Title: Quantum Theory Lecture - 092123

Speakers: Bindiya Arora, Dan Wohns

Collection: Quantum Theory 2023/24

Date: September 21, 2023 - 9:00 AM

URL: https://pirsa.org/23090046

Quantization of Klein-Gordon Theory

$$\int \frac{d^3p}{(2\pi)^3 2E_{\vec{p}}} = \frac{1}{2E_{\vec{p}}}\int \frac{d^4p}{(2\pi)^4}(2\pi)\,\delta(p^0 - E_{\vec{p}})$$

$$= \frac{1}{2E_{\vec{p}}}\int \frac{d^4p}{(2\pi)^4}\, 2\pi\, \delta(\underbrace{p^{0\,2} - \vec{p}^{\,2} - m^2}_{p^2 - m^2})\, 2E_{\vec{p}}\, \Theta(p^0)$$

$$\varphi(x) = \int \frac{d^3p}{(2\pi)^3 2E_{\vec{p}}}\left(a(\vec{p})\, e^{-ip\cdot x} + a^*(\vec{p})\, e^{+ip\cdot x}\right)\Big|_{p^0 = E_{\vec{p}}}$$

Quantization

1) Choose $\mathcal{L}$

$$\mathcal{L}_{KG} = \frac{1}{2}(\partial\varphi)^2 - \frac{1}{2}m^2\varphi^2$$

2) Compute π, H

$$\pi = \frac{\partial\mathcal{L}}{\partial\dot{\varphi}} = \dot{\varphi}$$

$$H = \int d^3x \left(\frac{1}{2}\dot{\varphi}^2 + \frac{1}{2}(\vec{\nabla}\varphi)^2 + \frac{1}{2}m^2\varphi^2\right)$$

3) Impose commutation relations

$$[q_a, p_b] = i\delta_{ab}$$

label

$$[q_a, q_b] = 0 = [p_a, p_b]$$

$$\boxed{\begin{gathered}[\hat{\varphi}(\vec{x}), \hat{\pi}(\vec{y})] = i\delta^{(3)}(\vec{x}-\vec{y}) \\ [\hat{\varphi}(\vec{x}), \hat{\varphi}(\vec{y})] = 0 = [\hat{\pi}(\vec{x}), \hat{\pi}(\vec{y})]\end{gathered}}$$

3) Impose commutation relations

$$[q_a, p_b] = i\delta_{ab}$$

label

$$[q_a, q_b] = 0 = [p_a, p_b]$$

$$\boxed{\begin{gathered}[\hat{\varphi}(\vec{x}), \hat{\pi}(\vec{y})] = i\delta^{(3)}(\vec{x}-\vec{y}) \\ [\hat{\varphi}(\vec{x}), \hat{\varphi}(\vec{y})] = 0 = [\hat{\pi}(\vec{x}), \hat{\pi}(\vec{y})]\end{gathered}} \xRightarrow{\text{+ classical solution}}$$

Schrödinger

ladder operators

$a(\vec{k}) \to a_{\vec{k}}$ annihilation

$a^*(\vec{k}) \to a^\dagger_{\vec{k}}$ creation

$$[a_{\vec{k}}, a^\dagger_{\vec{p}}] = (2\pi)^3 2E_{\vec{p}}\, \delta^{(3)}(\vec{p}-\vec{k})$$

$$[a_{\vec{k}}, a_{\vec{p}}] = 0 = [a^\dagger_{\vec{k}}, a^\dagger_{\vec{p}}]$$

ladder operators

$$\left\{\begin{aligned}
&a(\vec{k}) \to a_{\vec{k}} \quad \text{annihilation} \\
&a^*(\vec{k}) \to a^\dagger_{\vec{k}} \quad \text{creation} \\
&[a_{\vec{k}}, a^\dagger_{\vec{p}}] = (2\pi)^3 \, 2E_{\vec{p}} \, \delta^{(3)}(\vec{p}-\vec{k}) \\
&[a_{\vec{k}}, a_{\vec{p}}] = 0 = [a^\dagger_{\vec{k}}, a^\dagger_{\vec{p}}] \\
&H = \frac{1}{2}\int \frac{d^3k}{(2\pi)^3 2E_{\vec{k}}} \, E_{\vec{k}} \left(a^\dagger_{\vec{k}} a_{\vec{k}} + a_{\vec{k}} a^\dagger_{\vec{k}}\right)
\end{aligned}\right.$$

4) Impose ordering

$$\mathcal{L}'_{KG} = \mathcal{L}_{KG} + \underbrace{(\varphi\dot{\varphi} - \dot{\varphi}\varphi)}_{[\varphi, \pi]}$$

States

vacuum $|0\rangle$

$a_{\vec{k}}|0\rangle = 0$ for all $\vec{k}$

$\langle 0|0\rangle = 1$

$$H|0\rangle = \int \frac{d^3k}{(2\pi)^3 2E_{\vec{k}}} \frac{1}{2} E_{\vec{k}} \left(a_{\vec{k}} a^\dagger_{\vec{k}} + \underbrace{\cancel{a^\dagger_{\vec{k}} a_{\vec{k}}}}_{0}\right)|0\rangle$$

$$= \int d^3k \, \frac{1}{2} E_{\vec{k}} \, \delta(\vec{k}-\vec{k})|0\rangle$$

$$\to \infty |0\rangle \,!$$

States

vacuum $|0\rangle$

$$a_{\vec{k}}|0\rangle = 0 \quad \text{for all } \vec{k}$$

$$\langle 0|0\rangle = 1$$

$$H|0\rangle = \int \frac{d^3k}{(2\pi)^3 2E_{\vec{k}}} \frac{1}{2} E_{\vec{k}} \left(a_{\vec{k}} a^\dagger_{\vec{k}} + \cancelto{0}{a^\dagger_{\vec{k}} a_{\vec{k}}} \right) |0\rangle$$

$$= \int d^3k \, \frac{1}{2} E_{\vec{k}} \, \delta(\vec{k} - \vec{k}) \, |0\rangle$$

$$\rightarrow \infty |0\rangle \; !$$

$E_0 \to \infty$! not observable

UV

2 divergences

IR: $\delta(\vec{k}-\vec{k}) = \int d^3x\, e^{i\vec{p}\cdot\vec{x}}\Big|_{\vec{p}=0}$

$= \lim_{L\to\infty} \int_{-L}^{L} d^3x$

$=$ vol of space

$E_0 = \rho_0\, \mathrm{vol}$

$\to \infty$! not observable

divergences

$\delta(\vec{k}-\vec{k}) = \int d^3x\, e^{i\vec{p}\cdot\vec{x}}\Big|_{\vec{p}=0}$

