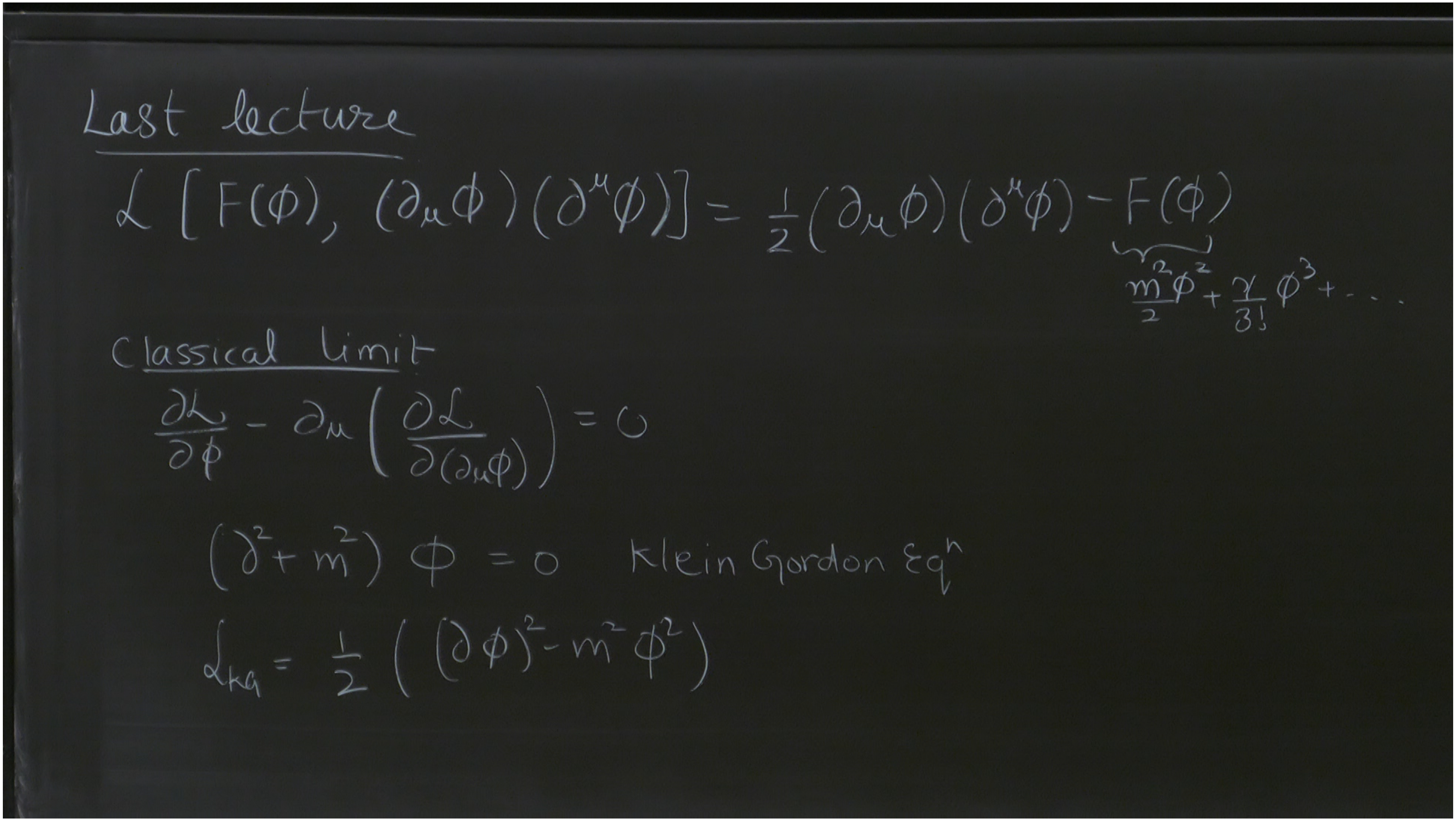
Title: Quantum Theory Lecture - 091823

Speakers: Bindiya Arora, Dan Wohns

Collection: Quantum Theory 2023/24

Date: September 18, 2023 - 10:45 AM

URL: https://pirsa.org/23090044



Last lecture

$$\mathcal{L}\left[F(\phi),\ (\partial_\mu\phi)(\partial^\mu\phi)\right] = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \underbrace{F(\phi)}_{\frac{m^2\phi^2}{2}+\frac{\gamma}{3!}\phi^3+\dots}$$

