Title: Petz map recovery in quantum many-body systems

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Series: Quantum Matter

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Abstract: We study the Petz map, which is a universal recovery channel of a tripartite quantum state upon erasing one party, in quantum many-body systems. The fidelity of the recovered state with the original state quantifies how much information shared by the two parties is not mediated by one of the party, and has a universal lower bound in terms of the conditional mutual information (CMI). I will study this quantity in two different contexts. First, in a CFT ground state, we show that the fidelity is universal, which means it only depends on the central charge and the cross ratio. We compute this universal function numerically and show that it is consistently better than the naive CMI bound. Secondly, we show that for two broad classes of the states, the CMI lower bound is saturated. These include stabilizer states (in any dimensions) and the ground state of 2+1D topological order.

Zoom link: https://pitp.zoom.us/j/92623435839?pwd=N1JIdkUwWHFkZGpqb1p1V3NKYy91QT09

Petz map recovery in quantum many-body systems

Shreya Vardhan, Annie Wei and Yijian Zou

ArXiv: 2307.14434 and work in progress

Quantum entanglement in many-body physics

- Quantum phase of matter characterized by different patterns of long-range entanglement

- Entanglement entropy is useful to characterize bipartite entanglement

$$|\psi\rangle_{AB} = \sum_i \sqrt{p_i}|i\rangle_A|i\rangle_B$$

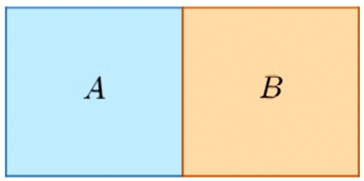
$$S = -\sum_i p_i \log p_i$$

- 1+1D CFT ground state: $S(A) = \frac{c}{3}\log\frac{l_A}{a} + O(1)$

- 2+1D topological order: $S(A) = \alpha l_A - \gamma$

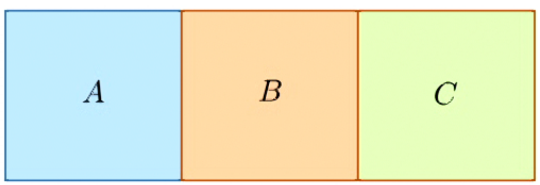
Operational interpretation of entanglement quantities

- Entanglement entropy quantifies how much Bell pair can be extracted from the state under local operations and classical communications (* asymptotically per state *)



$$\xrightarrow{\text{LOCC}} \left(\frac{1}{\sqrt{2}} \left(|01\rangle - |10\rangle \right) \right)^{\otimes N} \qquad N = S(A) = S(B)$$

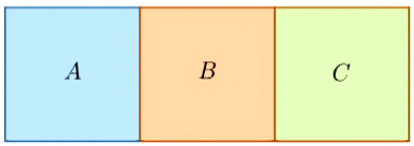
- Multipartite entanglement: operational meaning not clear in general.
- We will consider a tripartite entanglement quantity: conditional mutual information



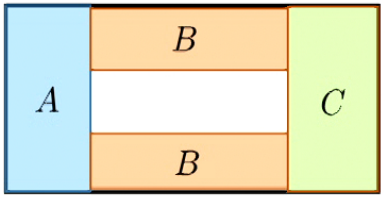
$$I(A:C|B) = S(AB) + S(BC) - S(B) - S(ABC)$$

CMI in quantum many-body systems

- 2+1D topological order ground state

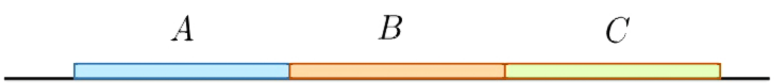


$$I(A:C|B) = 0$$



$$I(A:C|B) = 2\gamma$$

- 1+1D CFT ground state



$$I(A:C|B) = \frac{c}{3}\log\left(\frac{1}{1-\eta}\right) = \frac{c}{3}\eta + O(\eta^2)$$

$$\eta = \frac{L_A L_C}{L_{AB} L_{BC}}$$

Conditional mutual information

- Mutual information: $I(A:B) = S(A) + S(B) - S(AB) \geq 0$

  Quantifies how much correlations are shared between A and B

  Operational meaning: How close we can get to the state with local preparation

$$I(A:B) = \min_{\rho_A,\rho_B} S(\rho_{AB}||\rho_A \otimes \rho_B)$$

- Conditional mutual information: $I(A:C|B) = I(A:BC) - I(A:B) \geq 0$

  Quantifies how much correlations are shared between A and C **that is not mediated by B**