$= \lim_{L\to\infty} \int_{-L}^{L} d^3x$

$=$ vol of space

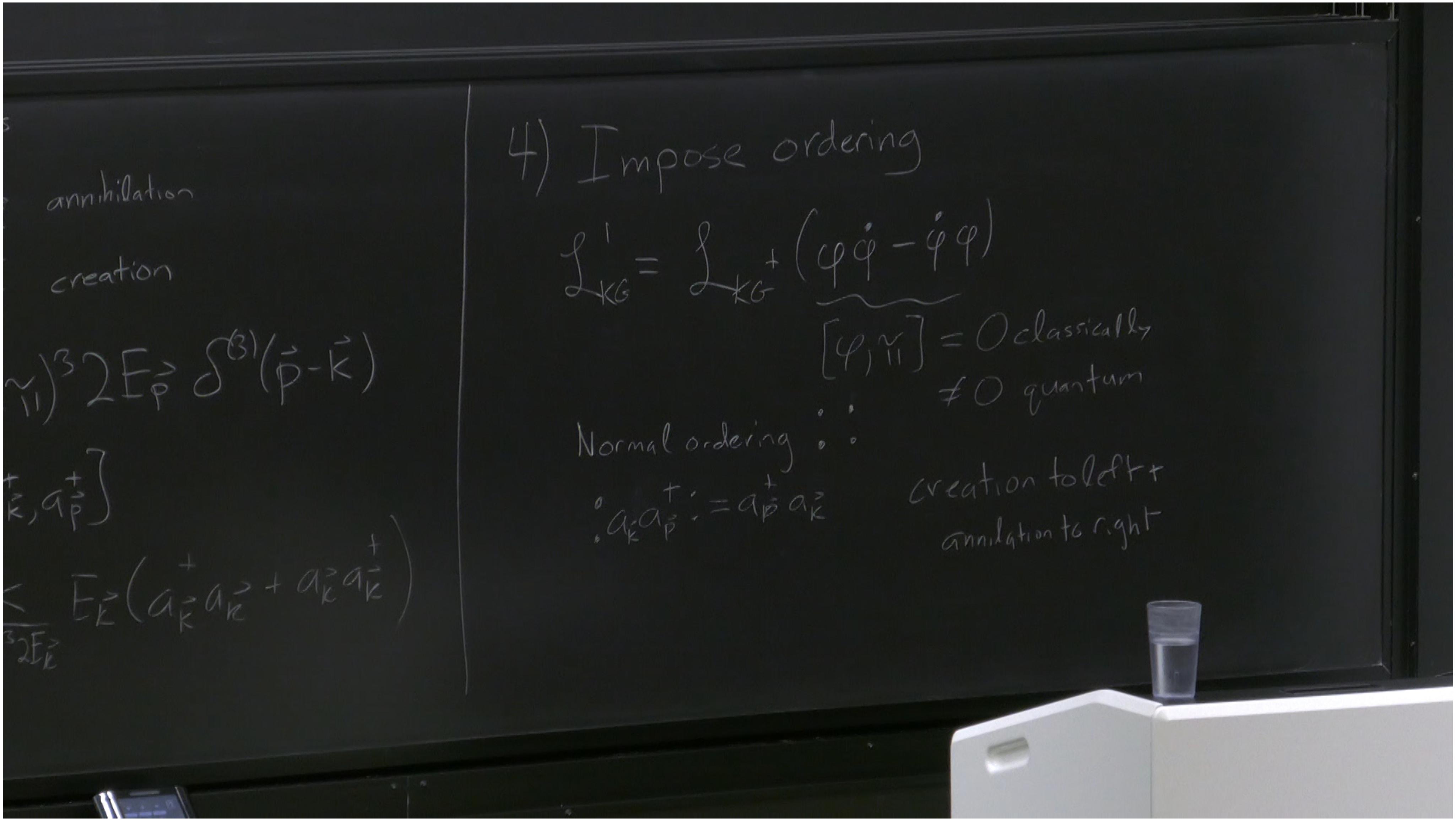
$E_0 = \rho_0 \text{vol}$

UV: $\rho_0 = \int d^3k\, \frac{1}{2} E_{\vec{k}}$

$= \lim_{\Lambda\to\infty} \int_0^{\Lambda} |k|^2 dk\, d\Omega\, \frac{1}{2}\sqrt{|k|^2+m^2}$

$\sim \lim_{\Lambda\to\infty} \Lambda^4$

$\rho_0 \to \infty$


annihilation
creation
4) Impose ordering
$\mathcal{L}'_{KG} = \mathcal{L}_{KG} + (\varphi\dot{\varphi} - \dot{\varphi}\varphi)$
$[\varphi, \pi] = 0$ classically
$\neq 0$ quantum
Normal ordering : :
$:a_{\vec{k}} a^{\dagger}_{\vec{p}}: = a^{\dagger}_{\vec{p}} a_{\vec{k}}$
creation to left + annilation to right

$E_0 \to \infty$! not observable

2 divergences

IR: $\delta(\vec{k}-\vec{k}) = \int d^3x \, e^{i\vec{p}\cdot\vec{x}} \Big|_{\vec{p}=0}$

$= \lim_{L\to\infty} \int_{-L}^{L} d^3x$

$=$ vol of space

$E_0 = \rho_0 \, \text{vol}$

UV: $\rho_0 = \int d^3k \, \frac{1}{2} E_{\vec{k}}$

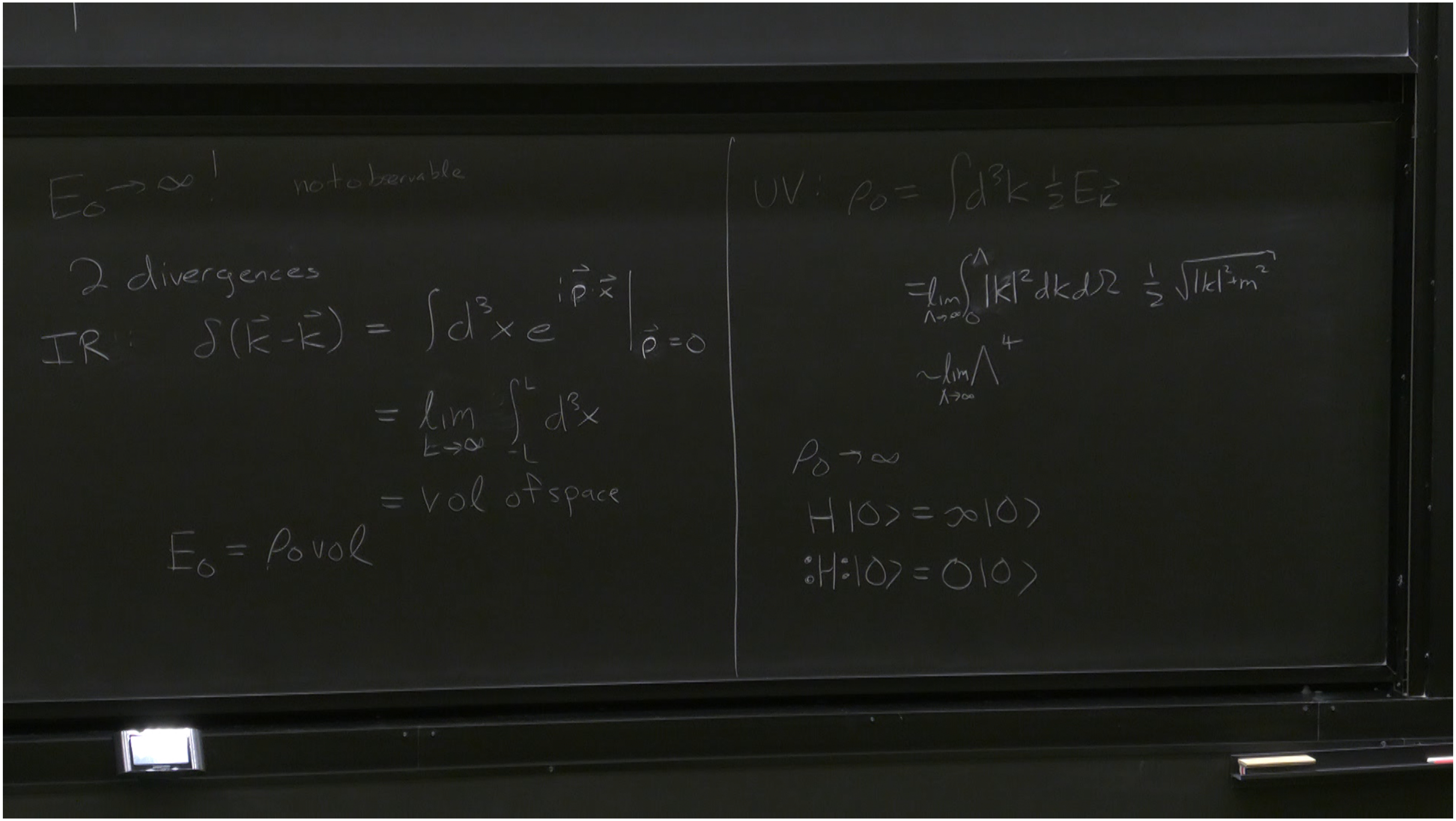
$= \lim_{\Lambda\to\infty} \int_0^{\Lambda} |k|^2 dk \, d\Omega \, \frac{1}{2} \sqrt{|k|^2+m^2}$

