Title: Quantum Theory Lecture - 092523

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LSZ (Lehmann, Symanzik, Zimmerman)

Reduction Formula

$$\langle f|S|i\rangle_H = \langle f; t=+\infty \,|\, i; t=-\infty\rangle_S$$

time evolution

$$|i\rangle = |\vec{k}_1, \vec{k}_2\rangle$$

$$|f\rangle = |\vec{k}_3, \vec{k}_4, \dots \vec{k}_N\rangle$$

$$\Omega| T\, \varphi(x_1) \cdots \varphi(x_N) |\Omega\rangle \xleftrightarrow{\text{Dyson's}} \langle 0| T\, \varphi_I(x_1) \cdots \varphi_I(x_m) |0\rangle$$

time-ordering

free vacuum

interaction picture

cting

um

Wick's Theorem

$$\langle 0| T\, \varphi_I(x_1)\, \varphi_I(x_2) |0\rangle$$

Feynman diagrams

Assumptions

- $[a_{\vec{k}}(t), a^{\dagger}_{\vec{p}}(t)] = (2\pi)^3 \, 2E_{\vec{p}} \, \delta^{(3)}(\vec{p}-\vec{k})$ Definition

- $|\Omega\rangle$ is Poincare invariant

- $\varphi(x)|\Omega\rangle$ is a one particle state for $t = \pm\infty$

  $\rightarrow \varphi$ is linear in $a^{\dagger}$

-

$$|i\rangle_H = a_1^\dagger(0)\, a_2^\dagger(0)|\Omega\rangle \qquad a_1^\dagger \equiv a_{\vec{k}_1}^\dagger$$

$$|f\rangle_H = a_3^\dagger(0)\, a_4^\dagger(0)|\Omega\rangle$$

$$\langle f|S|i\rangle = \langle\Omega|\, a_4(+\infty)\, a_3(+\infty)\, a_1^\dagger(-\infty)\, a_2^\dagger(-\infty)|\Omega\rangle$$

$$= \langle\Omega|\, T\, a_4(+\infty)\, a_3(+\infty)\, a_1^\dagger(-\infty)\, a_2^\dagger(-\infty)|\Omega\rangle$$

already time-ordered

$$T\,\mathcal{O}(t_1)\mathcal{O}(t_2) = \begin{cases} \mathcal{O}(t_1)\mathcal{O}(t_2) & \text{if } t_1 > t_2 \\ \mathcal{O}(t_2)\mathcal{O}(t_1) & \text{if } t_2 > t_1 \end{cases}$$

Latest to Left

Useful identity:

$$a_1^\dagger(+\infty) - a_1^\dagger(-\infty) = -i\int d^4x\, e^{-ik_1\cdot x}\underbrace{(\partial^2+m^2)\varphi(x)}_{=0 \text{ in free theory}} \equiv I_1^\dagger$$

<u>Proof:</u> $a_1^\dagger(+\infty) - a_1^\dagger(-\infty) = \int_{-\infty}^{\infty} dt\, \partial_0 a_1^\dagger(t)$

$$= -i\int d^4x\, \partial_0\left(e^{-ik_1\cdot x}\left[\partial_0\varphi(x) + iE_{\vec{k}_1}\varphi(x)\right]\right)$$

$$= -i\int d^4x\, e^{-ik_1\cdot x}\left(\partial_0^2 + E_{\vec{k}_1}^2\right)\varphi(x) \quad \text{others cancel}$$

$$\varphi(x) = \int \frac{d^3k}{(2\pi)^3 2E_{\vec{k}}} \left(a_{\vec{k}}(t)\, e^{-ik\cdot x} + a^{\dagger}_{\vec{k}}(t)\, e^{ik\cdot x} \right)$$

if φ is real

if boundary terms vanish $\partial_t a^{\dagger}_{\vec{k}}(\pm\infty) = 0$

$$a^{\dagger}_{p}(t) = -i \int d^3x \left(e^{-ik\cdot x}\, \partial_t \varphi(x) + i E_{\vec{k}}\, \varphi(x) \right)$$