  Operational meaning: **Petz map recovery** (this talk)

Outline of the talk

- Petz map recovery

- Petz map recovery fidelity in CFT
  - Numerical observations
  - Analytical treatment using the replica trick

- Petz map recovery fidelity in special many-body states
  - Stabilizer state
  - Ground state of topological order

Petz map recovery

$$\tilde{\rho}_{ABC} = \mathcal{N}_{B\to BC}(\rho_{AB})$$

Operational question: How close* the recovered state can be with the original state?

$$\max_{\mathcal{N}:B\to BC} F(\tilde{\rho}_{ABC}, \rho_{ABC}) \qquad F(\rho,\sigma) \equiv \mathrm{Tr}\left(\sqrt{\sqrt{\sigma}\rho\sqrt{\sigma}}\right)$$

*Distance measures of mixed states include fidelity, trace distance and (Renyi-) relative entropy
They bound each other without dimensionality dependence
We will mainly consider fidelity in this talk

Perfect recovery

- Theorem 1 (Petz): $\max_{\mathcal{N}:B\to BC} F(\tilde{\rho}_{ABC}, \rho_{ABC}) = 1$ if and only if $I(A:C|B) = 0$

$$\mathcal{N}^{\lambda}(\rho_{AB}) = \rho_{BC}^{1/2+i\lambda}\rho_B^{-1/2-i\lambda}\rho_{AB}\rho_B^{-1/2+i\lambda}\rho_{BC}^{1/2-i\lambda}, \quad \forall\lambda \in \mathcal{R}$$

- This map only relies on the density matrices on AB and BC

- Theorem 2 (Hayden, Winter): $I(A:C|B) = 0$ if and only if there exists a decomposition

$$\mathcal{H}_B = \bigoplus_j \mathcal{H}_{B_j^L} \otimes \mathcal{H}_{B_j^R} \qquad \rho_{ABC} = \sum_j p_j \rho_{AB_j^L} \otimes \rho_{B_j^R C}$$

- This form makes it clear that the correlations between A and C are mediated by B
- It also implies that the correlations between A and C are classical

Approximate recovery

- Theorem 3: (Sutter, Fawzi, Renner) $\max_{\mathcal{N}:B\to BC} F(\tilde{\rho}_{ABC}, \rho_{ABC}) \geq e^{-I(A:C|B)/2}$

$$\tilde{\rho}_{ABC}(\lambda) = \mathcal{N}^{\lambda}(\rho_{AB}) = \rho_{BC}^{1/2+i\lambda}\rho_{B}^{-1/2-i\lambda}\rho_{AB}\rho_{B}^{-1/2+i\lambda}\rho_{BC}^{1/2-i\lambda},$$

- The following two inequality holds:

$$-\overline{\log F}(\tilde{\rho}_{ABC}(\lambda), \rho_{ABC})) \geq I(A:C|B)/2$$ *(Average fidelity)*

$$\max_{\lambda} F(\tilde{\rho}_{ABC}(\lambda), \rho_{ABC}) \geq e^{-I(A:C|B)/2}$$ *(Best fidelity)*

- This means if the CMI is small, then approximate recovery is possible

Petz map recovery in quantum many-body systems

- The theorem guarantees that a lower bound of the recovery fidelity
- However, it does not say how much better we can go beyond the CMI bound
- For both 1+1D CFT ground state and 2+1D topological order ground state, the CMI bound contains universal information
- Question 1: Does the best/average recovery fidelity tell us more about universal information?
- Question 2: Does the best/average recovery fidelity tell us about the entanglement of the state?