Classical limit

$$\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right) = 0$$

$(\partial^2+m^2)\,\phi = 0$ Klein Gordon Eq^n

$$\mathcal{L}_{KG} = \frac{1}{2}\left((\partial\phi)^2 - m^2\phi^2\right)$$

Last lecture

$$\mathcal{L}\left[F(\phi),\, (\partial_\mu\phi)(\partial^\mu\phi)\right] = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \underbrace{F(\phi)}_{\frac{m^2}{2}\phi^2 + \frac{\lambda}{3!}\phi^3 + \dots}$$

Classical limit

$$\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right) = 0$$

$(\partial^2 + m^2)\,\phi = 0$ Klein Gordon Eq^n

$$\boxed{\mathcal{L}_{KG} = \frac{1}{2}\left((\partial\phi)^2 - m^2\phi^2\right)}$$

Scalar free field

$$\langle 0 | e^{-i\hat{H}t} | 0 \rangle = Z[J] = \int D\phi \; e^{i\int d^4x \left[\frac{1}{2}(\partial\phi^2) - \frac{m^2\phi^2}{2}\right]}$$

$$\boxed{(1, -1, -1, -1)}$$

$\underline{\underline{J(x)}}$ → source.

$J(x) \to J$

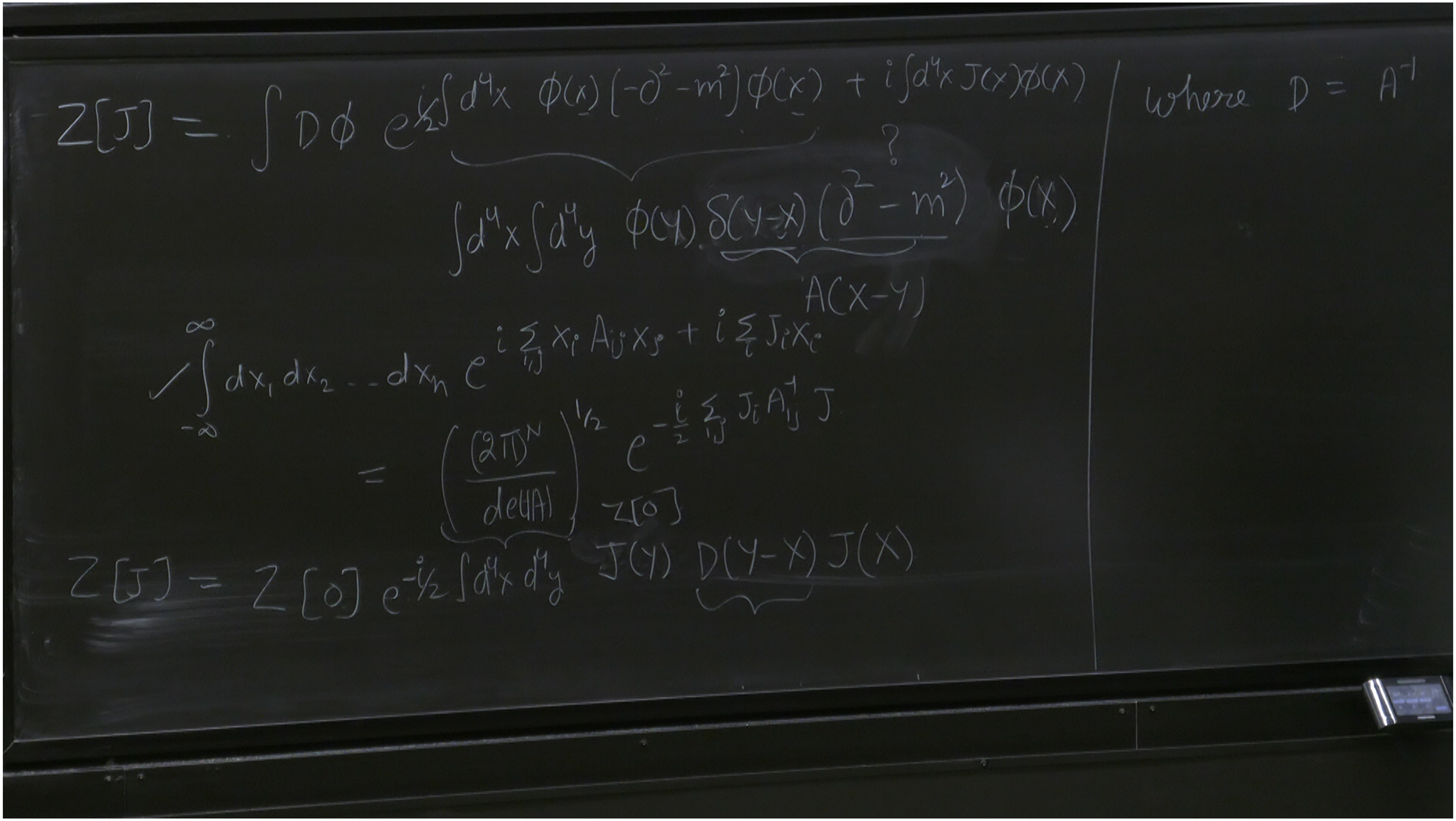
$\phi(x) \to \phi$

$$\mathcal{L} \longrightarrow \mathcal{L} + J(x)\phi(x)$$

$$Z[J] = \int D\phi \; e^{i\int d^4x \left[\underbrace{\frac{1}{2}(\partial\phi)^2} - \frac{m^2}{2}\phi^2 + J\phi\right]}$$

$$\int d^4x\, (\partial\phi)^2 = \int_{-\infty}^{\infty} d^4x\, (\partial_\mu\phi)(\partial^\mu\phi)$$

$$= \cancelto{0}{(\partial_\mu\phi)\phi}\Big|_{-\infty}^{\infty} - \int \phi\, \partial_\mu\partial^\mu\phi\, d^4x$$



$$Z[J] = \int D\phi \, e^{\frac{i}{2}\underbrace{\int d^4x \, \phi(x)\left(-\partial^2 - m^2\right)\phi(x)}_{\int d^4x \int d^4y \, \phi(y) \underbrace{\delta(y-x)\left(\partial^2 - m^2\right)}_{A(x-y)} \phi(x)} + i\int d^4x \, J(x)\phi(x)}$$

