Title: Quantum Theory Lecture - 090723

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Collection: Quantum Theory 2023/24

Date: September 07, 2023 - 10:45 AM

URL: https://pirsa.org/23090050

"arrow of time, something that distinguishes the past from the future, giving a direction to time"

Stephen W. Hawking

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Time Evolution

Time evolution signifies the change of a state occurring due to the passage of time

In Quantum Mechanics

- In contrast to the special relativity, time and position are not on equal standing

  Position → Operator   Time → Parameter

- The temporal progression of a state is determined by action of the Time Evolution Operator/Propagator

$$|\psi(\vec{r}, t)\rangle = U(t, t_0)|\psi(\vec{r}, t_0)\rangle$$

$|\Psi(t_0)\rangle \longrightarrow |\Psi(t)\rangle$

$U(t,t_0) \rightarrow$ time propagator

$$|\Psi(t)\rangle = U(t,t_0)\,|\Psi(t_0)\rangle$$

1) <u>Unitary</u> $\Rightarrow U^{\dagger}(t,t_0) = U(t,t_0)$

2) $U(t,t) \Rightarrow \mathbb{I}$ ✓

3) composition

$U(t_2,t_0) = U(t_2,t_1)\,U(t_1,t_0)$

4) Inverse of U

$U(t,t) = U(t,t_0)\,U(t_0,t)$

$\Rightarrow U(t,t_0) = U^{-1}(t_0,t)$

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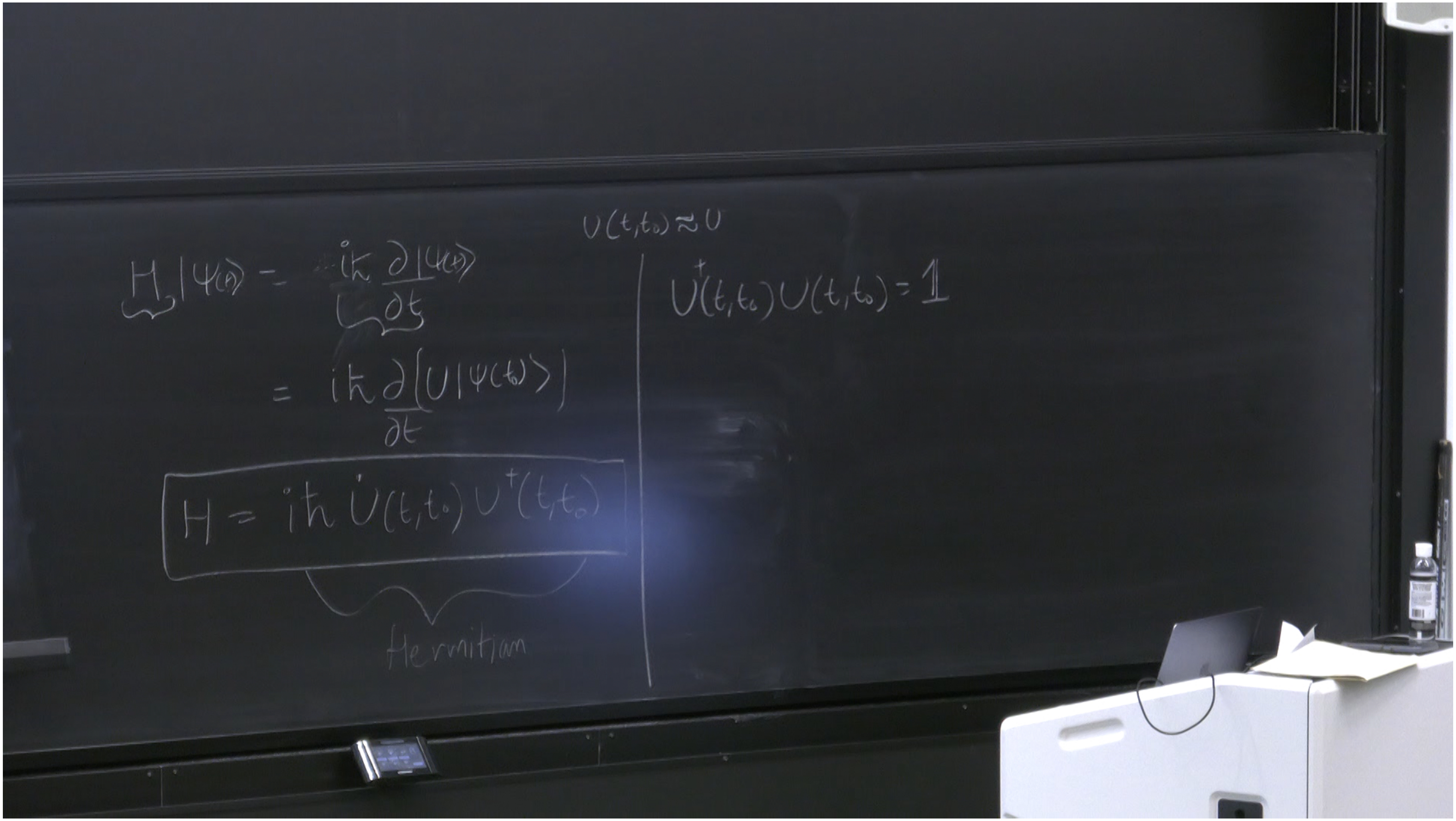
Equation of Motion for U

- U is an alternate way of writing the **Schrodinger Equation**, H and U must be related

- Derivative of state vector = **Hamiltonian** acting on the state vector

$$i\hbar\frac{\partial}{\partial t}U\left(t,t_0\right)=\hat{H}U\left(t,t_0\right)$$


U(t,t₀) ≈ U
H|ψ(t)⟩ = iħ ∂|ψ(t)⟩/∂t
= iħ ∂/∂t (U|ψ(t₀)⟩)
H = iħ U̇(t,t₀) U†(t,t₀)
Hermitian
U†(t,t₀) U(t,t₀) = 1

$$i\hbar \frac{\partial}{\partial t} U(t,t_0) = H U(t,t_0)$$

$$U(t,t_0) = 1 - \frac{i}{\hbar}\int_{t_0}^{t} dt_1\, H(t_1)\, U(t_1,t_0)$$

$$= 1 - \frac{i}{\hbar}\int_{t_0}^{t} dt_1\, H(t_1) + \left(\frac{-i}{\hbar}\right)^2 \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2\, H(t_1) H(t_2)\, U(t_2,t_0) + \dots$$