Perfect recovery

- Theorem 1 (Petz): $\max_{\mathcal{N}:B\to BC} F(\tilde{\rho}_{ABC}, \rho_{ABC}) = 1$ if and only if $I(A:C|B)=0$

$$\mathcal{N}^{\lambda}(\rho_{AB}) = \rho_{BC}^{1/2+i\lambda}\rho_B^{-1/2-i\lambda}\rho_{AB}\rho_B^{-1/2+i\lambda}\rho_{BC}^{1/2-i\lambda}, \quad \forall\lambda\in\mathcal{R}$$

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Main result: 1+1D CFT

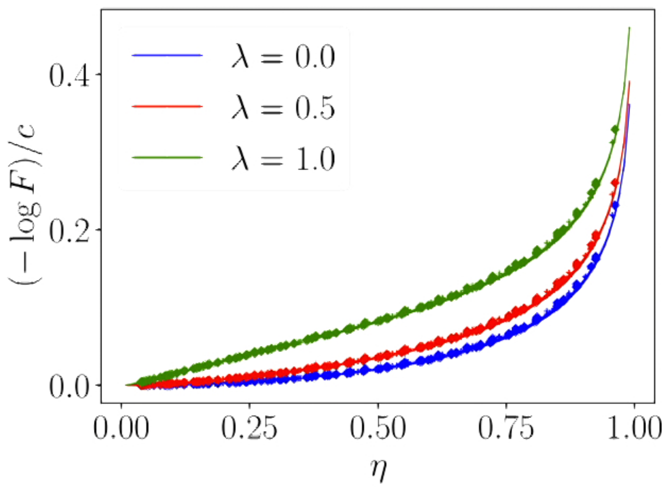
- The fidelity has a universal form $F(\tilde{\rho}_{ABC}(\lambda), \rho_{ABC}) = e^{-cf_\lambda(\eta)}$

- $f_\lambda(\eta)$ is theory independent

- Best recovery is always achieved by $\lambda = 0$. Fidelity is *significantly* better than CMI bound

- Average fidelity is better than the CMI bound by a constant factor

Theory independence



Different markers label three different models (Ising, tricritical Ising, free compactified boson)

Numerical data obtained using Periodic Uniform Matrix Product State for up to N=128

Approximate recovery

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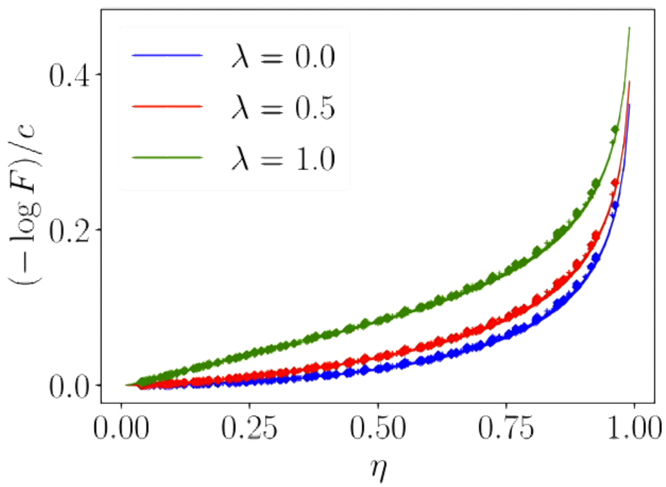
Theory independence