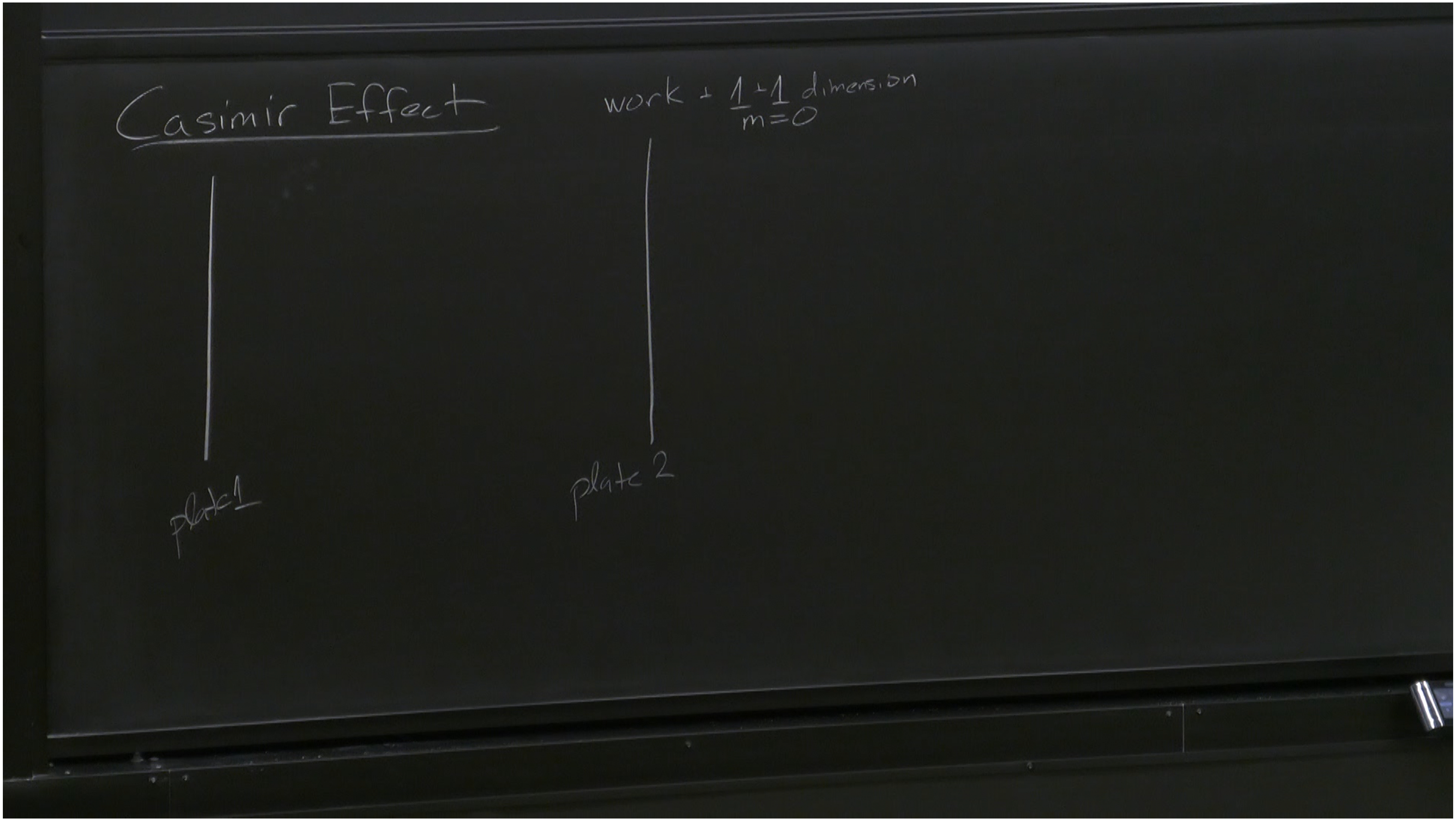
$\sim \lim_{\Lambda\to\infty} \Lambda^4$