Dyson series

Change → n-fold coupled integrals into uncoupled

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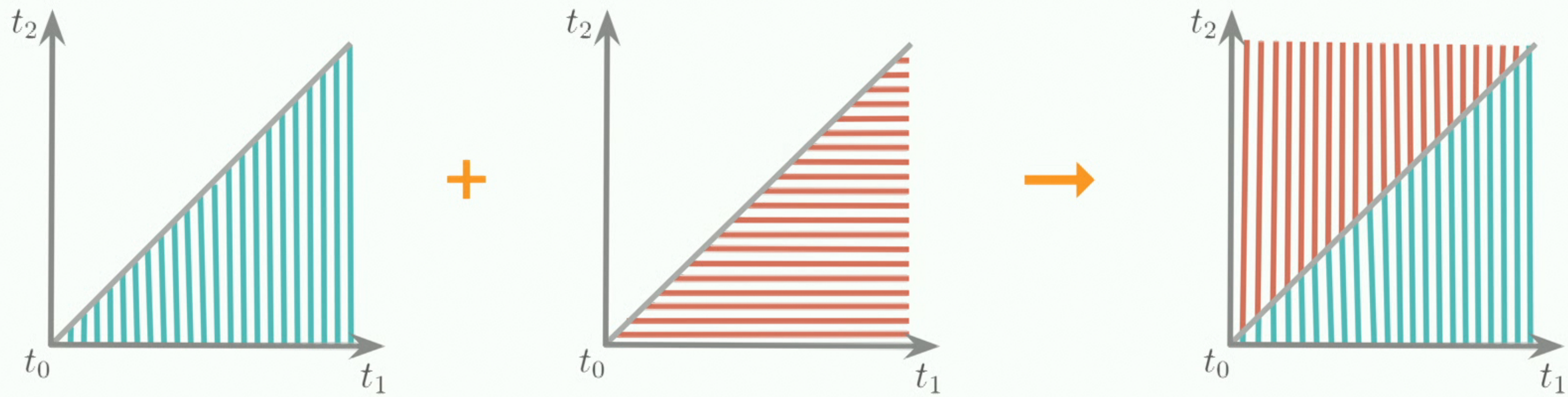
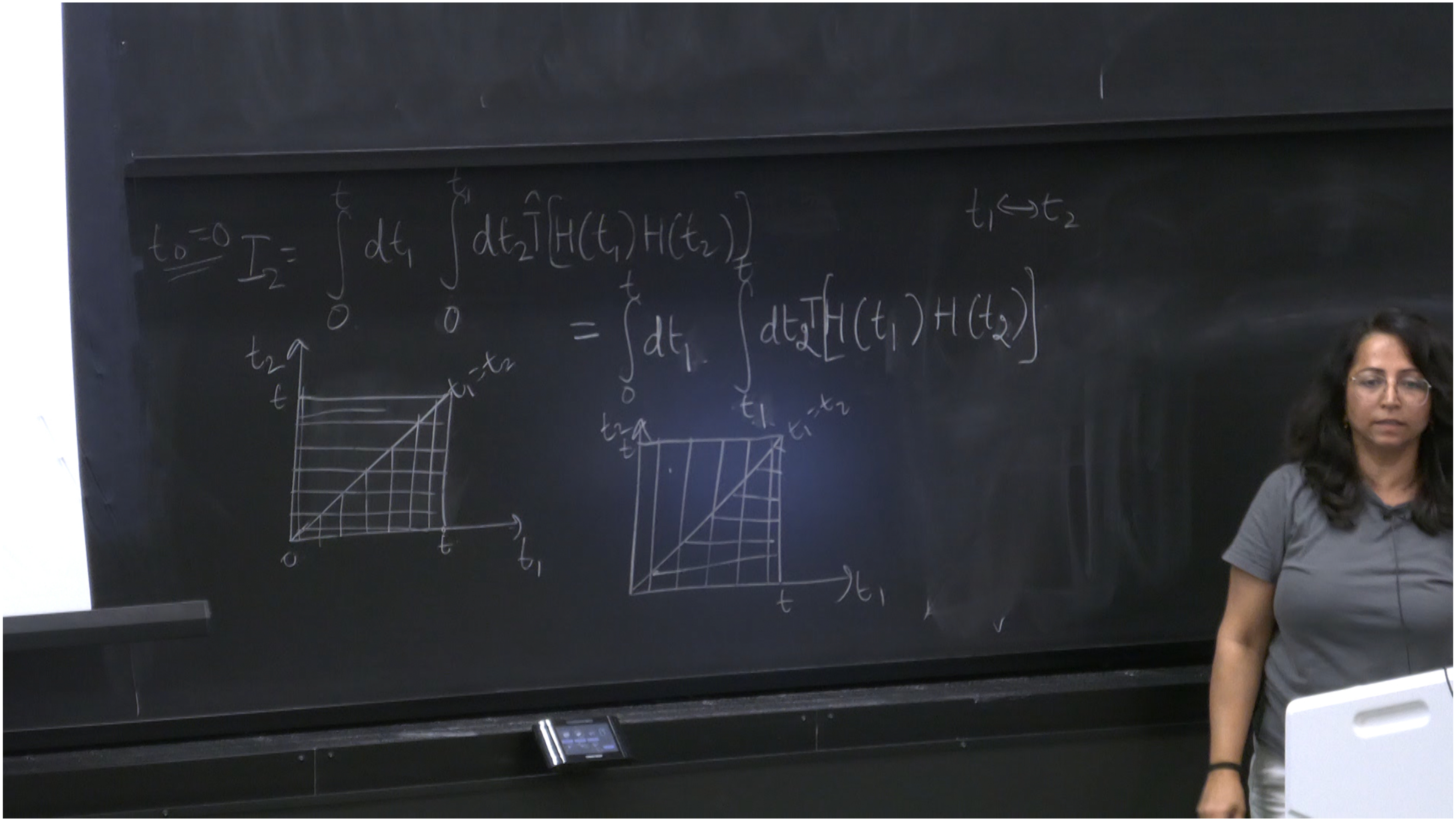
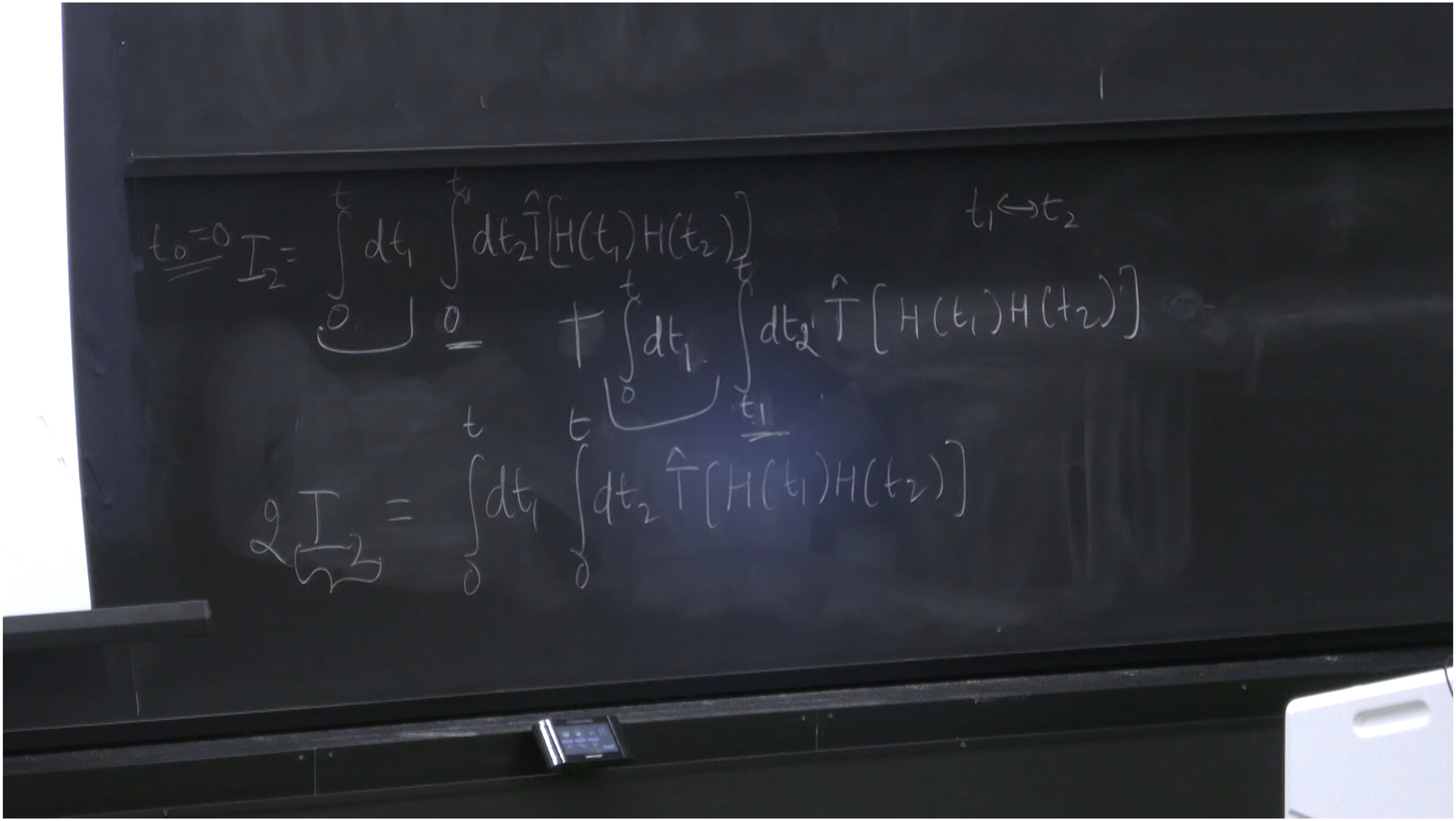
Modifying Integrals