for all t

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for all t

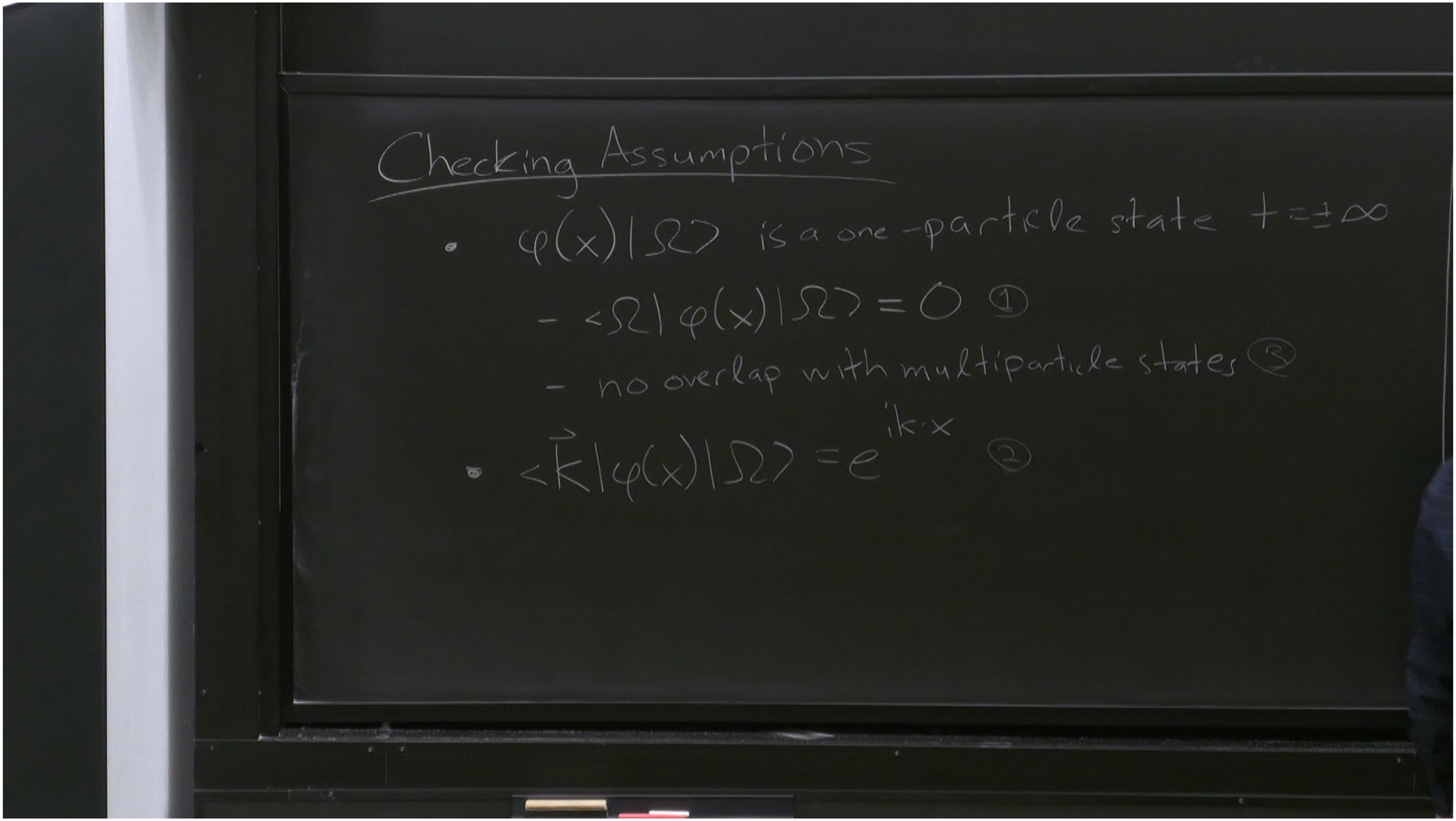
Left

useful identity

$$\langle f|S|i\rangle = \langle \Omega | T\, a_4(-\infty)\, a_3(-\infty)\, a_1^\dagger(+\infty)\, a_2^\dagger(+\infty) | \Omega \rangle + \dots + \langle \Omega | T\, I_4 I_3 I_1^\dagger I_2^\dagger | \Omega \rangle$$

$$\boxed{\langle f|S|i\rangle = \int \prod_{i=1}^{4} \left(d^4x_i \, e^{-i\lambda_i k_i \cdot x_i} \left(\partial_i^2 + m^2\right)\right) \langle \Omega | T\, \varphi_1 \varphi_2 \varphi_3 \varphi_4 | \Omega \rangle}$$

$$\lambda_i = \begin{cases} +1 & \text{initial} \\ -1 & \text{final} \end{cases} \qquad \partial_i^2 = \eta^{\mu\nu} \frac{\partial}{\partial x_i^\mu} \frac{\partial}{\partial x_i^\nu} \qquad \varphi_i = \varphi(x_i)$$


