

Title: Classical Physics Lecture - 092623

Speakers:

Collection: Classical Physics 2023/24

Date: September 26, 2023 - 9:00 AM

URL: <https://pirsa.org/23090097>

Neumann-type Green's function

problem:
$$\begin{cases} \Delta \varphi = -4\pi\rho & \text{in } R \\ \vec{s} \cdot \vec{\nabla} \varphi = -f & \text{at } \partial R \end{cases}$$

Gauss's law:
$$4\pi \int_R \rho = \oint_{\partial R} f$$

$$\left[\int_R \Delta \varphi = \int_R \vec{\nabla} \cdot (\vec{\nabla} \varphi) = \oint_{\partial R} \vec{s} \cdot \vec{\nabla} \varphi \right]$$

Green's identity:

$$\int_R \Phi(y) \Delta \Psi(y) = \int_R \Psi(y) \Delta \Phi(y) + \oint_{\partial R} \vec{s} \cdot (\Phi(y) \vec{\nabla} \Psi(y) - \Psi(y) \vec{\nabla} \Phi(y))$$

Rmk: $\varphi(x)$

$$\begin{cases} \Delta_y G_N(y, x) = \delta^3(y-x) \\ \vec{s} \cdot \vec{\nabla}_y G_N(y, x) = \frac{1}{\text{Area}(\partial R)} \end{cases}$$

$$\Phi(y) = \varphi(y) \quad \& \quad \Psi(y) = G_N(y, x)$$

Green's identity:

$$\varphi(x) = -4\pi \int_R \rho(y) G_N(y, x)$$

$$\begin{cases} \Delta_y G_N(y, x) = \delta^3(y-x) & \text{in } R \\ \vec{s} \cdot \vec{\nabla}_y G_N(y, x) = \frac{1}{\text{Area}(\partial R)} & \text{at } \partial R \end{cases}$$

$$\Phi(y) = \varphi(y) \quad \& \quad \Psi(y) = G_N(y, x)$$

Green's identity:

$$\varphi(x) = -4\pi \int_R \rho(y) G_N(y, x) + \oint_{\partial R} \frac{\varphi(y)}{\text{Area}(\partial R)} + \oint_{\partial R} f(y) G_N(y, x)$$

$$(\vec{\nabla}_y \Phi(y) \cdot \vec{\nabla} \Psi(y) - \Psi(y) \nabla^2 \Phi(y))$$

Remark: $\varphi(x)$ is a sol. $\langle \varphi \rangle_{\partial R} = \text{const.}$

iff $\varphi(x) + c$ is a sol. $\Rightarrow \langle \varphi \rangle_{\partial R}$ doesn't really matter

$$\begin{cases} \Delta_y G_N(y, x) = \delta^3(y-x) & y \in R \quad (x \in \dot{R}) \\ \vec{n} \cdot \vec{\nabla}_y G_N(y, x) = \frac{1}{\text{Area}(\partial R)} & y \in \partial R \end{cases}$$

$$\Phi(y) = \varphi(y) \quad \& \quad \Psi(y) = G_N(y, x)$$

Green's identity:

$$\varphi(x) = -4\pi \int_R \rho(y) G_N(y, x) + \oint_{\partial R} \frac{\varphi(y)}{\text{Area}(\partial R)} + \oint_{\partial R} f(y) G_N(y, x)$$

$$\Phi(y) \vec{\nabla} \Psi(y) - \Psi(y) \vec{\nabla} \Phi(y)$$

Remark: $\varphi(x)$ is a sol. $\langle \varphi \rangle_{\partial R} = \text{const.}$
 iff $\varphi(x) + c$ is a sol. $\Rightarrow \langle \varphi \rangle_{\partial R}$ doesn't really matter

Method of images

$$\Delta_y G_D(y, x) = \delta^3(y-x)$$

$$G_D(y, x) = 0$$

$$y \in \dot{R}$$

$$y \in \partial R$$

$$(x \in \dot{R})$$

General recipe

- Find a "special" sol for part with sources:

$$\Delta_y G_0(y, x) = \delta^3(y-x)$$

$$\left[\text{e.g. } G_0(y, x) = \frac{1}{4\pi |y-x|} \right]$$

- Look for a sol. of the homog. eq.

$$\Delta_y F(y, x) = 0$$

$$\text{such that } G_D(y, x) = G_0(y, x) + F(y, x)$$

satisfies the boundary condition

(y, x)

doesn't really matter

Method of images.