$$I_2 = \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \; H(t_1) H(t_2) \;\rightarrow\; I_2 = \frac{1}{2!} \int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 \; \hat{T}[H(t_1) H(t_2)]$$




t₀ = 0   I₂ = ∫₀ᵗ dt₁ ∫₀^{t₁} dt₂ T̂[H(t₁) H(t₂)]
t₁ ⟷ t₂
= ∫₀ᵗ dt₁ ∫_{t₁}ᵗ dt₂ T̂[H(t₁) H(t₂)]
t₁ = t₂


$t_0 = 0 \quad I_2 = \int_0^t dt_1 \int_0^{t_1} dt_2 \hat{T}[H(t_1)H(t_2)]$
$t_1 \leftrightarrow t_2$
$+ \int_0^t dt_1 \int_{t_1}^t dt_2 \hat{T}[H(t_1)H(t_2)]$
$2 I_2 = \int_0^t dt_1 \int_0^t dt_2 \hat{T}[H(t_1)H(t_2)]$

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A Simpler Form

$$U(t,t_0) = I + \sum_{n=1}^{\infty} \frac{1}{n!} \left(\frac{-i}{\hbar}\right)^n \int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 \ldots \int_{t_0}^{t} dt\ \hat{T}[H(t_1)\,H(t_2)\ldots H(t_n)]$$

***Dyson Series**

$\hat{T}$ = Time-ordering operator

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Case I : Time Independent H $U(t,t_0) = \exp^{-iHt/\hbar}$

Case II : Time Dependent And Commuting H $U(t,t_0) = \exp^{-\frac{i}{\hbar}\int H(t)dt}$

Case III : Time Dependent And Non Commuting H

$$U(t,t_0) = I + \sum_{n=1}^{\infty} \frac{1}{n!}\left(\frac{-i}{\hbar}\right)^n \int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 \ldots \int_{t_0}^{t} dt\ \hat{T}[H(t_1)\,H(t_2)\ldots H(t_n)]$$

Different Interpretations

Schrödinger's Representation

$$\langle \psi_S(t)| \, \hat{A}_S \, |\psi_S(t)\rangle = \overbrace{\underbrace{\langle \psi_S(t_0)|}_{\langle \psi_S(t_0)|} \, \hat{U}^\dagger(t,t_0)}^{\langle \psi_S(t)|} \, \overbrace{\hat{A}_S}^{\hat{A}_S} \, \overbrace{\hat{U}(t,t_0) \, \underbrace{|\psi_S(t_0)\rangle}_{|\psi_S(t_0)\rangle}}^{|\psi_S(t)\rangle}$$

$$\underbrace{\hat{U}^\dagger(t,t_0) \, \hat{A}_S \, \hat{U}(t,t_0)}_{\hat{A}_H(t)}$$

$$= \langle \psi_S(t_0)| \, \hat{A}_H(t) \, |\psi_S(t_0)\rangle$$

Heisenberg's Representation

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Schrödinger's Approach

The **state** of a system evolves with time

Schrodinger Equation

$$\frac{\hbar^2}{2m}\frac{\partial^2\psi(x,t)}{\partial x^2} + V(x)\psi(x,t) = i\hbar\frac{\partial\psi(x,t)}{\partial t}$$

Time Evolution

$$|\psi(x,t)\rangle = U(t,t_0)|\psi(x,t_0)\rangle$$

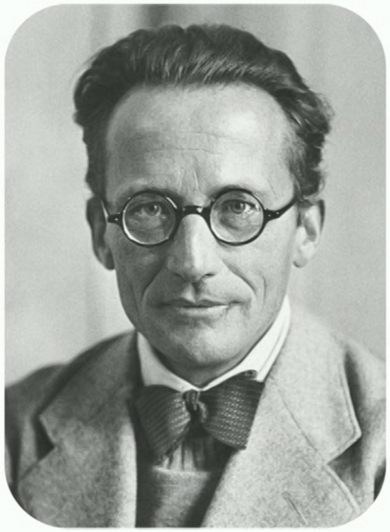
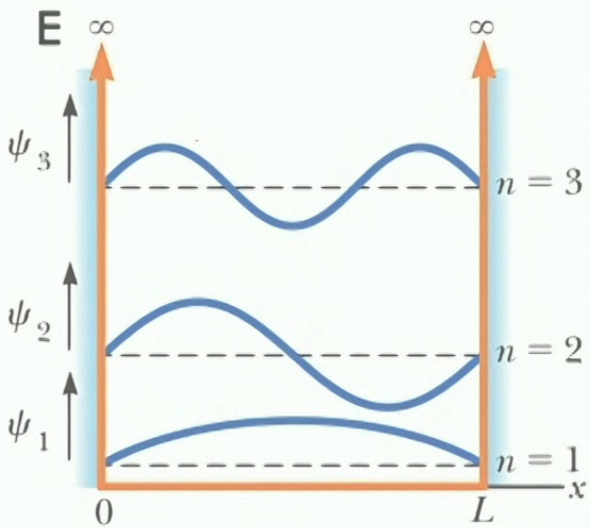


Image Source : OpenMind, www.bbvaopenmind.com/en/science/leading-figures/schrodinger-a-quantum-behind-the-secret-of-life/