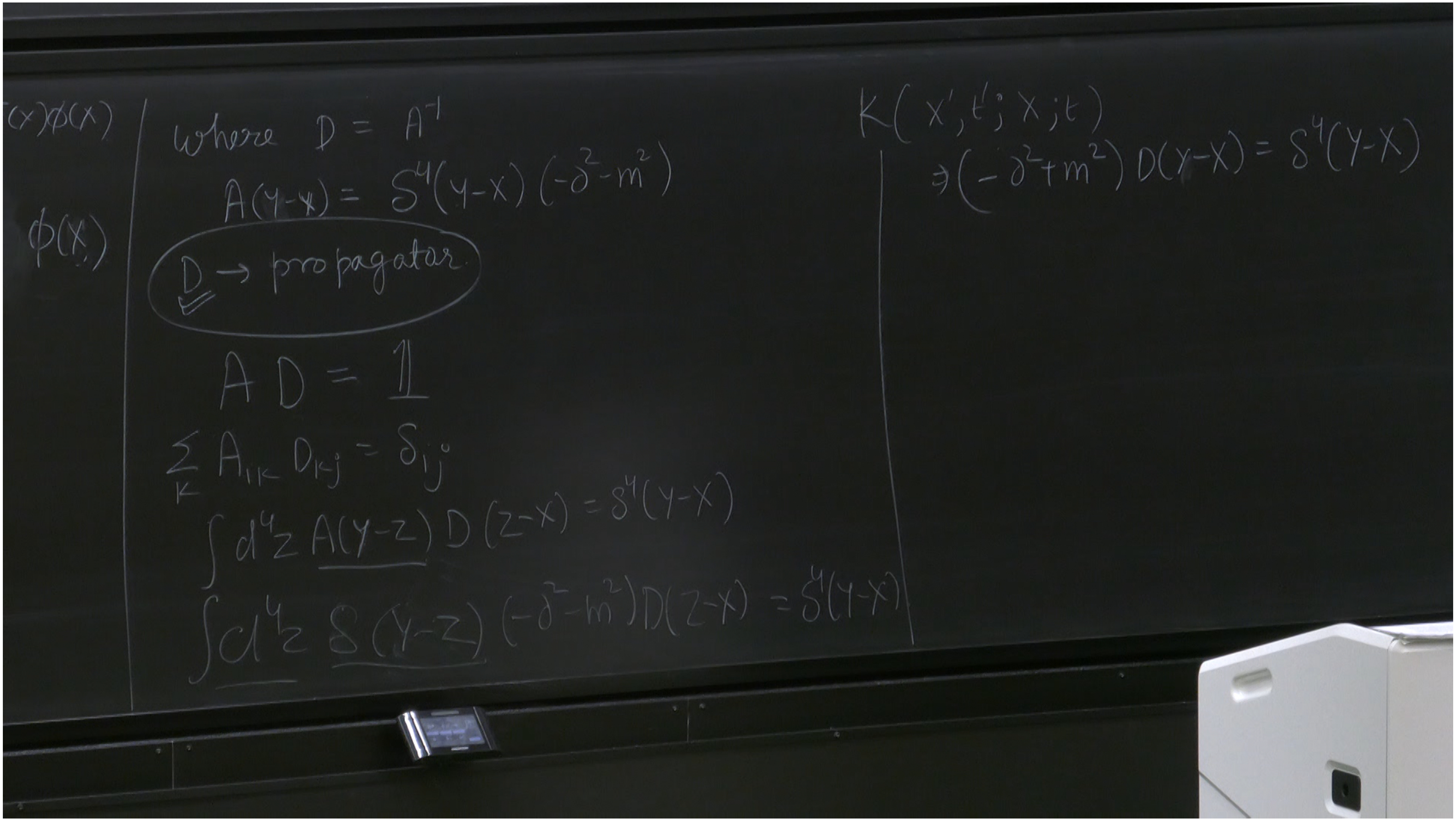
?

where $D = A^{-1}$

$$\checkmark \int_{-\infty}^{\infty} dx_1 \, dx_2 \ldots dx_n \, e^{i\sum_{i,j} x_i A_{ij} x_j + i\sum_i J_i x_i}$$

$$= \underbrace{\left(\frac{(2\pi)^N}{\det(A)}\right)^{1/2}}_{Z[0]} e^{-\frac{i}{2}\sum_{i,j} J_i A^{-1}_{ij} J}$$

$$Z[J] = Z[0] \, e^{-i/2 \int d^4x \, d^4y \, J(y) \underbrace{D(y-x)} J(x)}$$



$(x)\phi(x)$

$\phi(x)$

where $D = A^{-1}$

$$A(y-x) = \delta^4(y-x)(-\partial^2 - m^2)$$

$\underline{\underline{D}} \to$ propagator.