Checking Assumptions
• $\varphi(x)|\Omega\rangle$ is a one-particle state $t=\pm\infty$
- $\langle\Omega|\varphi(x)|\Omega\rangle = 0$ ①
- no overlap with multiparticle states ③
• $\langle\vec{k}|\varphi(x)|\Omega\rangle = e^{ik\cdot x}$ ②

① $\langle\Omega|\varphi(x)|\Omega\rangle$

$$\varphi(x) = e^{i\hat{P}\cdot x}\,\varphi(0)\,e^{-i\hat{P}\cdot x}$$

4-momentum generate spacetime translation

$$\langle\Omega|\varphi(x)|\Omega\rangle = \langle\Omega|e^{i\hat{P}\cdot x}\,\varphi(0)\,e^{-i\hat{P}\cdot x}|\Omega\rangle$$

$= \langle\Omega|\varphi(0)|\Omega\rangle$ constant, independent of x

$= v$

Fix by shifting field $\tilde{\varphi}(x) = \varphi(x) - v$

$$\langle\Omega|\hat{\varphi}(x)|\Omega\rangle = 0$$

Checking Assumptions

- $\varphi(x)|\Omega\rangle$ is a one-particle state $t = \pm\infty$
  - $\langle\Omega|\varphi(x)|\Omega\rangle = 0$ ① ✓ with shift
  - no overlap with multiparticle states ③
- $\langle\vec{k}|\varphi(x)|\Omega\rangle = e^{ik\cdot x}$ ②

$$\underline{\langle \vec{k} | \varphi(x) | \Omega \rangle}$$

$$\langle \vec{k} | \varphi(x) | \Omega \rangle = \langle \vec{k} | e^{i\hat{P}\cdot x} \varphi(0) e^{-i\hat{P}\cdot x} | \Omega \rangle$$

$$= e^{ik\cdot x} \underbrace{\langle \vec{k} | \varphi(0) | \Omega \rangle} \quad \leftarrow \text{constant, independent of spacetime}$$

$$= Z e^{ik\cdot x} \quad \text{claim: Lorentz-invariant}$$

Lorentz-invariant function of $k^2 = m^2$

Resale $\tilde{\varphi}(x) = Z\varphi(x)$ $\qquad \langle \vec{k} | \tilde{\varphi}(x) | \Omega \rangle = e^{ik\cdot x}$

$$\lim_{t\to\pm\infty} \langle \Psi | A_1^\dagger(t) | \Omega \rangle = \int d^3p \, (\cdots\cdots) \, e^{i(p^0 - k_1^0)t}$$

E of multi: $p^0 = (\vec{p}^{\,2} + M^2)^{1/2}$ invariant mass of multiparticle state

$k_1^0 = (\vec{p}^{\,2} + m^2)^{1/2}$ mass of particle

free $M \geq 2m > m$

physically 1 + multiparticle spread out differently as $t \to \pm\infty$

Overlap $\to 0$

Drop ~ moving forward

coupling constant

Example: $\mathcal{L}_{\text{naïve}} = \frac{1}{2}(\partial\varphi)^2 - \frac{1}{2}m^2\varphi^2 - \frac{g}{3!}\varphi^3$

Must shift + rescale (+ rename)