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Numerical data obtained using
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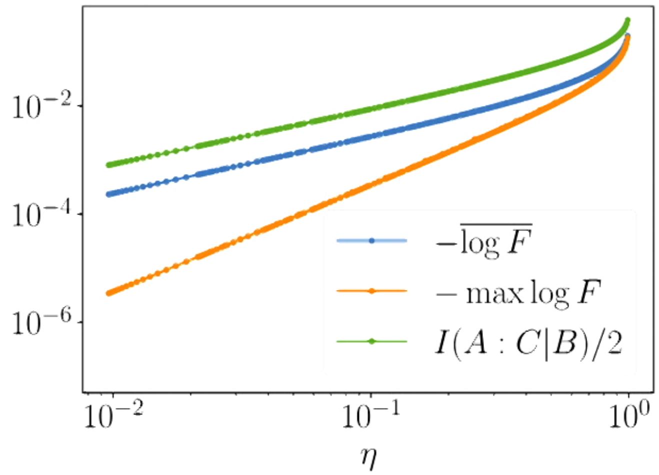
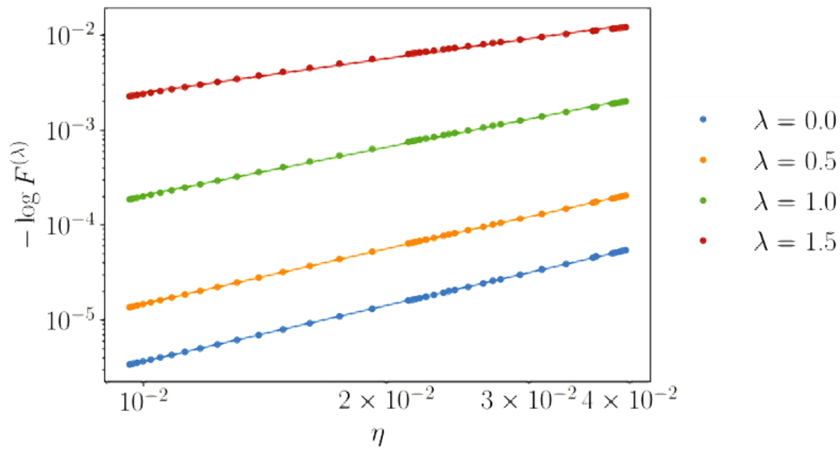
Long B limit



$$-\overline{\log F} \approx 0.055c\eta + O(\eta^2)$$
$$-\max \log F \approx 0.07c\eta^2 + O(\eta^4)$$

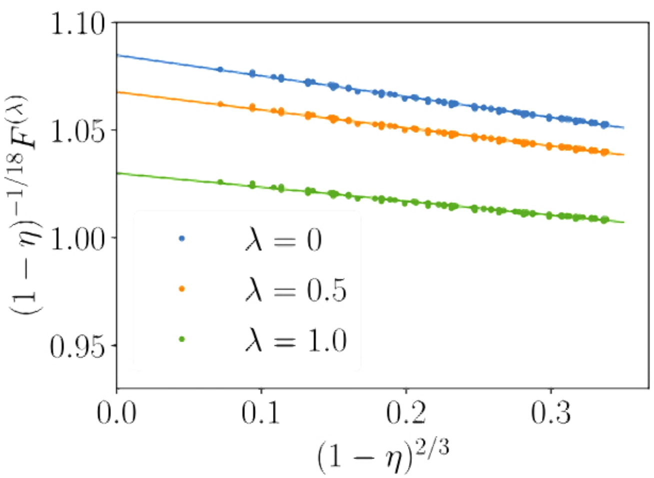


$$-\log F_\lambda = O(\eta^p)$$

Exponent decreases as $|\lambda|$ increases

$$I(A:C|B) = \frac{c}{3}\log\left(\frac{1}{1-\eta}\right)$$

Short B limit



$$-\frac{1}{c}\log F = \frac{1}{9}\log\frac{1}{1-\eta} + \text{const} + O((1-\eta)^{2/3}) \quad (\eta \to 1)$$

Different λ differ by the constant term

We will prove this using a replica trick

Replica trick

$$F(\rho,\sigma) \equiv \mathrm{Tr}\left(\sqrt{\sqrt{\sigma}\rho\sqrt{\sigma}}\right) = \mathrm{Tr}((\rho\sigma)^{1/2})$$