$$AD = \mathbb{1}$$

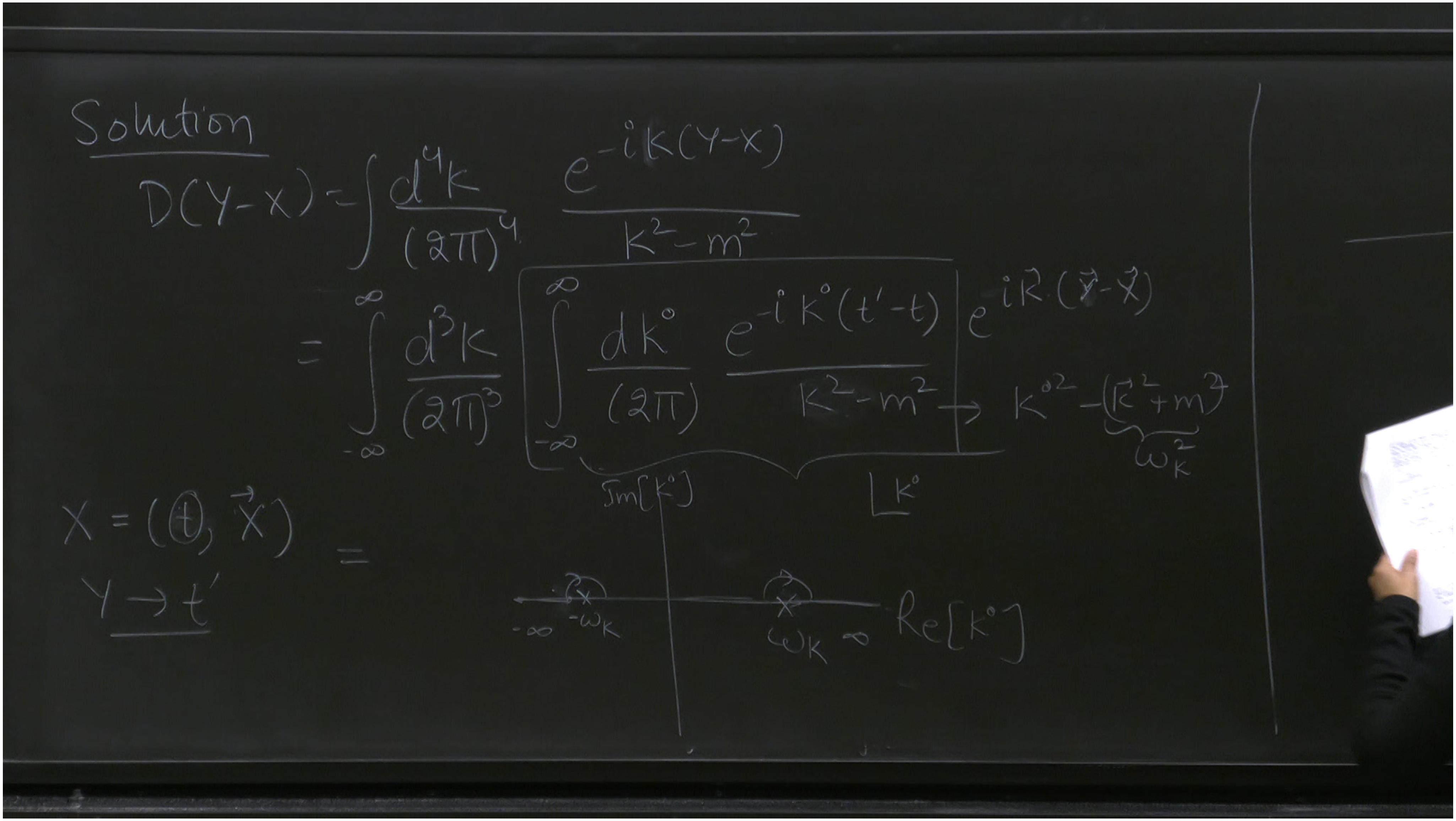
$$\sum_k A_{ik} D_{kj} = \delta_{ij}$$

$$\int d^4z \, \underline{A(y-z)} \, D(z-x) = \delta^4(y-x)$$

$$\int \underline{d^4z} \, \underline{\delta(y-z)} \, (-\partial^2 - m^2) D(z-x) = \delta^4(y-x)$$