If we are lucky (\equiv if the shape of R is simple enough) we can

build
$$F(y, x) = \sum_{\alpha} q_{\alpha} G_0(y, x_{\alpha}), \quad x_{\alpha} \in \mathbb{R}^3 \setminus R$$

↑ image charges' charge. ↑ image charges' position.

WAVE EQ

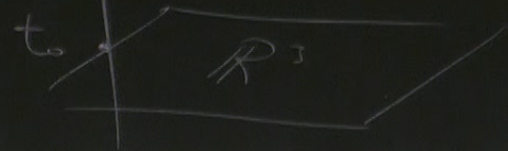
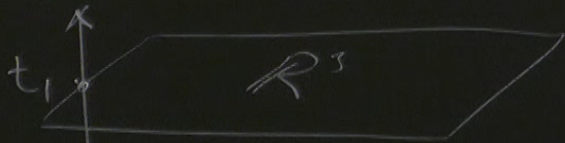
$$\square \varphi(t, \vec{x}) = -4\pi \rho(\vec{x}, t)$$

$$\varphi(t_0, \vec{x}) = \phi_0(\vec{x})$$

$$\partial_t \varphi(t_0, \vec{x}) = \pi_0(\vec{x})$$

initial
conditions

(or final cond
if $t_0 \leadsto t_1$)



$x_\alpha \in \mathbb{R}^3 \setminus \mathbb{R}$
eg charges' position
charge.

WAVE EQ

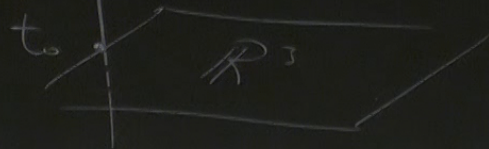
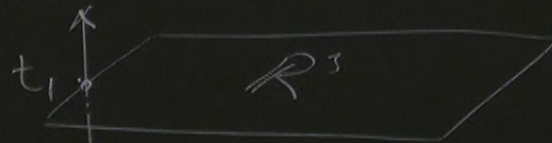
$$\square \varphi(t, \vec{x}) = -4\pi \rho(\vec{x}, t)$$

$$\varphi(t_0, \vec{x}) = \phi_0(\vec{x})$$

$$\partial_t \varphi(t_0, \vec{x}) = \pi_0(\vec{x})$$

initial
conditions

(or final cond
if $t_0 \rightsquigarrow t_1$)



Goal: use Green's functions to write down sol of $\varphi(t, \vec{x})$ as integrals of ρ, ϕ_0, π .

initial conditions

(or final cond if $t_0 \rightarrow t_1$)

$$\int_M \Phi(y) \square \Psi(y) = \int_M \Psi(y) \square \Phi(y) + \int_{\mathbb{R}^3} \Phi(\vec{y}, t) \partial_t \Psi(\vec{y}, t) \Big|_{t_0}^{t_1} - \Psi(\vec{y}, t) \partial_t \Phi(\vec{y}, t) \Big|_{t_0}^{t_1}$$

$$\Phi = \varphi$$

$$\Psi = G = \begin{cases} 0 & \text{at } t_1 \\ \neq & \text{at } t_0 \end{cases}$$

↑
initial v.p.

initial bound value pb

$$\left\{ \begin{array}{l} \square_y G_R(y, x) = \delta^4(y-x) \\ G_R(y, x) = 0 \end{array} \right.$$

↑ "retarded"