Stationary States



- All observables are independent of time
- Stationary states are the **eigenstates** of the systems

$$\psi_n(x) = \sqrt{\frac{2}{L}}\,\sin\left(\frac{n\pi x}{L}\right)$$

$$E_n = \frac{n^2\pi^2\hbar^2}{2mL^2}$$

Image Source : Unknown

$\langle \psi_S(t) | A |$

$$|\psi(t)\rangle = U(t,t_0)\,|\psi(t_0)\rangle \qquad H|n\rangle = E_n|n\rangle$$

$$|\psi(t)\rangle = \sum_n c_n\,|n\rangle$$

$$|n(t)\rangle = e^{-iHt/\hbar}\,|n\rangle$$

$$= e^{-iE_n t/\hbar}\,|n(0)\rangle$$

$$\langle n|n\rangle = \langle n|n\rangle \Rightarrow \text{stationary state}$$

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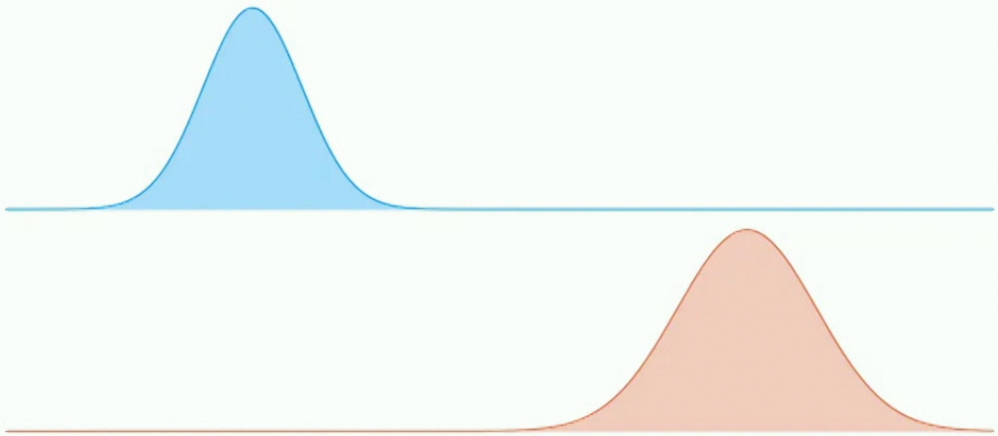
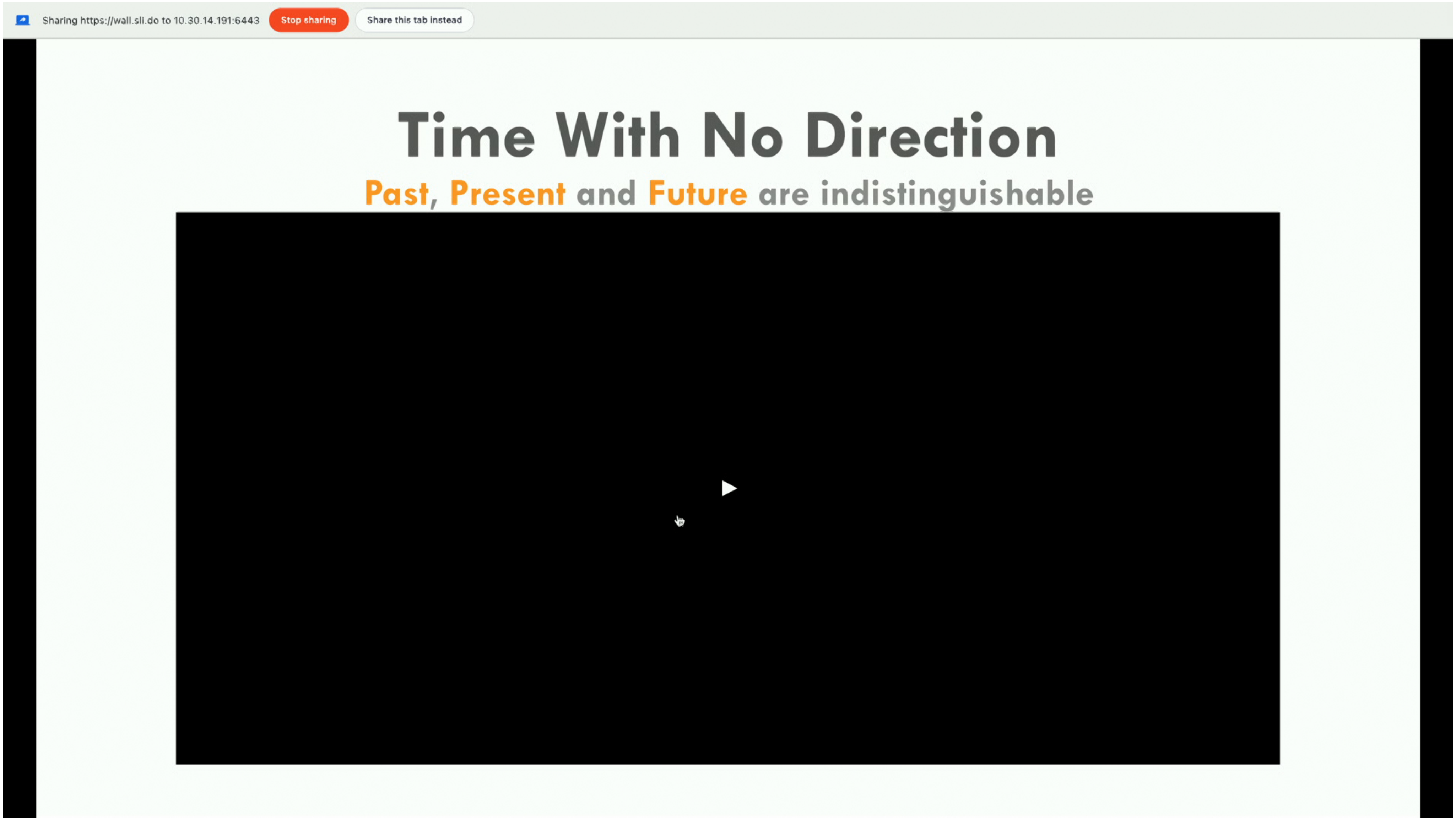
Gaussian Wave Packets



$$\psi(x,t) = \sum_{n=0,\infty} c_n\,\psi_n(x)\,\mathrm{e}^{-\mathrm{i}\,E_n\,t/\hbar}$$

$$c_n = \int_0^L \psi_n^*(x)\,\psi(x,0)\,dx \qquad \psi(x,0) = ae^{-(x-b)^2/2c^2}$$

Image Source : The product of two Gaussian is Gaussian. Is it?, forem.julialang.org/mroavi/the-product-of-two-gaussian-pdfs-is-gaussian-is-it-agg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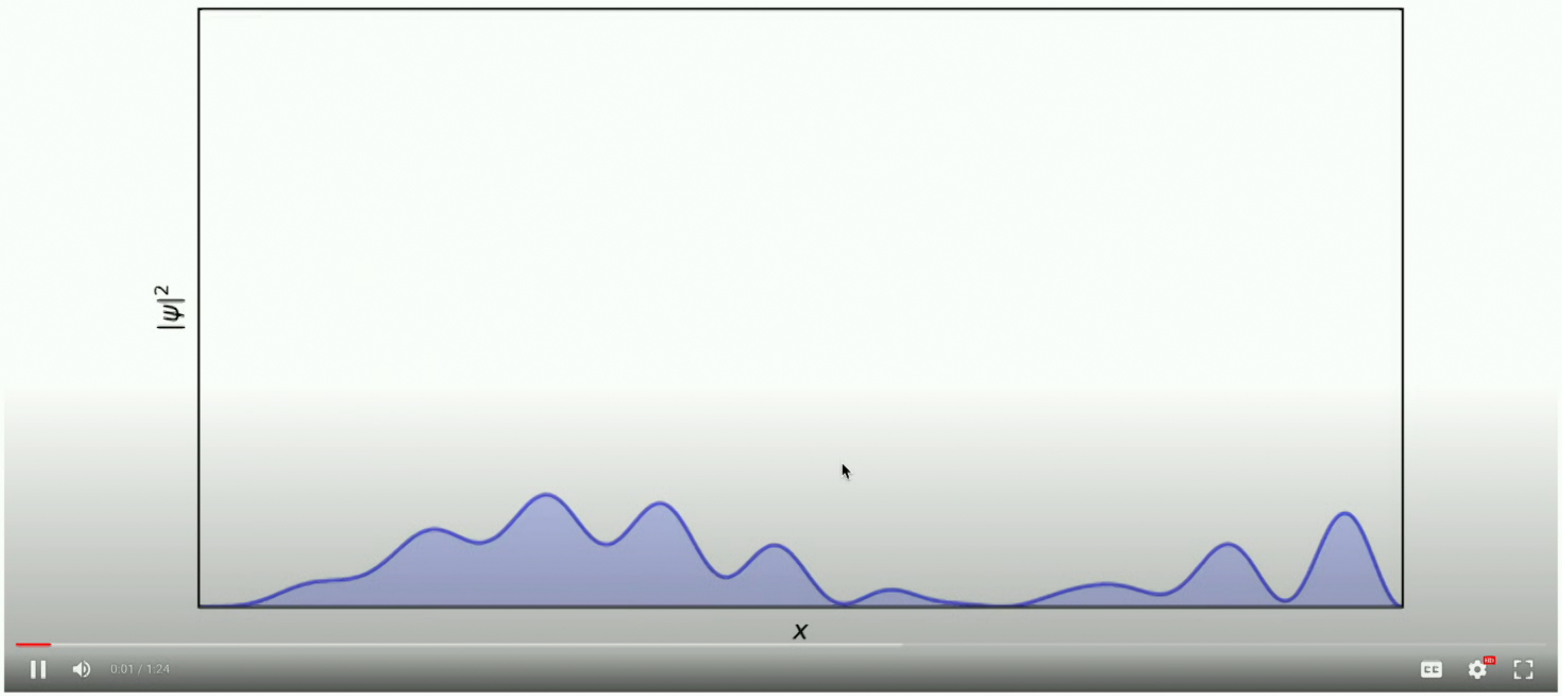
Time With No Direction
Past, Present and Future are indistinguishable