We want to compute: $F^{(\lambda)} = \mathrm{Tr}\left[\left((\rho_{BC}^{\frac{1}{2}-\frac{i\lambda}{2}}\rho_B^{-\frac{1}{2}+\frac{i\lambda}{2}}\rho_{AB}\rho_B^{-\frac{1}{2}+\frac{i\lambda}{2}}\rho_{BC}^{\frac{1}{2}-\frac{i\lambda}{2}})\rho_{ABC}\right)^{\frac{1}{2}}\right]$

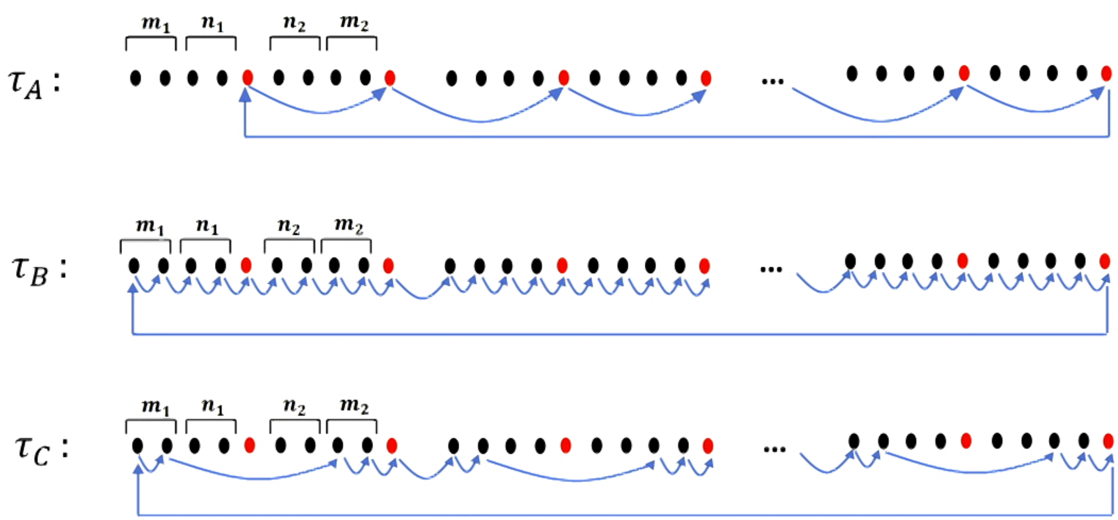
Replica version: $F_{k,n_1,n_2,m_1,m_2} = \mathrm{Tr}[(\rho_{BC}^{m_1}\rho_B^{n_1}\rho_{AB}\rho_B^{n_2}\rho_{BC}^{m_2}\rho_{ABC})^k]$

$$m_1 \to \frac{1}{2}-\frac{i\lambda}{2},\quad n_1 \to -\frac{1}{2}+\frac{i\lambda}{2},\quad m_2 \to \frac{1}{2}+\frac{i\lambda}{2},\quad n_2 \to -\frac{1}{2}-\frac{i\lambda}{2},\qquad k \to \frac{1}{2}$$