$y, x \in M$
if $y^0 > x^0$
↑
time coord

→ G

final bvp:

$$\left\{ \begin{array}{l} \square_y G_A(y, x) = \delta^4(y-x) \\ G_A(y, x) = 0 \end{array} \right.$$

if $y^0 < x^0$

$$4\pi \int_M \rho(\vec{y}) G_R(\vec{y}, x) + \int_{\mathbb{R}^3 \times \{t_0\}} \left(\pi_0(\vec{y}) G_R(t_0, \vec{y}, x) - \phi_0(\vec{y}) \frac{\partial}{\partial y^0} G_R(t_0, \vec{y}, x) \right)$$

bulk source
 initial conditions

but with $t_0 \rightarrow \Delta t$,
& sign swapped

Retarded Green's f.

$$G(y, x) \equiv D(r), \quad r^\mu = x^\mu - y^\mu$$

fulk. $\square_y G(y, x) = \delta^4(y-x)$

iff $\square D(r) = \delta^4(r)$

Fourier: $D(r) = \int d^4k e^{-ik_\mu r^\mu} \hat{D}(k)$

$$\hat{D}(k) = \frac{1}{(2\pi)^4} \int d^4r e^{+ik_\mu r^\mu} D(r)$$

iff $\hat{D}(k) = -\frac{1}{(2\pi)^4} \frac{1}{k_\mu k^\mu}$

Pf

$$\square D(r) = \int d^4k e^{ik_\mu r^\mu} \delta^4(r)$$

for $\int d^4k e^{ik_\mu r^\mu}$

$$D(r) = -$$

$$k^\mu = (\omega$$

Pf

$$\square D(r) = \int d^4k \underbrace{(-ik_\tau)(ik^\tau)}_{= \frac{1}{(2\pi)^4}} e^{-ik_\tau r^\tau} \hat{D}(k)$$

$$\delta^4(r)$$

for $\int d^4k e^{ik_\tau r^\tau} = (2\pi)^4 \delta^4(r)$, $\nabla_\mu e^{-ik_\nu r^\nu} = -ik_\mu e^{-ik_\nu r^\nu}$

$$D(r) = -\frac{1}{(2\pi)^4} \int d^4k \frac{e^{+ik_\tau r^\tau}}{k^\tau k_\tau} = \frac{1}{(2\pi)^4} \int d\vec{k} e^{i\vec{k}\vec{r}} \int d\omega \frac{e^{-i\omega t}}{\omega^2 - \vec{k}^2}$$

↑ poles!

$\hat{D}(k)$

$e^{ik_\tau r^\tau} D(r)$

$$k^\tau = (\omega, \vec{k})$$

iff $\square D(r) = \delta^4(r)$

Fourier: $D(r) = \int d^4k e^{-ik_\mu r^\mu} \hat{D}(k)$

$$\hat{D}(k) = \frac{1}{(2\pi)^4} \int d^4r e^{+ik_\mu r^\mu} D(r)$$

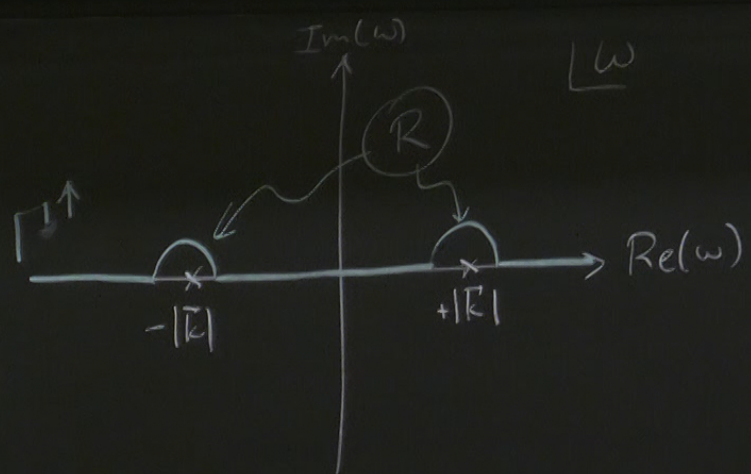
iff $\hat{D}(k) = -\frac{1}{(2\pi)^4} \frac{1}{k_\mu k^\mu}$ (naive)