Time With No Direction

Past, Present and Future are indistinguishable



Interaction With Environment

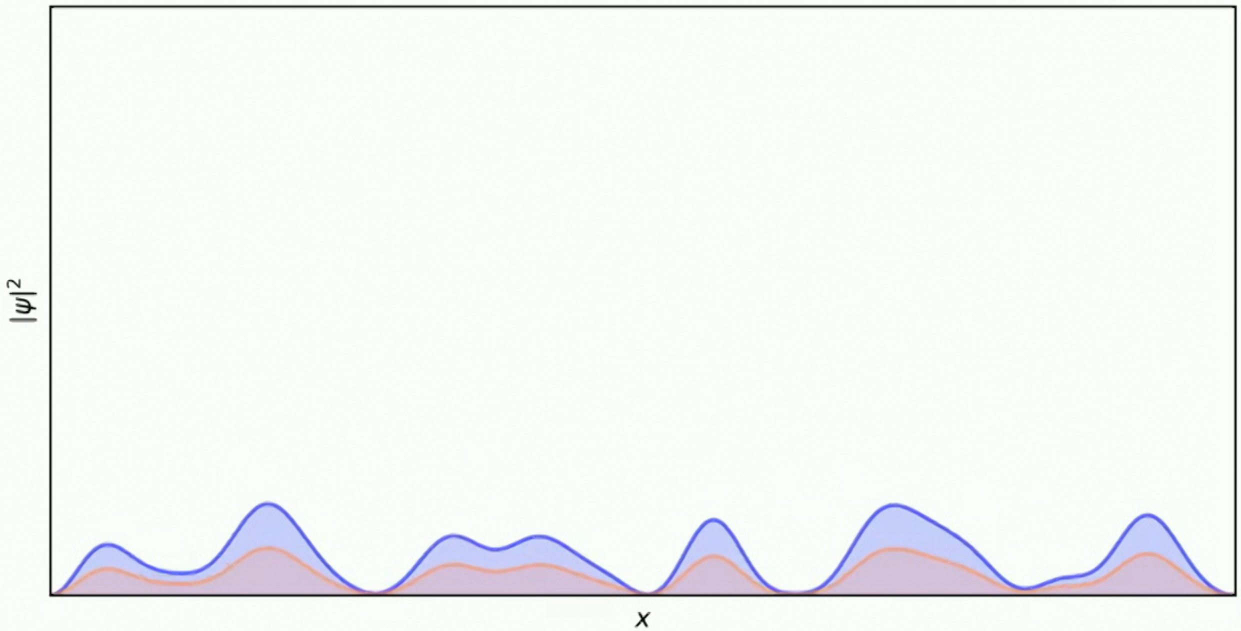
Why excited states of atoms have finite lifetime even though they are eigen states

$|\psi\rangle_s$ Not the complete description of the system

$|\psi\rangle_s \otimes |E\rangle$ System interact with environment and dissipates to give time a direction

Time With Direction

The arrow of time emerges



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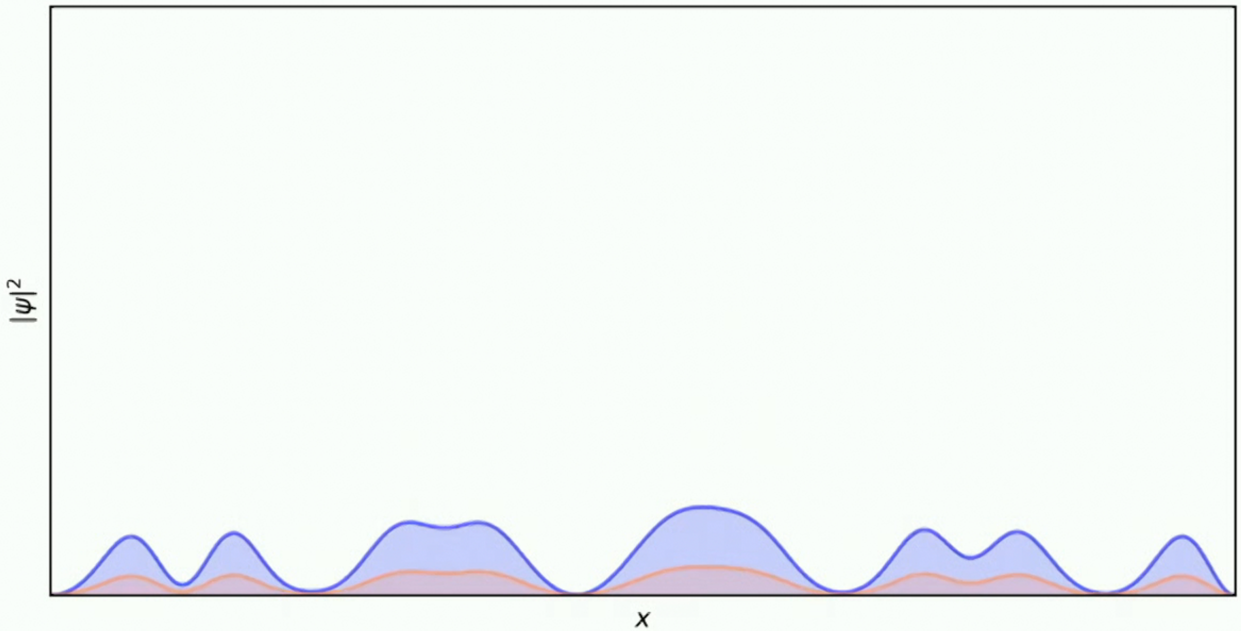
Time With Direction