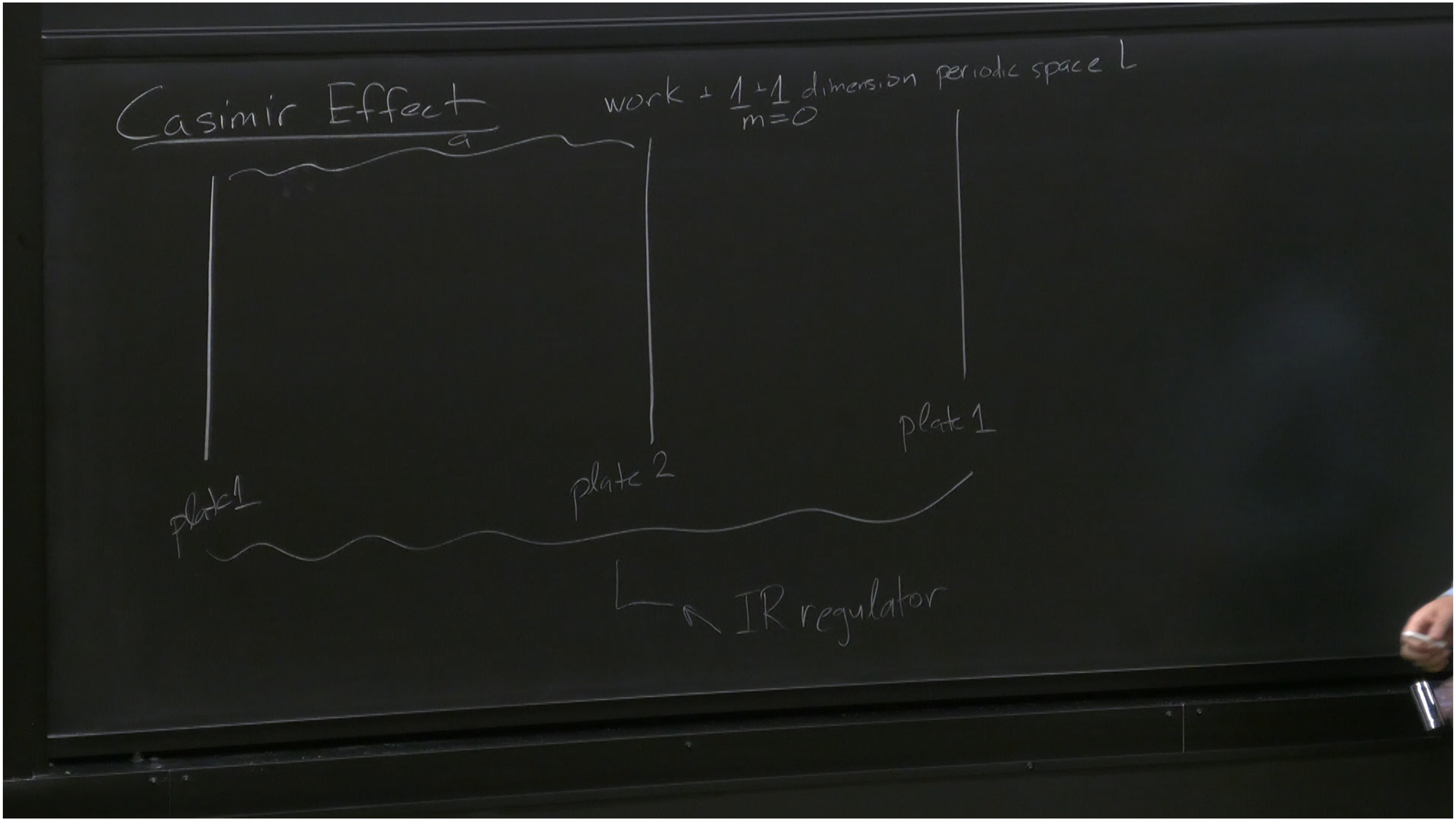
$\rho_0 \to \infty$

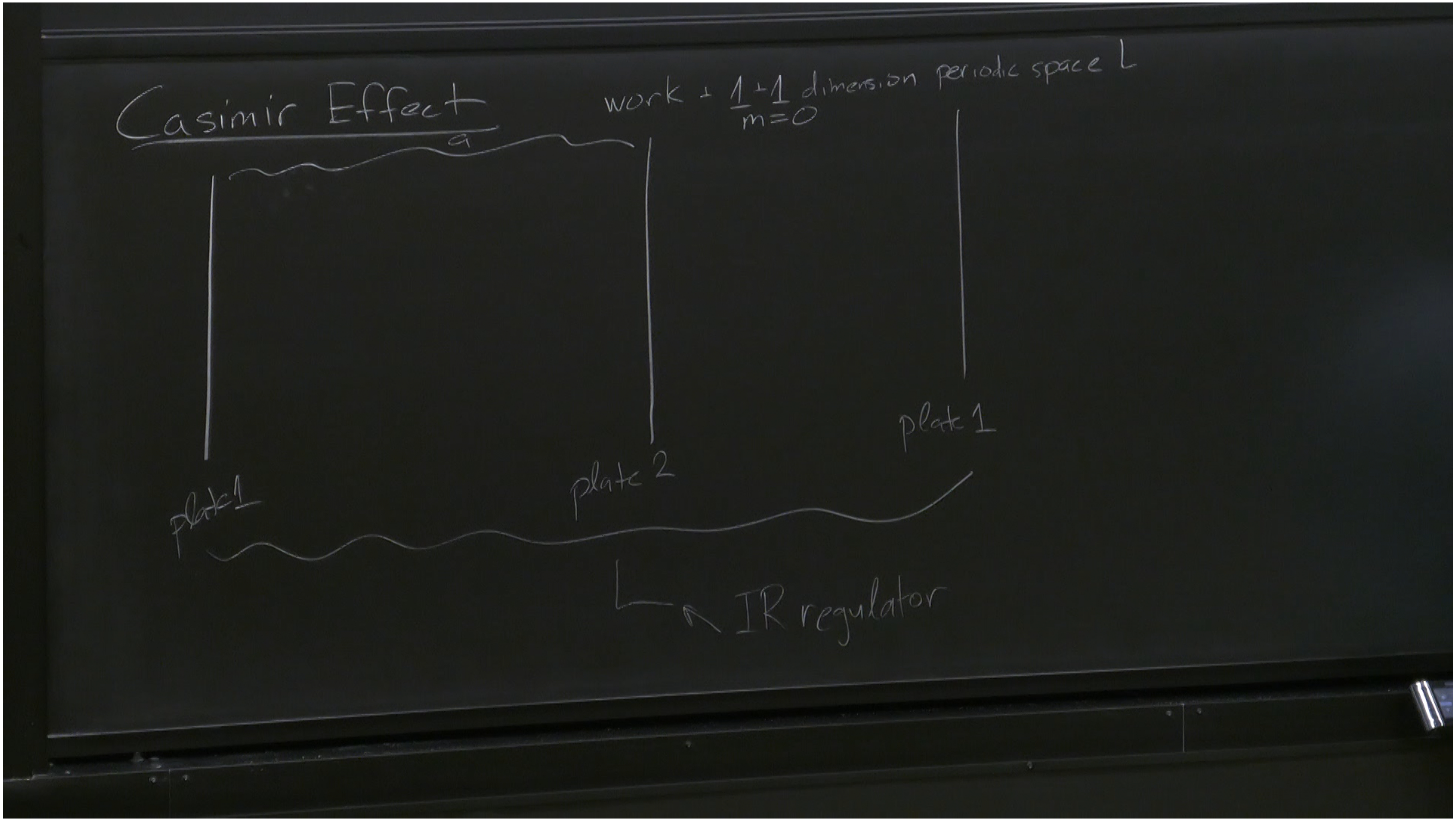
$H|0\rangle = \infty|0\rangle$

$:H:$


$E_0 \to \infty!$ not observable
2 divergences
IR: $\delta(\vec{k}-\vec{k}) = \int d^3x\, e^{i\vec{p}\cdot\vec{x}}\Big|_{\vec{p}=0}$
$= \lim_{L\to\infty} \int_{-L}^{L} d^3x$
$=$ vol of space
$E_0 = \rho_0 \,\mathrm{vol}$
UV: $\rho_0 = \int d^3k\, \frac{1}{2} E_{\vec{k}}$
$= \lim_{\Lambda\to\infty} \int_0^{\Lambda} |k|^2 dk\, d\Omega\, \frac{1}{2}\sqrt{|k|^2+m^2}$
$\sim \lim_{\Lambda\to\infty} \Lambda^4$
$\rho_0 \to \infty$
$H|0\rangle = \infty|0\rangle$
$:H:|0\rangle = 0|0\rangle$

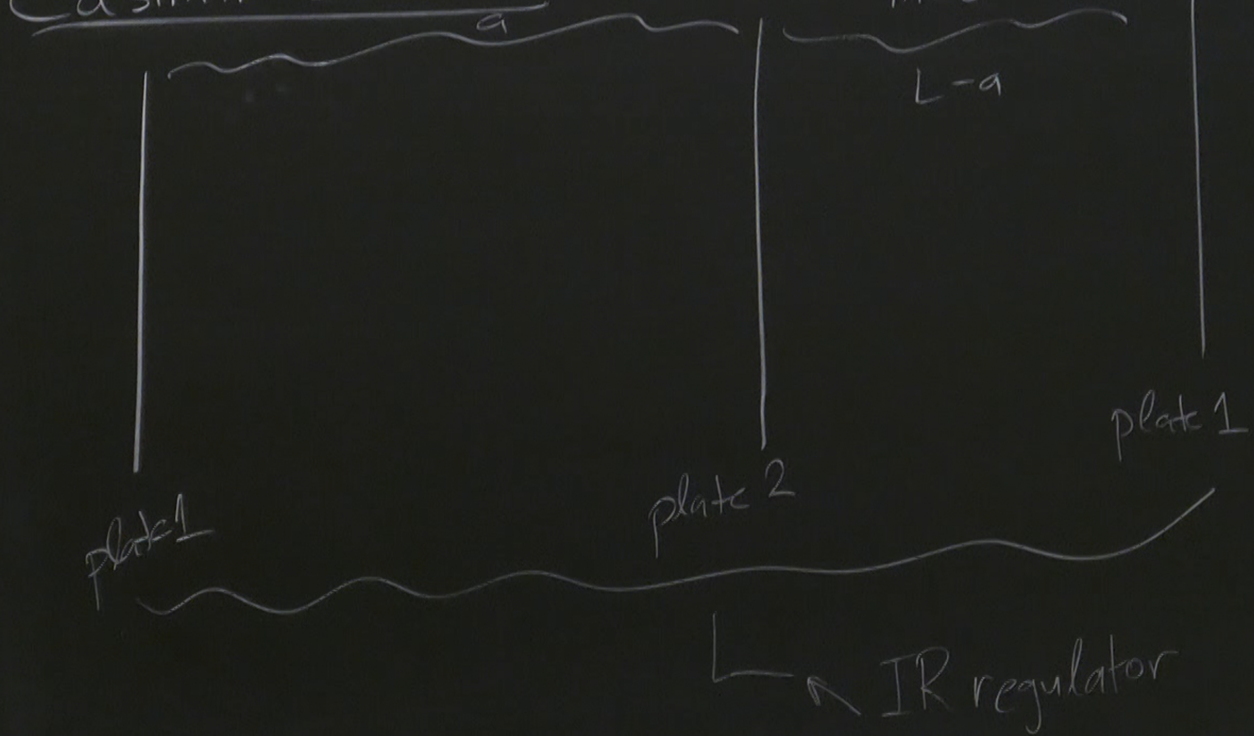

Casimir Effect
work + 1+1 dimension
m=0
plate 1
plate 2


Casimir Effect
work + 1+1 dimension periodic space L
m=0
a
plate1
plate 2
plate 1
L
IR regulator