for $\int d^4k e^{ik_\mu r^\mu} = (2\pi)^4 \delta^4(r)$

$$D(r) = -\frac{1}{(2\pi)^4} \int d^4k \frac{e^{+ik_\mu r^\mu}}{k^\mu k_\mu}$$

$$k^\mu = (\omega, \vec{k})$$

$$\frac{1}{\omega^2 - k^2} = \frac{1}{(2\pi)^4} K_{\mu} K_{\nu}$$

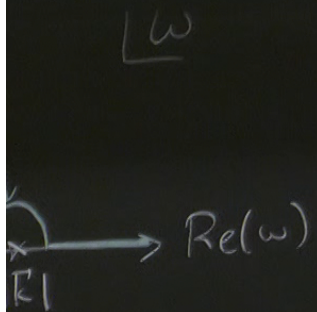


$$\omega^2 - k^2 = (\omega - |k|)(\omega + |k|)$$

Prescription "above the poles" \rightarrow (Retarded)

Contour \curvearrowright

$$\int_{\Gamma} d\omega \frac{e^{-i\omega t}}{\omega^2 - |k|^2}$$

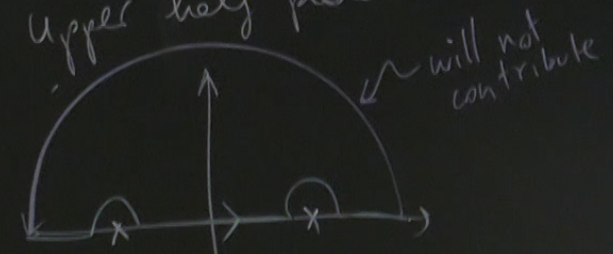


$|k|)(\omega + |k|)$
 over the poles \rightarrow (Retarded)
 \uparrow

$$I = \int_{\Gamma} d\omega \frac{e^{-i\omega t}}{\omega^2 - |k|^2}$$

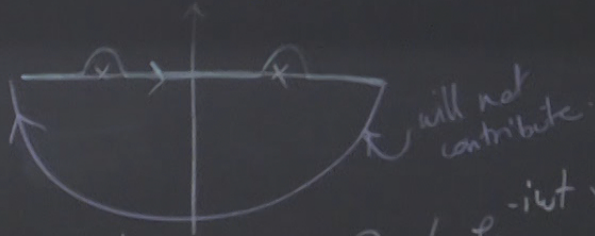
$t < 0$

$e^{-i\omega t} \rightarrow 0$ if $\text{Im}(\omega) > 0$
 \Rightarrow we can close contour in upper half plane:



$$I = \int_{\text{closed contour}} \frac{e^{-i\omega t}}{\omega^2 - |k|^2} = 0 \text{ bc it encircles no poles}$$

$t > 0$ $e^{-i\omega t} \rightarrow 0$ if $\text{Im}(\omega) < 0$.



$$I = \int_{\Gamma} d\omega \frac{e^{-i\omega t}}{\omega^2 - k^2} = -2\pi i \underset{\text{clockwise}}{\text{Res}} \left(\frac{e^{-i\omega t}}{\omega^2 - k^2} \right) = -2\pi \frac{\sin(|\vec{k}|t)}{|\vec{k}|}$$

$\omega > 0$

in

not contribute

encircles poles.

$$D(t, \vec{r}) = - \frac{\Theta(t)}{(2\pi)^3} \int d^3K e^{i\vec{K}\vec{r}} \frac{\sin(Kt)}{K}$$

clock counter

$$\Theta(t) = \begin{cases} 1 & \text{if } t > 0 \\ 0 & \text{if } t \leq 0 \end{cases}$$

Align our spherical coords for \vec{K} along \vec{r} ,

$$\vec{K} = (k, \theta, \psi), \quad \vec{K} \cdot \vec{r} = k r \cos\theta$$

↑ positive

$$= - \frac{\Theta(t)}{(2\pi)^{3/2}} \int_0^\infty dk \int_0^\pi d\theta \int_0^{2\pi} d\psi k^2 \sin\theta e^{ikr \cos\theta} \frac{\sin(kt)}{k}$$

=

$$D(t, \vec{r}) = - \frac{\Theta(t)}{(2\pi)^3} \int d^3k e^{i\vec{k}\cdot\vec{r}} \frac{\sin(kRt)}{k}$$