The arrow of time emerges



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Heisenberg's Approach

The operators (observables) evolves with time

Heisenberg's Equation of Motion

$$\frac{dA_H(t)}{dt} = -\frac{i}{\hbar}[A_H(t), H]$$

where $A_H = U^\dagger A U$

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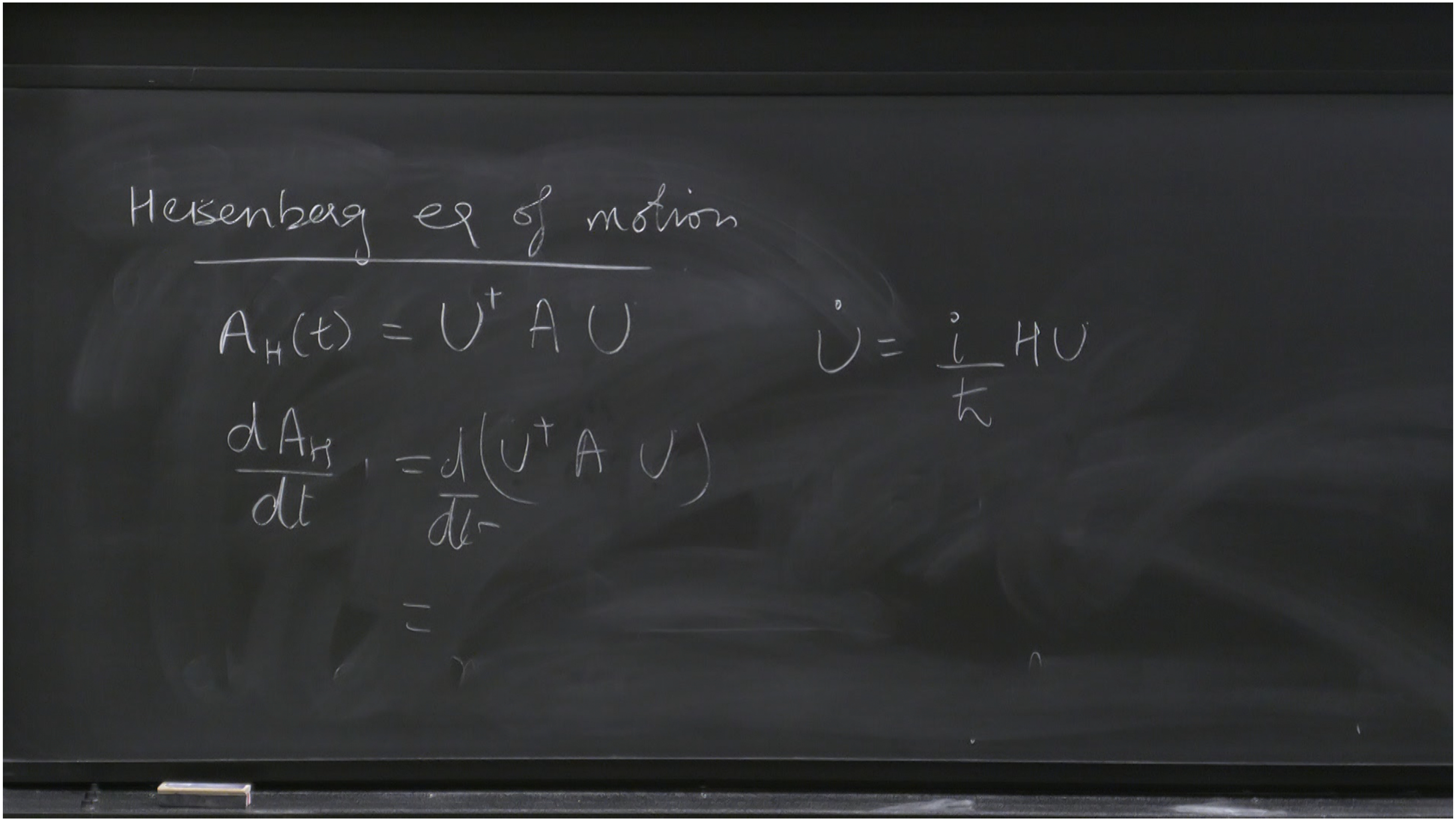
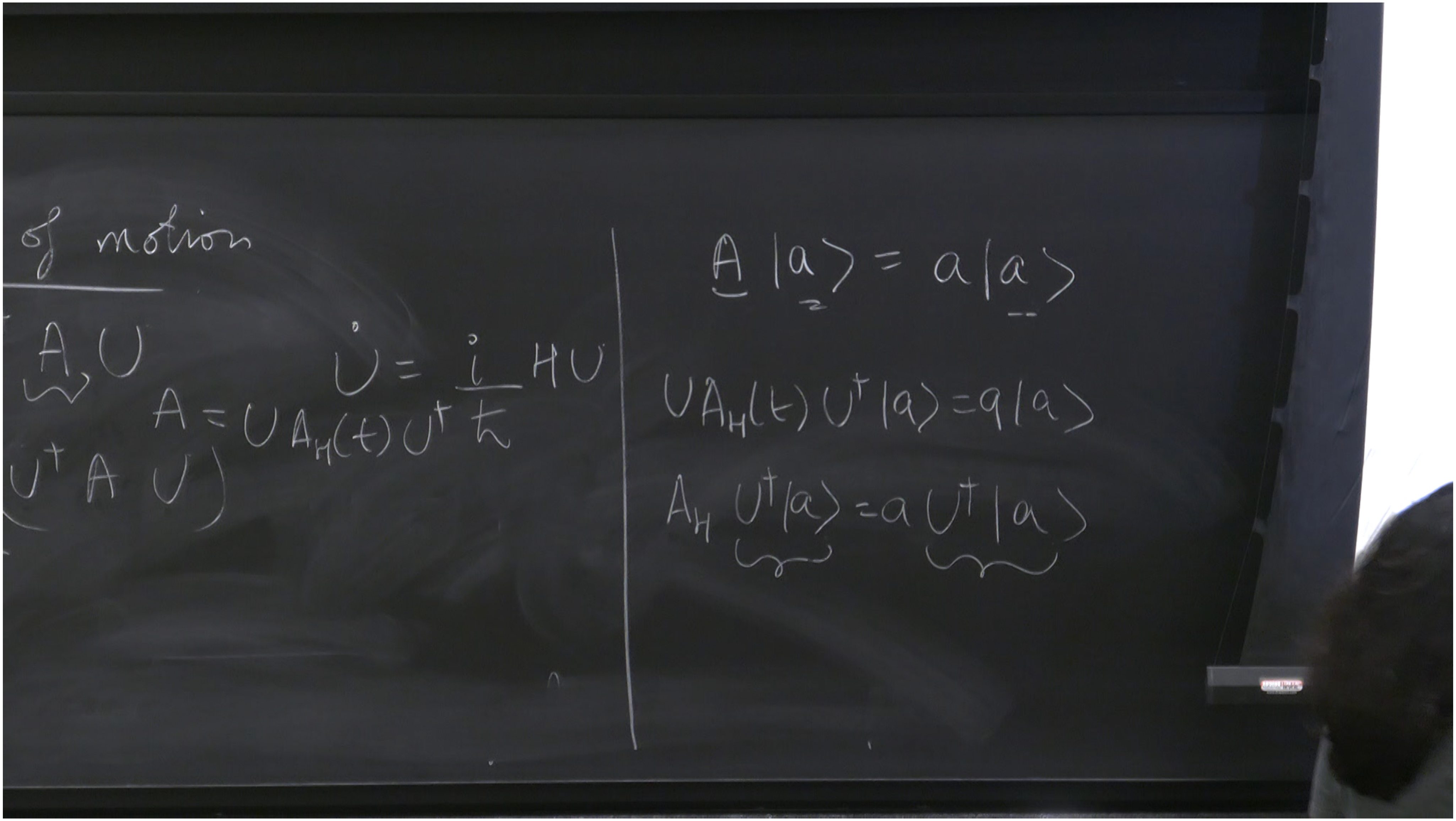
Image Source : The Conversation, Explainer: Heisenberg's Uncertainty Principle, theconversation.com/explainer-heisenbergs-uncertainty-principle-7512