Casimir Effect
work : 1+1 dimension periodic space L
m=0
a
plate 1
plate 2
plate 1
L
IR regulator

Casimir Effect

work + 1+1 dimension periodic space L
$m=0$

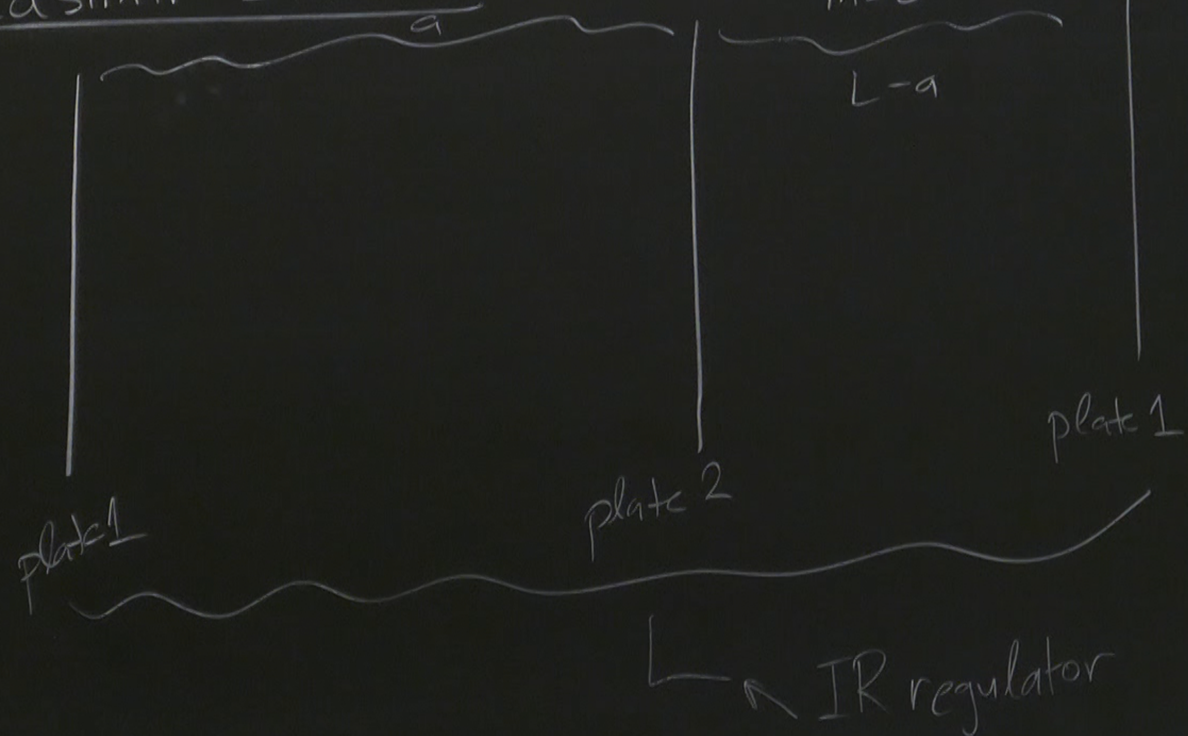


$$E_k = k = \omega$$

$$\omega_n = \frac{n\pi}{}$$

Casimir Effect

work in $\underset{m=0}{1+1}$ dimension periodic space L

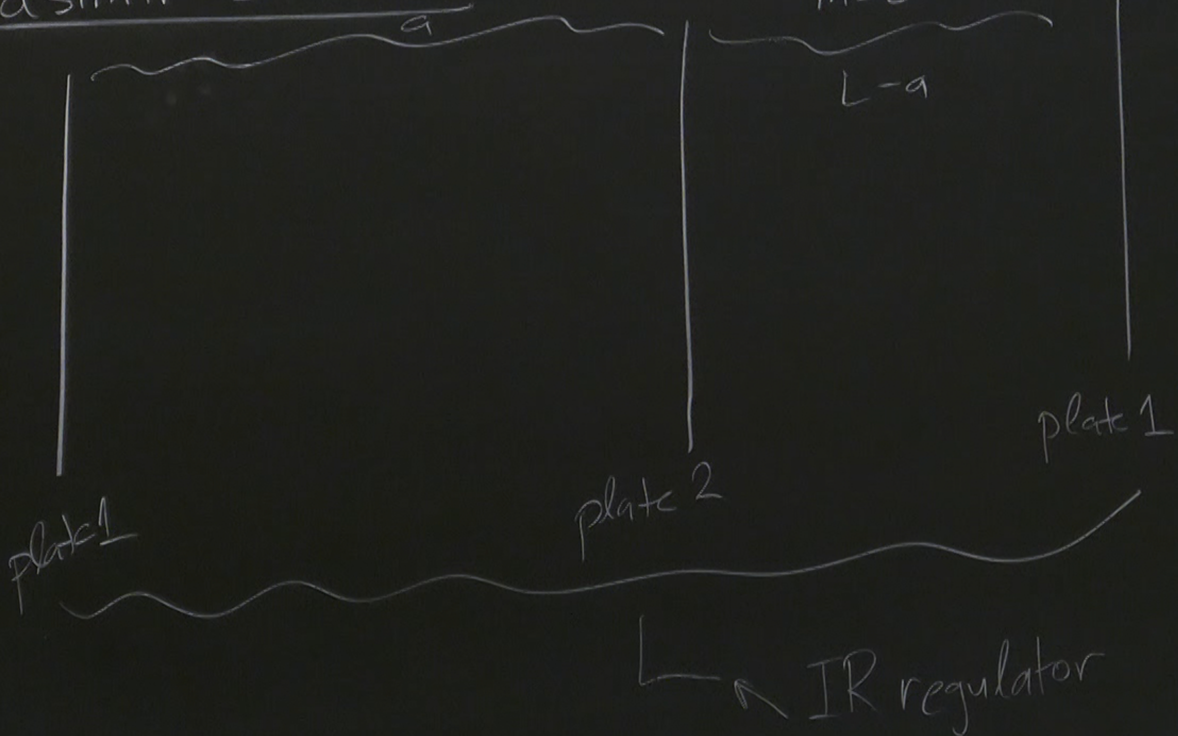


$$E_k = k = \omega$$

$$\omega_n = \frac{n\pi}{r} \quad \text{in a box of size } r$$

Casimir Effect

work in 1+1 dimension periodic space L
$m=0$



$$E_k = k = \omega$$

$$\omega_n = \frac{n\pi}{r} \quad \text{in a box of size } r$$

$$E(r) = \sum$$

Casimir Effect

work : 1+1 dimension periodic space L
$m=0$

a

$L-a$

plate 1

plate 2

plate 1

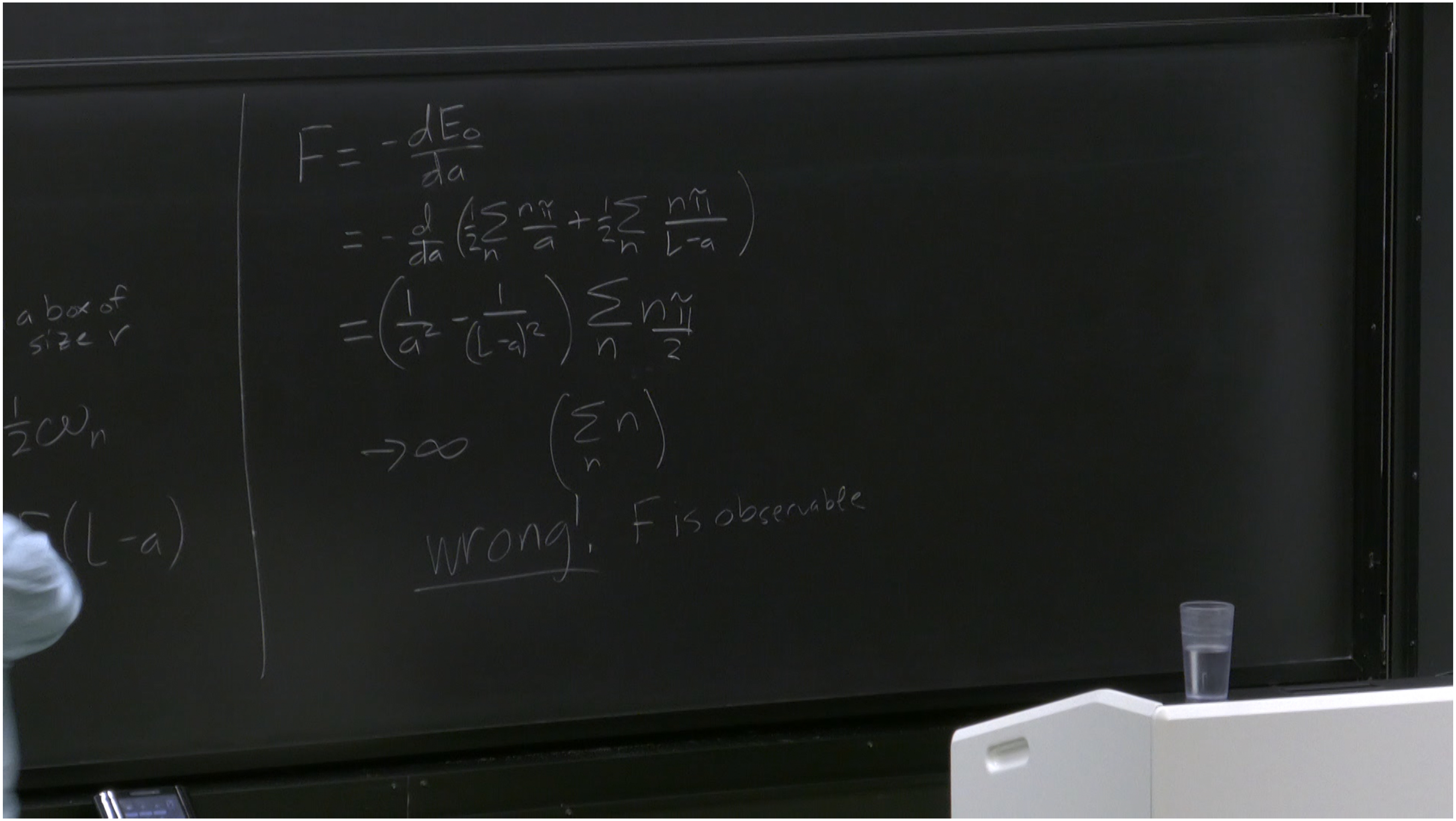
L ← IR regulator

$$E_k = k = \omega$$

$$\omega_n = \frac{n\pi}{r} \quad \text{in a box of size } r$$

$$E(r) = \sum_n \frac{1}{2}\omega_n$$

$$E_0 = E(a) + E(L-a)$$


a box of
size r
$\frac{1}{2}\omega_n$
$(L-a)$