$\Theta(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}$

Align our spherical coords for \vec{k} along \vec{r} , $\vec{k} = (k, \theta, \psi)$, k positive

(1+d=4)

$$= - \frac{\Theta(t)}{(2\pi)^{3/2}} \int_0^\infty dk \int_0^\pi d\theta \int_0^{2\pi} d\psi k^2 \sin\theta e^{ikr \cos\theta} \frac{\sin(kt)}{k}$$

$$= + \frac{\Theta(t)}{4\pi^2} \int_0^\infty dk k \sin(kt) \left(\frac{e^{-ikt} - e^{ikt}}{ikt} \right) = - \frac{\Theta(t)}{8\pi^2}$$

closed contour

$$= \begin{cases} 1 & \text{if } t > 0 \\ 0 & \text{if } t \leq 0 \end{cases}$$

$[0, 2\pi]$

θ, ψ positive, $\vec{k} \cdot \vec{r} = k p \cos \theta$

$$D_R(t, \vec{r}) = -\frac{1}{4\pi|\vec{r}|} \Theta(t) \delta(t - |\vec{r}|)$$

both $t, p > 0$

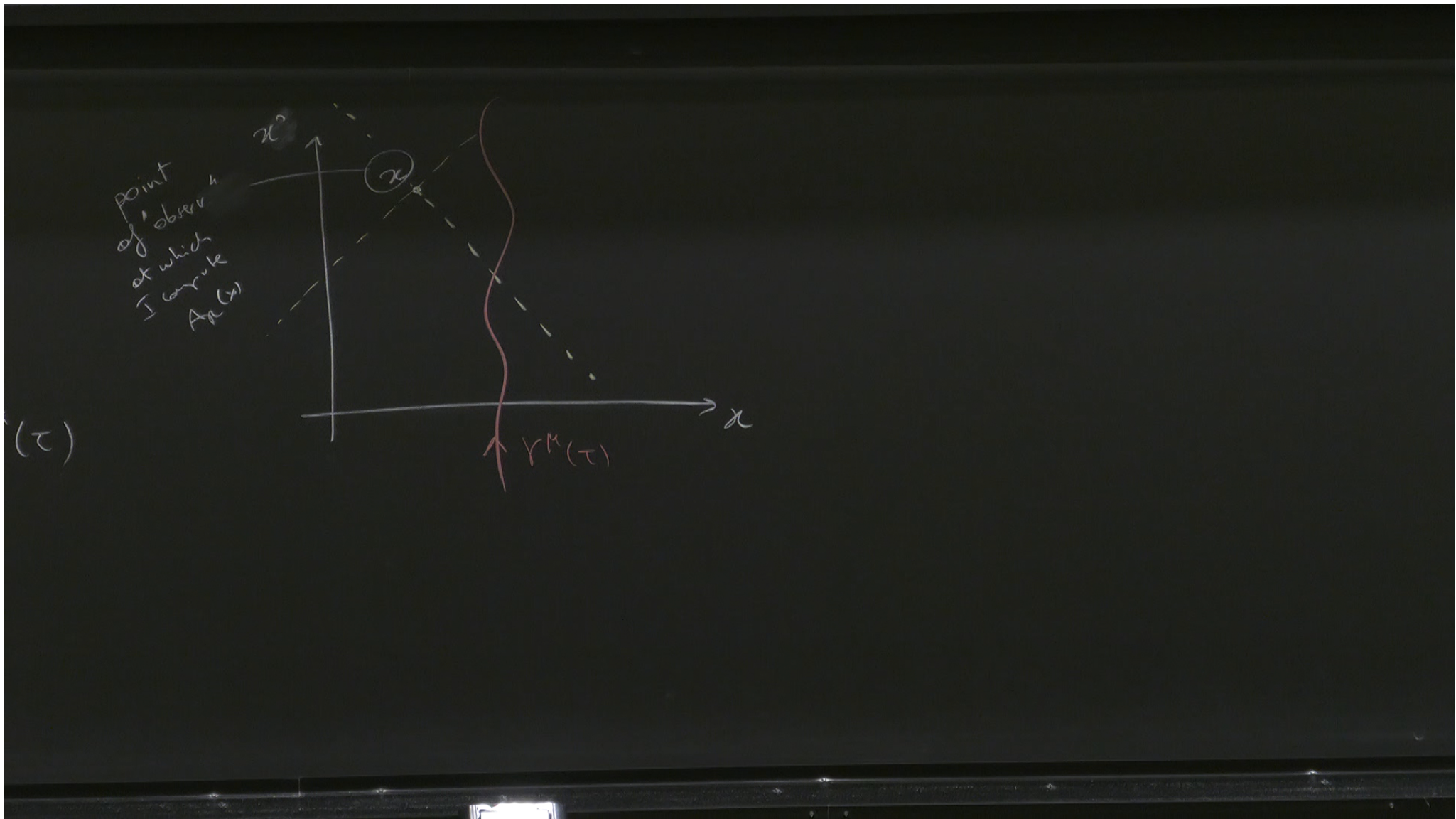
$$= -\frac{\Theta(t)}{8\pi^2 p} \int_{-\infty}^{\infty} dk (e^{ikt} - e^{-ikt}) e^{-ikp} = -\frac{\Theta(t)}{4\pi p} (\delta(t-p) + \delta(t+p))$$