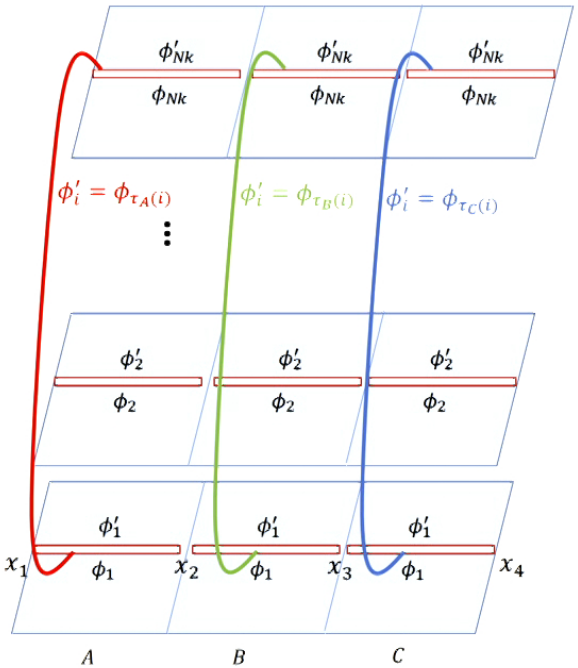
Twist operation formulation

$$F_{k,n_1,n_2,m_1,m_2} = \mathrm{Tr}[(\rho_{BC}^{m_1}\rho_B^{n_1}\rho_{AB}\rho_B^{n_2}\rho_{BC}^{m_2}\rho_{ABC})^k]$$

$$F_{k,n_1,n_2,m_1,m_2} = \mathrm{Tr}((\rho_{ABC})^{\otimes Nk}\tau_A\tau_B\tau_C), \quad N = (m_1+m_2+n_1+n_2+2), \tau_i \in S_{Nk}$$



Twist operator correlation functions



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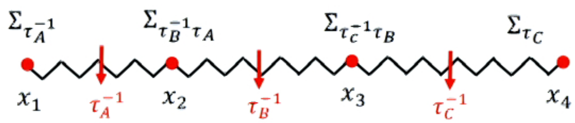


$$F_{k,n_1,n_2,m_1,m_2} = \langle \Sigma_{\tau_A^{-1}}(x_1)\Sigma_{\tau_B^{-1}\tau_A}(x_2)\Sigma_{\tau_C^{-1}\tau_B}(x_3)\Sigma_{\tau_C}(x_4)\rangle$$

Independence of the length scale

- Four point function of primary operators depends on the scaling dimensions and OPE
- Scaling dimensions of twist operators are completely determined by the cycle decomposition

$$\Delta_\tau = \frac{c}{12}\sum_{i=1}^{m}\left(n_i - \frac{1}{n_i}\right)$$

If τ has m cycles and each cycle length n_i

- For the Petz map fidelity, the scaling dimensions all go to zero in the replica limit.
- Thus, it is a function of only cross ratio (UV finiteness).

Independence of the operator content

The manifold is *genus zero* by using the *Riemann-Hurwitz* formula

$$g = \frac{1}{2}\sum_i (n_i - 1) - Nk + 1$$

Thus, we can write down a conformal transformation to map the manifold to a sphere

Under conformal transformation $g_{\mu\nu} \to e^{2\Phi} g_{\mu\nu}$, CFT partition function $Z \to e^{-cS_L[\Phi]} Z$

The sphere has no operation insertions, thus the partition function is 1. Thus,

$$F_{k,n_1,n_2,m_1,m_2} = \langle \Sigma_{\tau_A^{-1}}(x_1) \Sigma_{\tau_B^{-1}\tau_A}(x_2) \Sigma_{\tau_C^{-1}\tau_B}(x_3) \Sigma_{\tau_C}(x_4) \rangle = e^{-cS_L[\Phi]_{m_1,m_2,n_1,n_2,k}}$$

(even if one cannot explicitly work out the map)

OPE limit

- Let length of B to be very small, then we can use the OPE

$$\Sigma_{\tau_B^{-1}\tau_A}(x_2)\Sigma_{\tau_C^{-1}\tau_B}(x_3) \to (x_3 - x_2)^{\frac{c}{9}}\Sigma_{\tau_C^{-1}\tau_A}(x_3) + (x_3 - x_2)^{\frac{2}{3}+\frac{c}{9}} L_{-\frac{2}{3}}\Sigma_{\tau_C^{-1}\tau_A}(x_3)$$

$$\Delta_{\tau_C^{-1}\tau_A} = \frac{c}{9}$$

(Fractional descendant)

- Thus $F = e^{-cf(\eta)}$ $\quad f(\eta) = -\frac{1}{9}\log\frac{1}{1-\eta} + \mathrm{const} + O((1-\eta)^{2/3}) \quad (\eta \to 1)$

*The other limit (A, C small) does not work due to non-commutativity of OPE limit and replica limit