Heisenberg eq of motion
$A_H(t) = U^\dagger A U$
$\dot{U} = \frac{i}{\hbar} H U$
$\frac{dA_H}{dt} = \frac{d}{dt}(U^\dagger A U)$
=


of motion
A U
U̇ = i/ħ H U
A = U A_H(t) U†
U† A U
A |a> = a |a>
U A_H(t) U† |a> = a |a>
A_H U† |a> = a U† |a>

Spin Precession On Bloch Sphere

A representation of given quantum state of qubit

$$H = \omega S_z$$

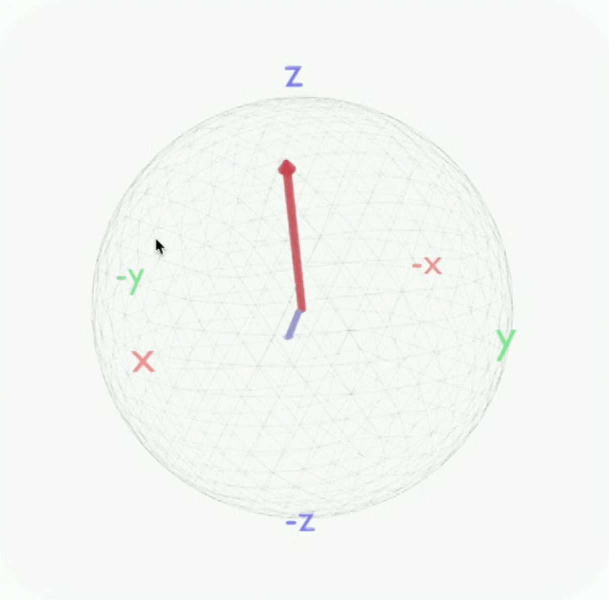
Schrödinger's Approach

t=0 **t=π/ω**

$$|+\rangle_Z \rightarrow |+\rangle_Z$$

$$|+\rangle_X \rightarrow |+\rangle_X$$

$$|+\rangle_Y \rightarrow |-\rangle_Y$$



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THANK YOU..!

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n_0

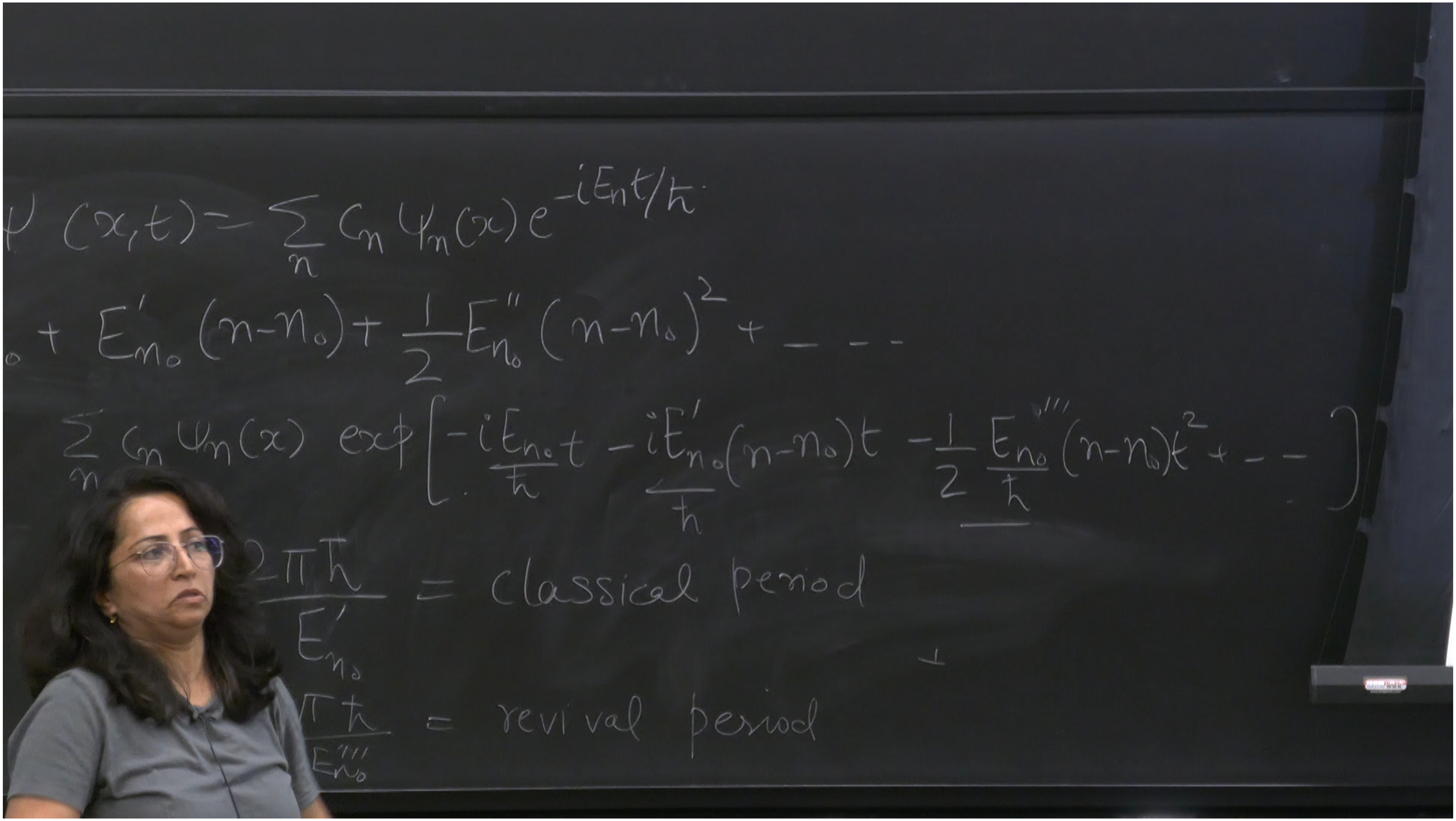
$$\psi(x,t) = \sum_n c_n \psi_n(x) e^{-iE_n t/\hbar}$$