$K(x', t'; x; t)$

$$\Rightarrow (-\partial^2 + m^2) D(y-x) = \delta^4(y-x)$$



Solution

$$D(y-x) = \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik(y-x)}}{k^2-m^2}$$

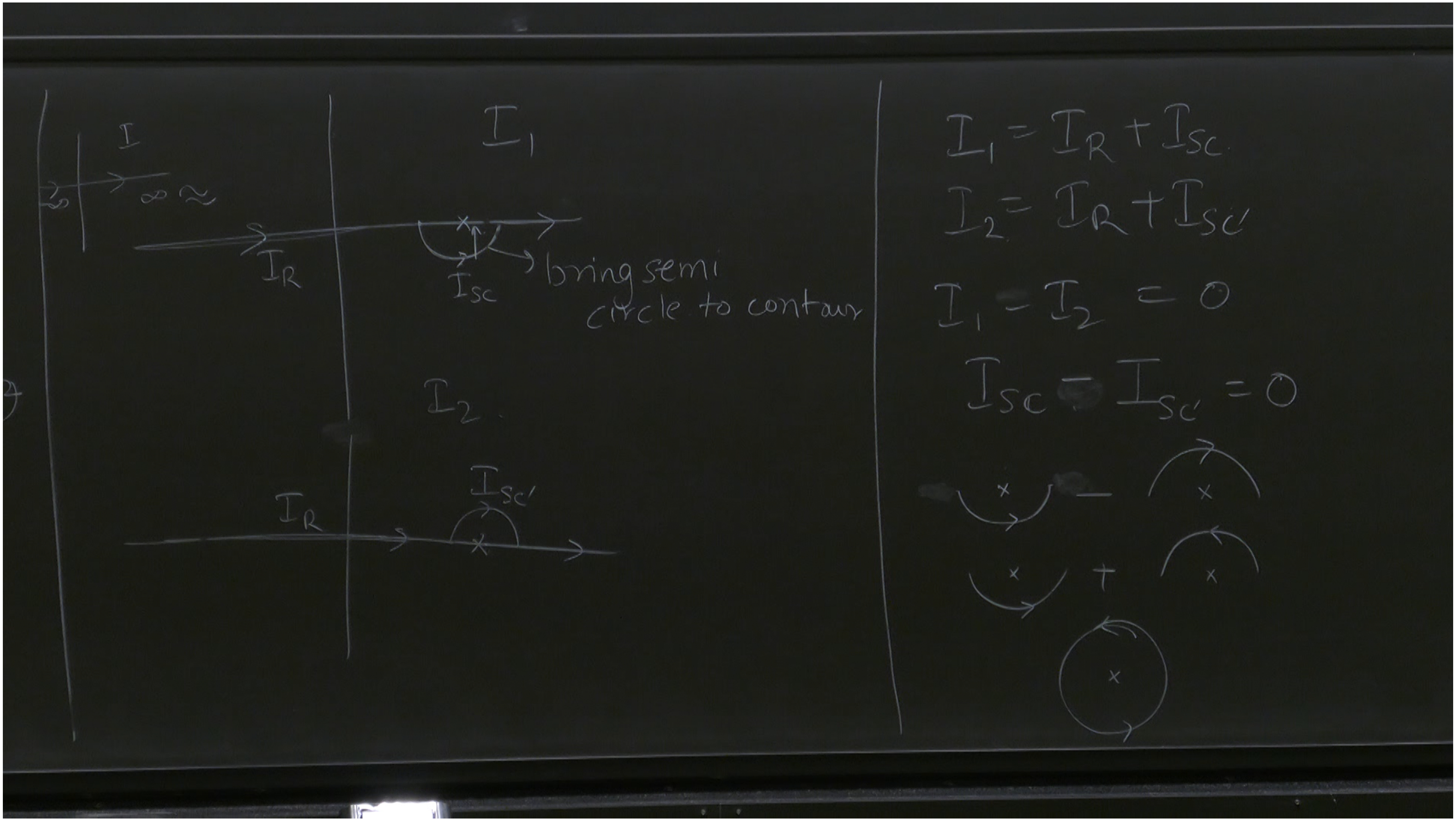
$$= \int_{-\infty}^{\infty} \frac{d^3k}{(2\pi)^3} \underbrace{\boxed{\int_{-\infty}^{\infty} \frac{dk^0}{(2\pi)} \frac{e^{-ik^0(t'-t)}}{k^2-m^2}}} e^{i\vec{k}\cdot(\vec{y}-\vec{x})}$$