$$D_R(t, \vec{r}) = -\frac{1}{4\pi|\vec{r}|} \Theta(t) \delta(t - |\vec{r}|) = -\frac{1}{2\pi} \Theta(t) \delta(r_p - r)$$

both $t, p > 0$

$$t) e^{-ikp} = -\frac{\Theta(t)}{4\pi p} (\delta(t-p) + \delta(t+p))$$

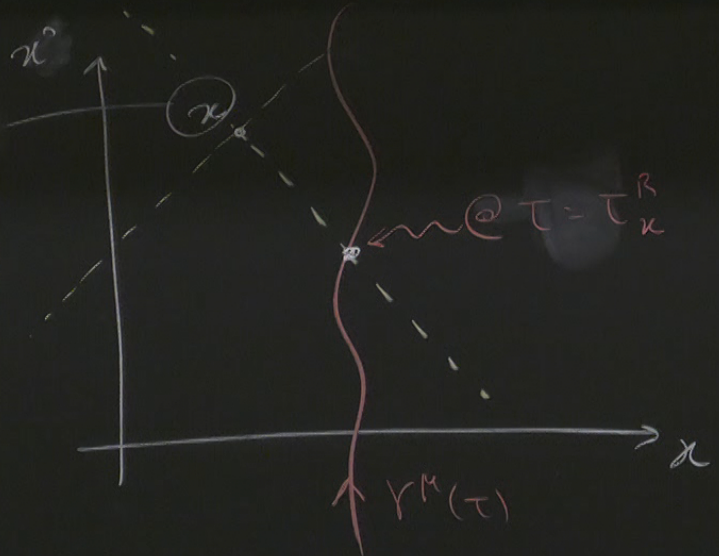
$$\begin{aligned}
 A_R^\mu(x) &= -4\pi \int d^4y \quad \dot{z}^\mu(y) G_R(y,x) \\
 &= -4\pi \int d^4y \quad D_R(x-y) \dot{z}^\mu(y) \\
 &\stackrel{\text{cov. expr. of } D_R}{=} 2 \int d^4y \quad \Theta(x^0-y^0) \delta((x-y)^2) \dot{z}^\mu(y) \\
 &\stackrel{\text{integrate } \int d^4y \delta^4(y-\gamma)}{=} 2q \int d\tau \quad \Theta(x^0-\gamma^0(\tau)) \delta((x-\gamma(\tau))^2) \dot{z}^\mu(\tau)
 \end{aligned}$$



$$\begin{aligned}
 A_R^\mu(x) &= -4\pi \int d^4y \ j^\mu(y) G_R(y,x) \\
 &= -4\pi \int d^4y \ D_R(x-y) j^\mu(y) \\
 &\stackrel{\text{cov. expr. of } D_R}{=} 2 \int d^4y \ \Theta(x^0-y^0) \delta((x-y)^2) j^\mu(y) \\
 &\stackrel{\text{integrate } \int d^4y \ \delta^4(y-x)}{=} 2q \int d\tau \ \underbrace{\Theta(x^0-\gamma^0(\tau)) \delta((x-\gamma(\tau))^2)}_{\text{selects the only point at which the past lightcone of } x \text{ intercepts } \gamma^\mu(\tau)} u^\mu(\tau)
 \end{aligned}$$

point of ...