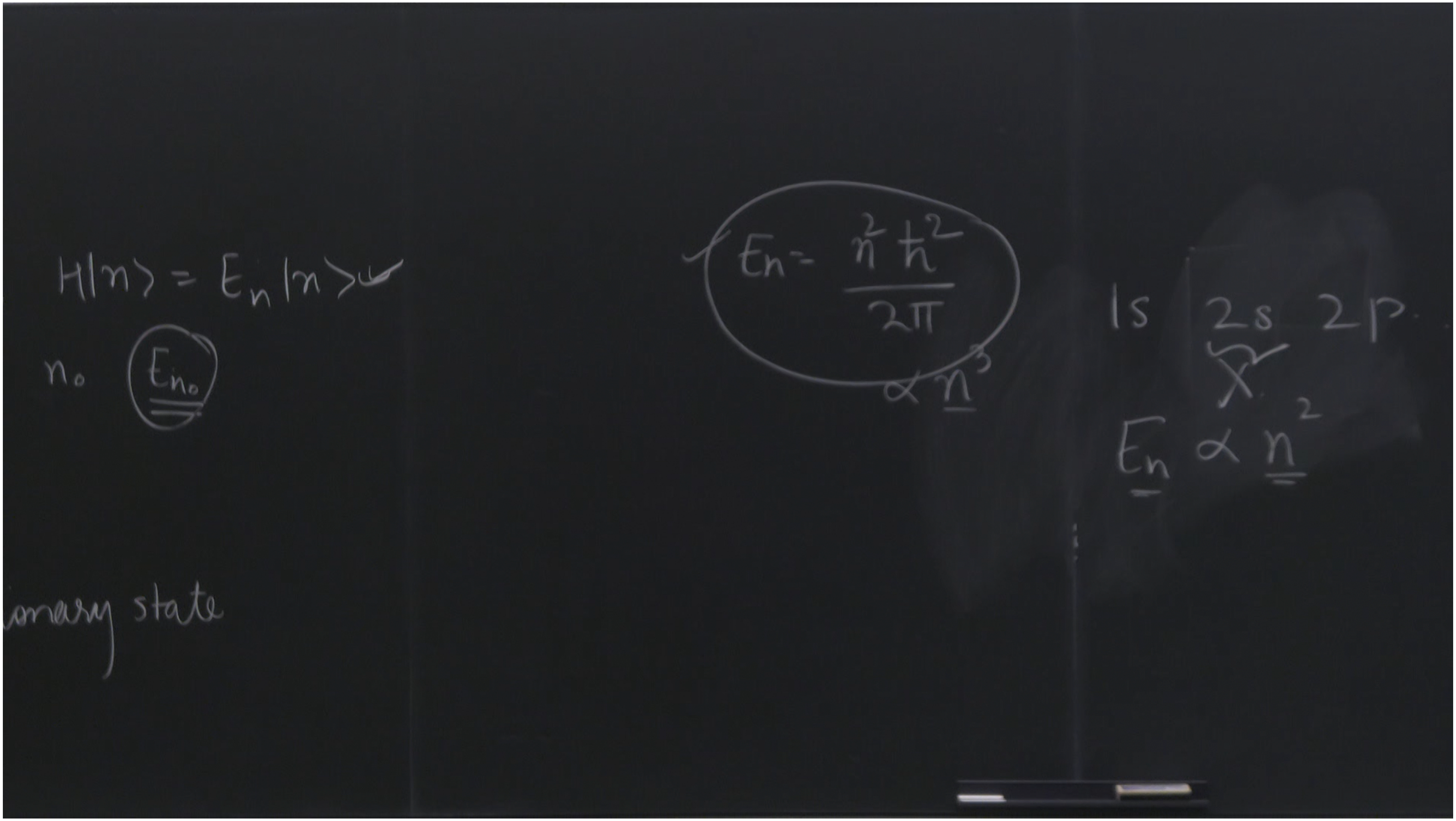
$$E_n = E_{n_0} + E'_{n_0}(n-n_0) + \frac{1}{2}E''_{n_0}(n-n_0)^2 + \dots$$

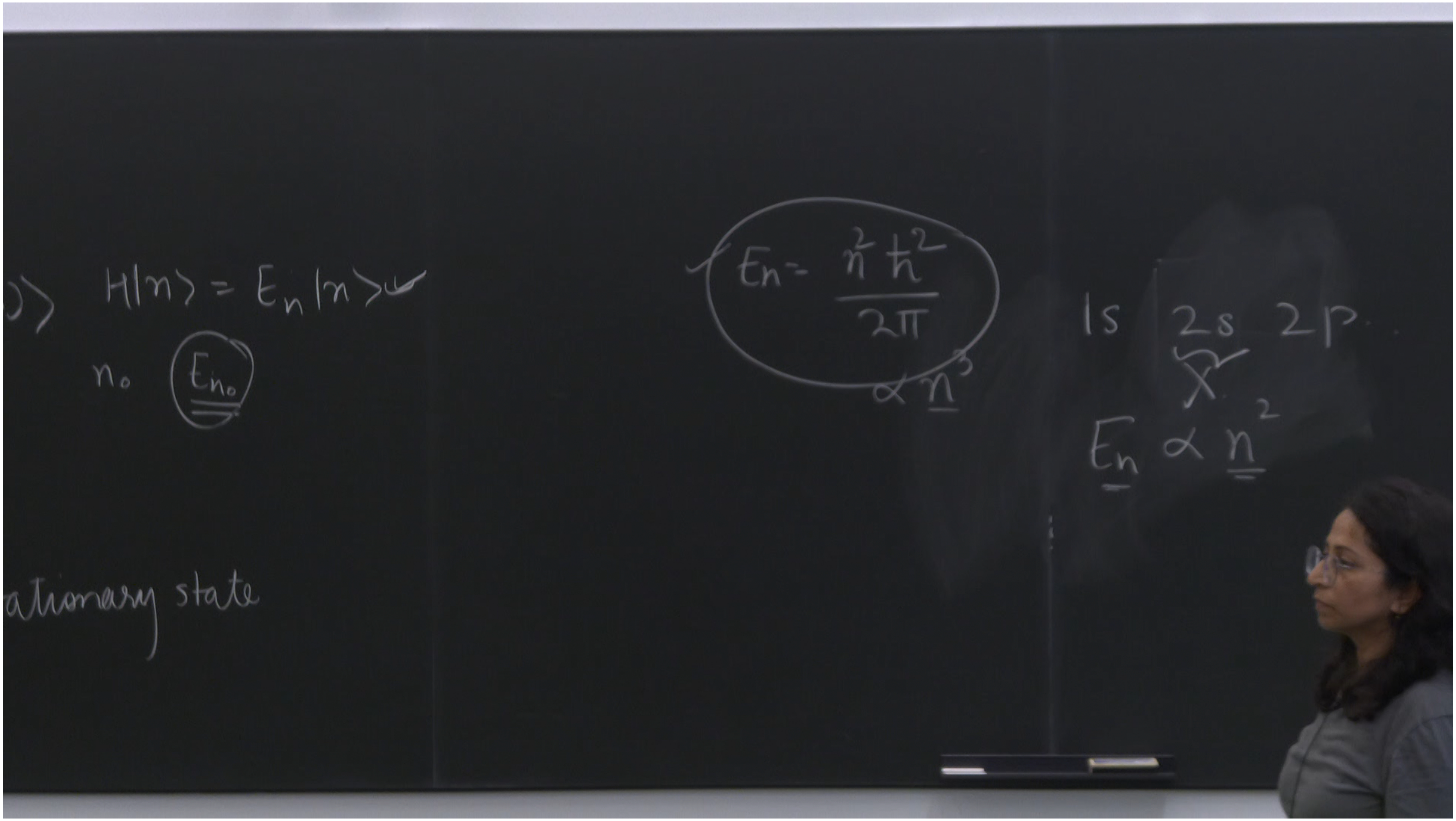
$$\psi(x,t) = \sum_n c_n \psi_n(x) \exp\left[-i\frac{E_{n_0}}{\hbar}t - i\frac{E'_{n_0}}{\hbar}(n-n_0)t - \frac{1}{2}\frac{E''_{n_0}}{\hbar}(n-n\right.$$

$$T_{cl} = \frac{2\pi\hbar}{E'_{n_0}} = \text{classical period}$$

$$T_{rev} = \frac{2\pi\hbar}{E''_{n_0}} = \text{revival period}$$


= classical period
= revival period


H|n> = E_n|n>
n_0 E_{n_0}
onary state
E_n = n^2 ħ^2 / 2π
∝ n^3
1s 2s 2p
E_n ∝ n^2


H|n> = E_n|n>
n₀ E_n₀
ationary state
E_n = n² ħ² / 2π
∝ n³
1s 2s 2p
E_n ∝ n²