$$k^2-m^2 \rightarrow k^{0\,2} - \underbrace{(\vec{k}^2+m^2)}_{\omega_k^2}$$

$x = (t, \vec{x})$

$y \rightarrow t'$

$=$

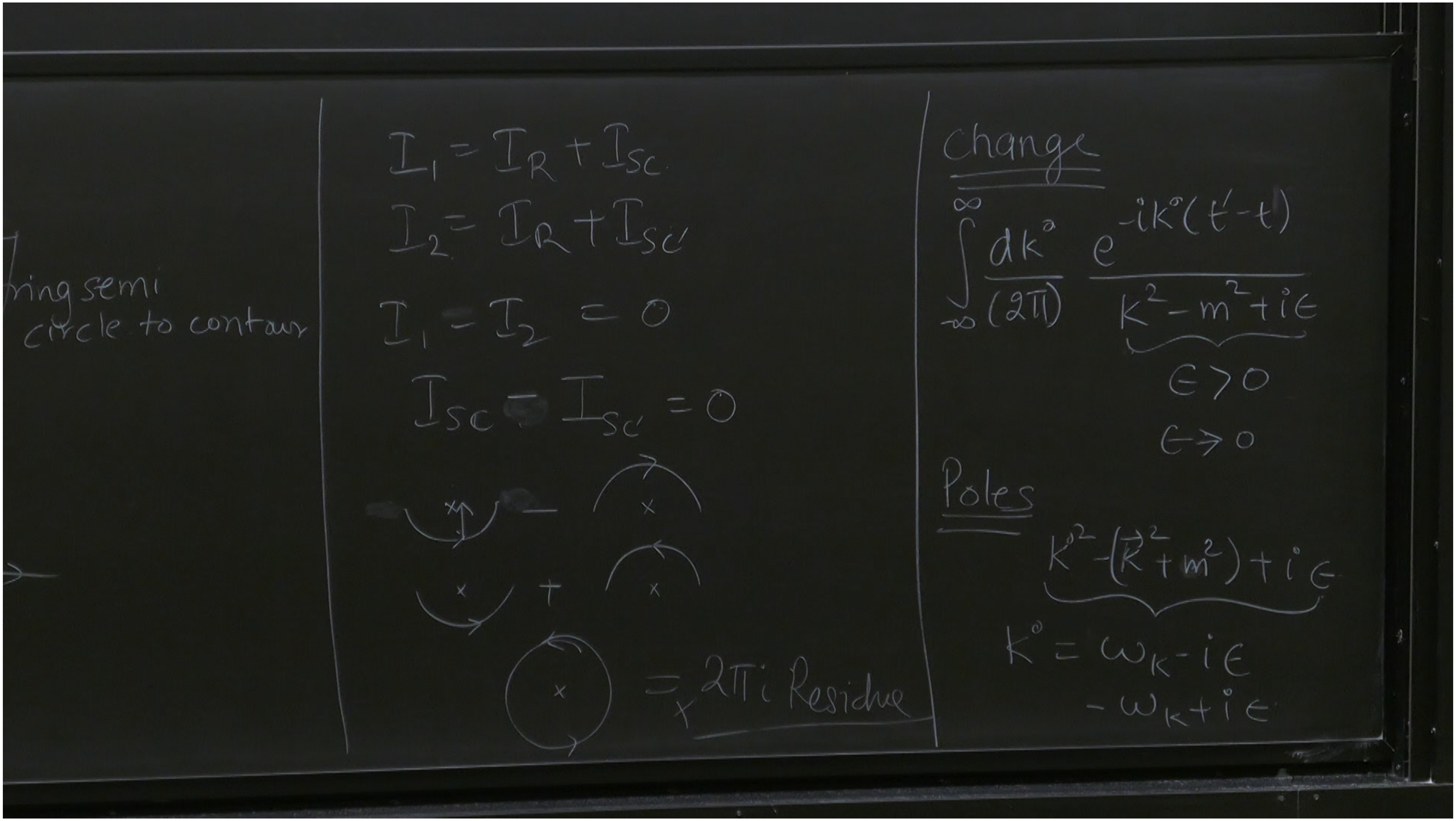


$I_1 = I_R + I_{sc}$

$I_2 = I_R + I_{sc'}$

$I_1 - I_2 = 0$

$I_{sc} - I_{sc'} = 0$



ring semi
circle to contour

$$I_1 = I_R + I_{SC}$$

$$I_2 = I_R + I_{SC'}$$

$$I_1 - I_2 = 0$$