Exact result for special classes of states

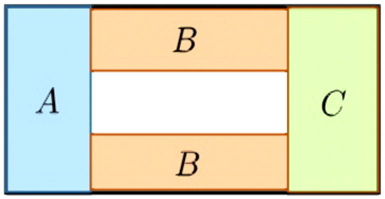
- Stabilizer states – CMI bound is saturated for all λ

$$\tilde{\rho}_{ABC}(\lambda) = \mathcal{N}^{\lambda}(\rho_{AB}) = \rho_{BC}^{1/2+i\lambda}\rho_{B}^{-1/2-i\lambda}\rho_{AB}\rho_{B}^{-1/2+i\lambda}\rho_{BC}^{1/2-i\lambda},$$

Independent of λ

$$F(\tilde{\rho}_{ABC}, \rho_{ABC}) = e^{-I(A:C|B)/2}$$

- 2+1D topological order ground state (Assuming entanglement bootstrap)



$$I(A:C|B) = 2\gamma$$

$$F(\tilde{\rho}_{ABC}, \rho_{ABC}) = e^{-I(A:C|B)/2} = \frac{1}{\sqrt{\sum_a d_a^2}}$$

The toric code is an example for both cases

Stabilizer state: proof outline

- Stabilizer group $G = \{g_1^{c_1} g_2^{c_2} \cdots g_n^{c_n} | c_i \in \{0,1\}\}$ $[g_i, g_j] = 0$
- Stabilizer state $\rho = \prod_i \frac{1+g_i}{2}$ $S(\rho) = N - n$
- Since each term is a projector, $\rho^a \propto \rho, \forall a$
- The recovered state has all stabilizers supported on AB and BC, but not jointly supported on ABC.

$$\tilde{\rho}_{ABC}(\lambda) = \mathcal{N}^{\lambda}(\rho_{AB}) = \rho_{BC}^{1/2+i\lambda} \rho_B^{-1/2-i\lambda} \rho_{AB} \rho_B^{-1/2+i\lambda} \rho_{BC}^{1/2-i\lambda},$$

- The number of missing stabilizer is exactly CMI, which equals $-\log F$

Topological order: proof outline

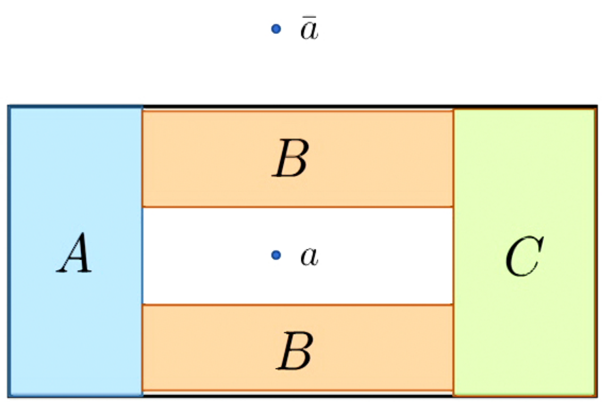
ρ^a_{ABC} are mutually orthogonal

ρ_{AB}, ρ_{BC} independent of the anyon insertion

$$\tilde{\rho}_{ABC} = \sum_a \frac{d_a^2}{D^2} \rho^a_{ABC} \quad I(A:C|B)_{\tilde{\rho}} = 0$$

Doing Petz map on $\tilde{\rho}_{ABC}$ we get exact recovery

$$\rho_{ABC} = \rho^{a=I}_{ABC} \qquad F(\rho_{ABC}, \tilde{\rho}_{ABC}) = \frac{1}{D} = e^{-\gamma}$$



Open questions

- Analytic form of the universal function in CFT
- More understanding about entanglement in CFT
    - When B is large, how close is ABC to a Markov chain?
    - How quantum the correlation between A and C is?
- Implications on measurement induced phase transitions (stabilizer v.s. Haar random)
- In what cases the Petz map gives more information about the universal properties than CMI?