$F = -\frac{dE_0}{da}$
$= -\frac{d}{da}\left(\frac{1}{2}\sum_n \frac{n\pi}{a} + \frac{1}{2}\sum_n \frac{n\pi}{L-a}\right)$
$= \left(\frac{1}{a^2} - \frac{1}{(L-a)^2}\right)\sum_n \frac{n\pi}{2}$
$\to \infty \quad \left(\sum_n n\right)$
wrong! F is observable

$$= -\frac{d}{da}\left(\frac{1}{2}\sum_n \frac{n\pi}{a} + \frac{1}{2}\sum_n \frac{n\pi}{L-a}\right)$$

$$= \left(\frac{1}{a^2} - \frac{1}{(L-a)^2}\right)\sum_n \frac{n\pi}{2}$$

$$\to \infty \qquad \left(\sum_n n\right)$$

wrong! F is observable

Mistake: high frequency modes do not interact with plates!

$\omega \lesssim \Lambda$ interact with plates $\qquad n_{max} = \lfloor \Lambda r \rfloor$ ← floor function

↖ UV regulator

$$E(r) = \frac{1}{2}\sum_{n=0}^{n_{max}} \frac{n\pi}{r} + \text{constant}$$

$$= \frac{1}{2}\,\frac{n_{max}(n_{max}+1)\pi}{2r}$$

Mistake: high frequency modes do not interact with plates!

$\omega < \pi \Lambda$ interact with plates $\quad n_{max} = \lfloor \Lambda r \rfloor \leftarrow$ floor function

