R)$.



$$\delta(f(x)) = \frac{\delta(x-x_0)}{f'(x_0)}$$

if $f(x)$ has a unique zero at x_0

$$\delta\left((x-\gamma(\tau))^2\right) = \frac{\delta(\tau-\tau^R_x)}{2(x^\mu-\gamma^\mu(\tau^R_x))\,\dot\gamma_\mu(\tau^R_x)}$$

$\tau = \tau^R_x$

$$\to A^R_\mu(x) = \left.\frac{q\,u^\mu(\tau)}{\underbrace{(x^\nu-\gamma^\nu(\tau))}_{\text{lighlike} \sim (|\vec r|,\,\vec r)}\,u_\nu(\tau)}\right|_{\tau=\tau^R_x}$$

x

$-x_0)$

(x_0)

$$\varphi_R(x) = \left.\frac{q}{(1-\vec v\cdot\vec n)|\vec r|}\right|_{ret}, \quad \vec A_R(x) = \left.\frac{q\vec v}{(1-\vec v\cdot\vec n)|\vec r|}\right|_{ret}$$

$$\vec n \equiv \frac{\vec r}{|r|}$$