point
of observⁿ
at which
I write
 $A_n(x)$



$x^\mu(\tau)$

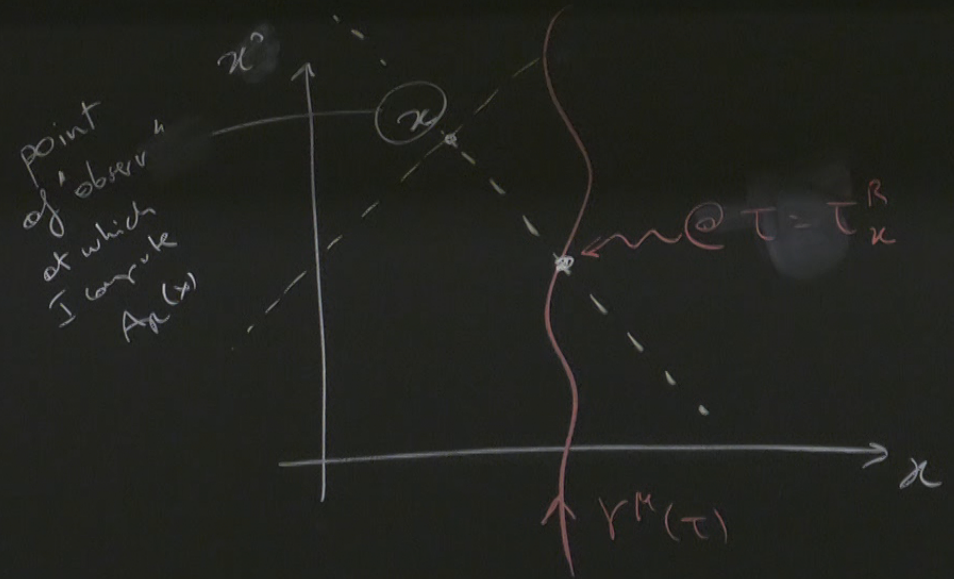
$$j^r(y) G_R(y, x)$$

$$D_R(x-y) j^r(y)$$

$$\textcircled{1} (x^0 - y^0) \delta((x-y)^2) j^r(y)$$

$$\textcircled{2} (x^0 - \gamma^0(\tau)) \delta((x - \gamma(\tau))^2) \mathcal{U}^r(\tau)$$

selects the only point
at which the past lightcone
of x intercepts $\gamma^r(\tau)$.
This happens at $\gamma^r(\tau_x^R)$.



$$\delta(f(x)) = \frac{\delta(x - x_0)}{f'(x_0)}$$

if $f(x)$ has
a unique zero at x_0 .

$$\delta((x - \gamma(\tau))^2) = \frac{\delta(\tau - \tau_x^R)}{2(x^\mu - \gamma^\mu(\tau_x^R)) \dot{\gamma}_\mu(\tau_x^R)}$$

$$\rightarrow A_\mu^R(x) = \frac{q u^\mu(\tau)}{\underbrace{(x^\nu - \gamma^\nu(\tau)) u_\nu(\tau)}_{\text{lightlike} \sim (|\vec{r}|, \vec{r})}} \Big|_{\tau = \tau_x^R}$$

$$\varphi_R(x) = \frac{q}{(1 - \vec{v} \cdot \vec{n}) |\vec{r}|} \Big|_{\text{ret}}, \quad \vec{A}_R(x) = \frac{q \vec{v}}{(1 - \vec{v} \cdot \vec{n}) |\vec{r}|} \Big|_{\text{ret}}$$

$$\vec{n} \equiv \frac{\vec{r}}{|\vec{r}|}$$