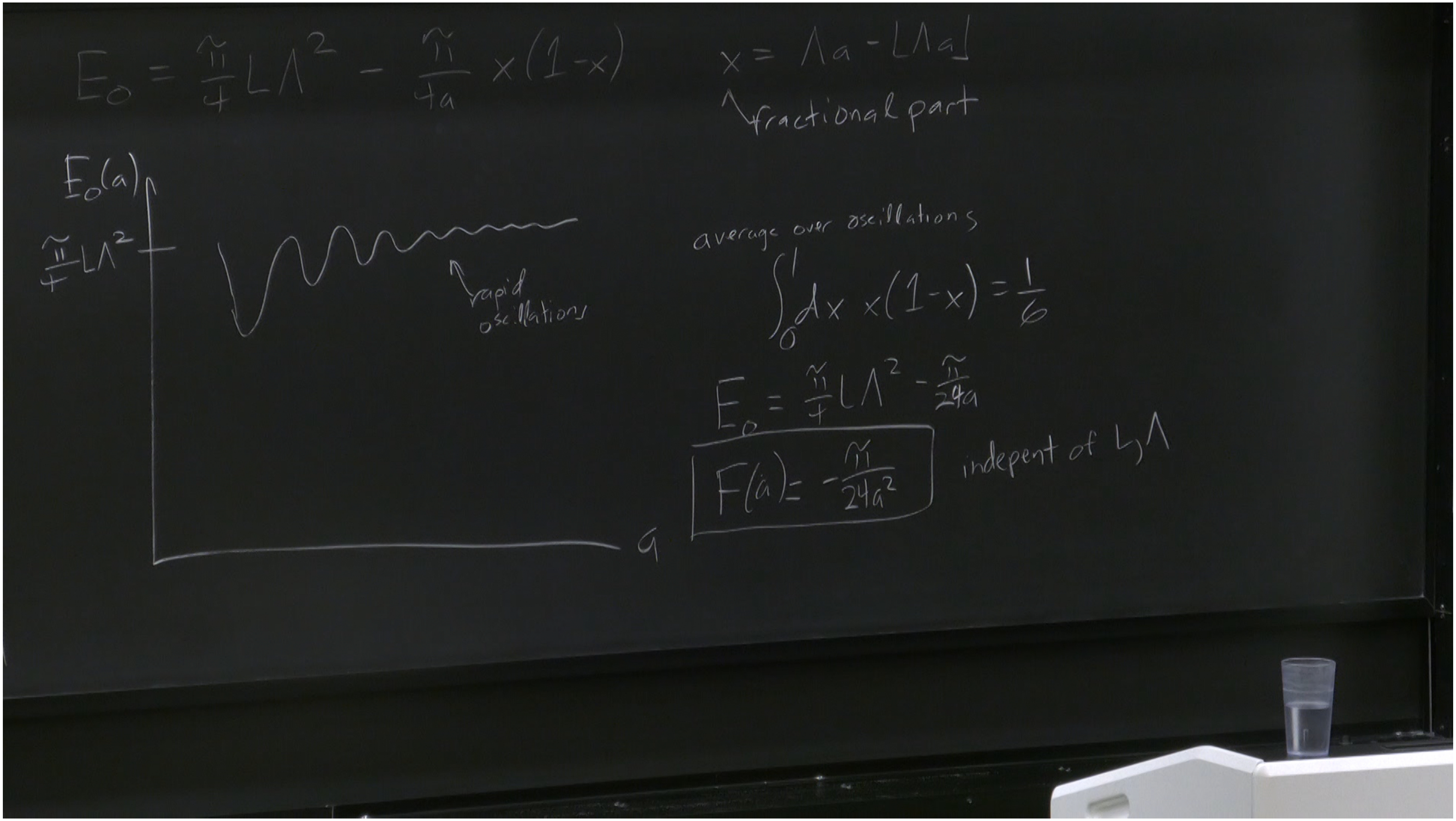
Λ UV regulator

$$E(r) = \frac{1}{2}\sum_{n=0}^{n_{max}} \frac{n\pi}{r} + \text{constant}$$

$$= \frac{1}{2}\,\frac{n_{max}(n_{max}+1)\pi}{2r}$$

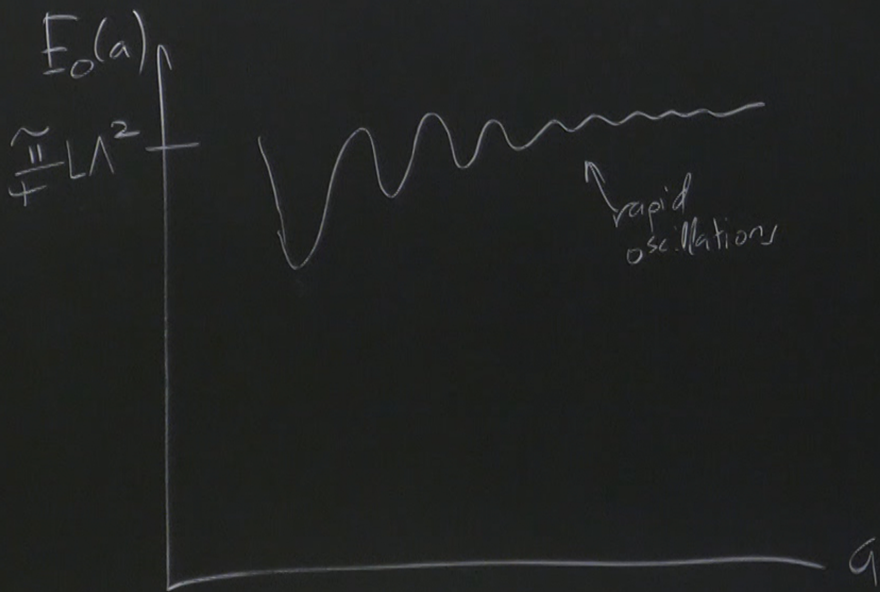
algebra

$E_0 =$


$E_0 = \frac{\tilde{\pi}}{4} L \Lambda^2 - \frac{\tilde{\pi}}{4a} x(1-x)$
$x = \Lambda a - \lfloor \Lambda a \rfloor$
fractional part
$E_0(a)$
$\frac{\tilde{\pi}}{4} L \Lambda^2$
rapid oscillations
a
average over oscillations
$\int_0^1 dx\, x(1-x) = \frac{1}{6}$
$E_0 = \frac{\tilde{\pi}}{4} L \Lambda^2 - \frac{\tilde{\pi}}{24a}$
$F(a) = -\frac{\tilde{\pi}}{24a^2}$
indepent of L, Λ

$$E_0 = \frac{\tilde{\pi}}{4} L\Lambda^2 - \frac{\tilde{\pi}}{4a} x(1-x) \qquad x = \Lambda a - \lfloor \Lambda a \rfloor$$

fractional part



average over oscillations

$$\int_0^1 dx\, x(1-x) = \frac{1}{6}$$

$$E_0 = \frac{\tilde{\pi}}{4} L\Lambda^2 - \frac{\tilde{\pi}}{24a}$$

$$\boxed{F(a) = -\frac{\tilde{\pi}}{24a^2}}$$

indepent of L, Λ

$\to \infty |0\rangle$,

Particles

1-particle state $|\vec{k}\rangle = a^\dagger_{\vec{k}}|0\rangle$

$$:H:|\vec{k}\rangle = \sqrt{\vec{k}^2+m^2}\,|\vec{k}\rangle$$

$$\langle\vec{k}|\vec{p}\rangle = (2\pi)^3\, 2E_{\vec{k}}\, \delta^{(3)}(\vec{p}-\vec{k}) \quad \leftarrow \text{Lorentz-invariant}$$

$$|\vec{k}_1,\vec{k}_2\rangle = a^\dagger_{\vec{k}_1} a^\dagger_{\vec{k}_2}|0\rangle = a^\dagger_{\vec{k}_2} a^\dagger_{\vec{k}_1}|0\rangle = |\vec{k}_2,\vec{k}_1\rangle \quad \text{bosons}$$

Interacting fields

$$\Phi(\vec{x},t) = \int \frac{d^3p}{(2\pi)^3 2E_{\vec{p}}} \left(a_{\vec{p}}(t)\, e^{-ip\cdot x} + a^{\dagger}_{\vec{p}}(t)\, e^{+ip\cdot x} \right)$$

Heisenberg

Heisenberg picture

$$\mathcal{O}_H(t) = e^{iHt}\,\mathcal{O}_S\,e^{-iHt}$$

$$a_{\vec{k}}(t) = e^{-iE_{\vec{k}}t}\,a_{\vec{k}}$$

$$\varphi(x) = \int \frac{d^3k}{(2\pi)^3 2E_{\vec{k}}}\left(a_{\vec{k}}\,e^{-ik\cdot x} + a^\dagger_{\vec{k}}\,e^{+ik\cdot x}\right)$$