new term from shift

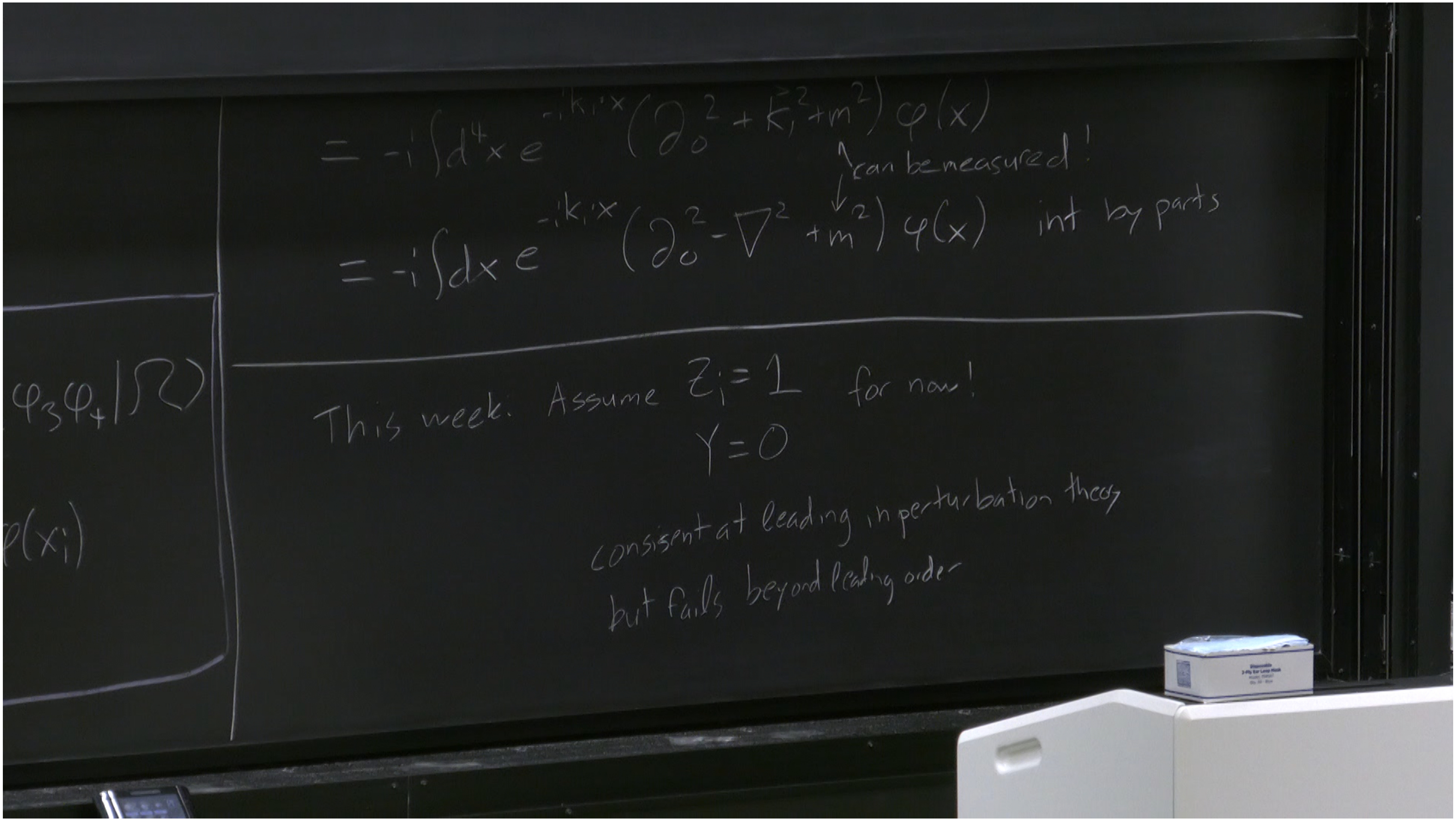
$$\mathcal{L} = \frac{1}{2}Z_\varphi(\partial\varphi)^2 - \frac{1}{2}Z_m m^2\varphi^2 - Z_g\frac{g}{3!}\varphi^3 + Y\varphi$$

Renormalization is required!

more next week!

$\langle\Omega|\varphi(x)|\Omega\rangle = 0$

$\langle k|\varphi(x)|\Omega\rangle = e^{ikx}$ put constraints on Z_i, Y


$= -i\int d^4x\, e^{-ik_i\cdot x}(\partial_0^2 + \vec{k}_i^2 + m^2)\varphi(x)$
can be measured!
$= -i\int dx\, e^{-ik_i\cdot x}(\partial_0^2 - \nabla^2 + m^2)\varphi(x)$ int by parts
$\varphi_3\varphi_4|\Omega\rangle$
$\varphi(x_i)$
This week: Assume $Z_i = 1$ for now!
$Y = 0$
consisent at leading in perturbation theory
but fails beyond leading order