Heisenberg

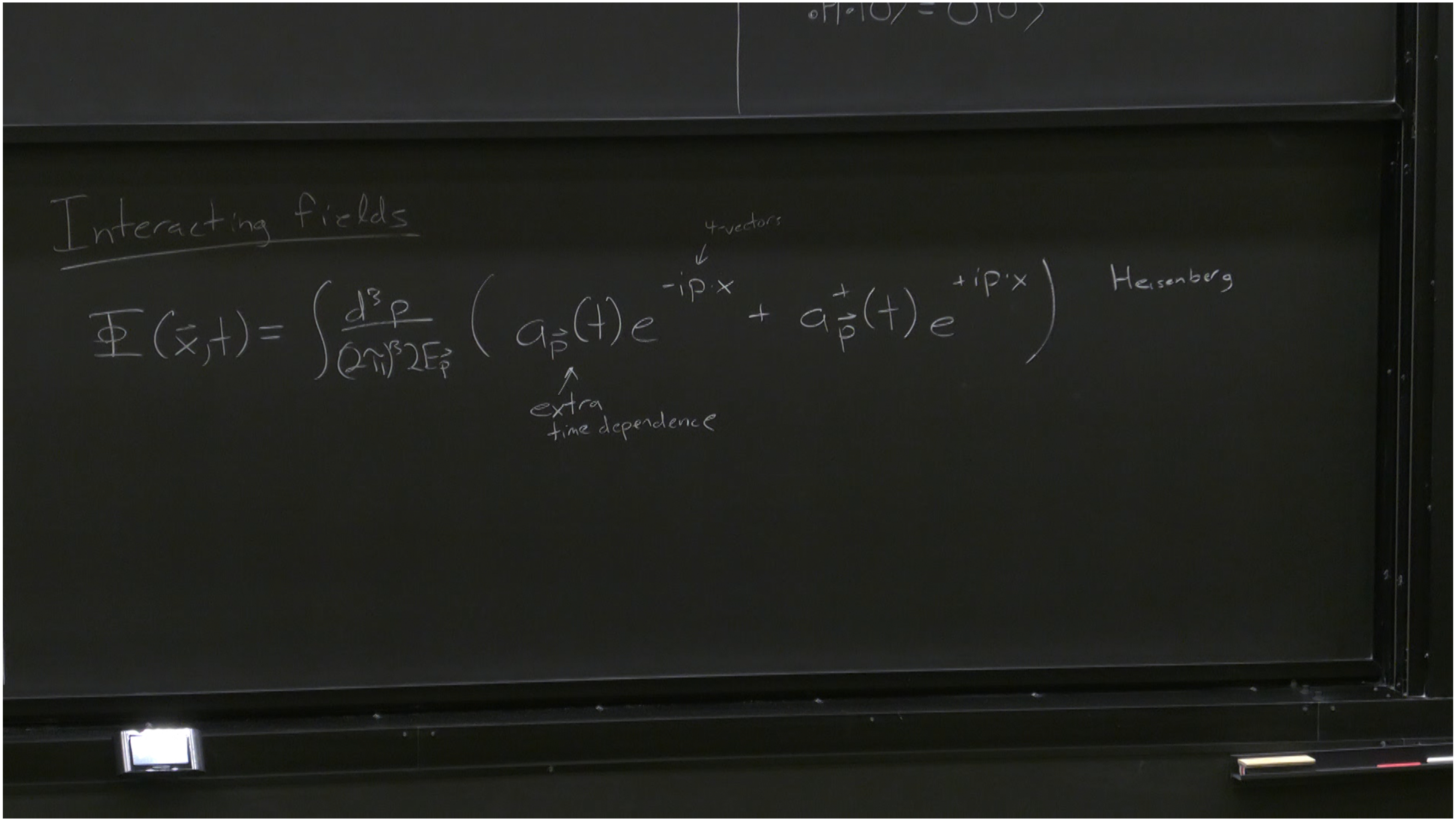
time-indendepent

Interacting fields

$$\Phi(\vec{x},t) = \int \frac{d^3p}{(2\pi)^3 2E_{\vec{p}}} \left(a_{\vec{p}}(t)\, e^{-ip\cdot x} + a^{\dagger}_{\vec{p}}(t)\, e^{+ip\cdot x} \right)$$

Heisenberg

extra time dependence


Interacting fields
4-vectors
Heisenberg
$\Phi(\vec{x},t) = \int \frac{d^3p}{(2\pi)^3 2E_{\vec{p}}} \left(a_{\vec{p}}(t)\, e^{-ip\cdot x} + a^{+}_{\vec{p}}(t)\, e^{+ip\cdot x} \right)$
extra time dependence

Heisenberg picture

$$O_H(t) = e^{iHt} O_S e^{-iHt}$$

$$a_{\vec{k}}(t) = e^{-iE_{\vec{k}}t} a_{\vec{k}}$$

$$\varphi(x) = \int \frac{d^3k}{(2\pi)^3 2E_{\vec{k}}} \left(a_{\vec{k}} e^{-ik\cdot x} + a^\dagger_{\vec{k}} e^{+ik\cdot x} \right)$$

Heisenberg

time-indendepent

Heisenberg picture

$$\mathcal{O}_H(t) = e^{iHt}\,\mathcal{O}_S\,e^{-iHt}$$

$$a_{\vec{k}}(t) = \underline{e^{-iE_{\vec{k}}t}}\;a_{\vec{k}}$$

free Klein-Gordon

$$\varphi(x) = \int \frac{d^3k}{(2\pi)^3 2E_{\vec{k}}}\left(a_{\vec{k}}\, e^{-ik\cdot x} + a^\dagger_{\vec{k}}\, e^{+ik\cdot x} \right)$$

Heisenberg

time-indendepent

$$\varphi(\vec{x}) = \int \frac{d^3k}{(2\pi)^3 2E_{\vec{k}}}\ ($$

$\circ P | \circ |0\rangle = O|0\rangle$

Interacting fields $\quad \mathcal{L} = \mathcal{L}_{KG} + \mathcal{L}_{int}$

4-vectors

$$\Phi(\vec{x},t) = \int \frac{d^3p}{(2\pi)^3 2E_{\vec{p}}} \left(a_{\vec{p}}(t)\, e^{-ip\cdot x} + a^{\dagger}_{\vec{p}}(t)\, e^{+ip\cdot x} \right)$$

Heisenberg

extra
time dependence

$$\Phi(\vec{x},t) = \int \frac{d^3p}{(2\pi)^3 2E_{\vec{p}}} \left(a_{\vec{p}}(t)\, e^{-ip\cdot x} + a^{\dagger}_{\vec{p}}(t)\, e^{+ip\cdot x} \right)$$

Heisenberg

extra time dependence

$$[a_{\vec{p}}(t), a_{\vec{k}}(t)] = (2\pi)^3\, 2E_{\vec{k}}\, \delta(\vec{p}-\vec{k})$$

$$|i\rangle = |\vec{k}_1, \vec{k}_2\rangle$$

$$|f\rangle = |\vec{k}_3, \vec{k}_4\rangle$$

$$\langle f | S | i \rangle \longrightarrow \langle \Omega | T\, \Phi(x_1) \cdots \Phi(x_4) | \Omega \rangle$$

↑ time evolution

$$\Phi(\vec{x},t) = \int \frac{d^3p}{(2\pi)^3 2E_{\vec{p}}} \left(a_{\vec{p}}(t)\, e^{-ip\cdot x} + a^{\dagger}_{\vec{p}}(t)\, e^{+ip\cdot x} \right)$$ Heisenberg

extra time dependence

$$[a_{\vec{p}}(t), a^{\dagger}_{\vec{k}}(t)] = (2\pi)^3\, 2E_{\vec{k}}\, \delta(\vec{p}-\vec{k})$$

$$\varphi(\vec{x}) = \int \frac{d^3k}{(2\pi)^3 2E_{\vec{k}}} \left(a_{\vec{k}}\, e^{+i\vec{k}\cdot\vec{x}} + a^{\dagger}_{\vec{k}}\, e^{-i\vec{k}\cdot\vec{x}} \right)$$

Heisenberg picture

$$O_H(t) = e^{iHt}\, O_S\, e^{-iHt}$$

$$a_{\vec{k}}(t) = e^{-iE_{\vec{k}}t}\, a_{\vec{k}}$$

free Klein-Gordon

$$\varphi(x) = \int \frac{d^3k}{(2\pi)^3 2E_{\vec{k}}} \left(a_{\vec{k}}\, e^{-ik\cdot x} + a^\dagger_{\vec{k}}\, e^{+ik\cdot x} \right)$$

Heisenberg

time-indendepent