$$I_{SC} - I_{SC'} = 0$$

$= 2\pi i$ Residue

Change

$$\int_{-\infty}^{\infty} \frac{dk^0}{(2\pi)} \frac{e^{-ik^0(t'-t)}}{\underbrace{k^2 - m^2 + i\epsilon}}$$

$\epsilon > 0$

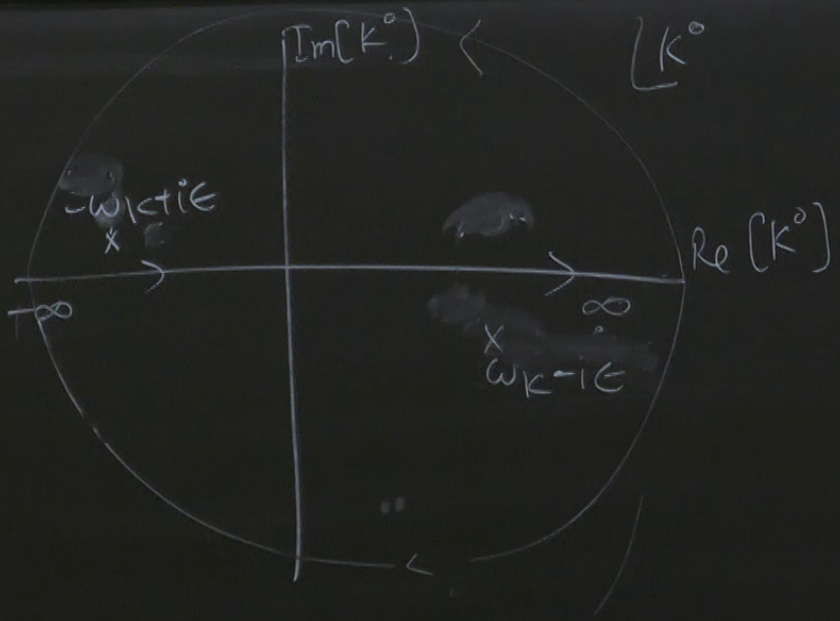
$\epsilon \to 0$

Poles

$$\underbrace{k^{0\,2} - (\vec{k}^2 + m^2) + i\epsilon}$$

$k^0 = \omega_k - i\epsilon$

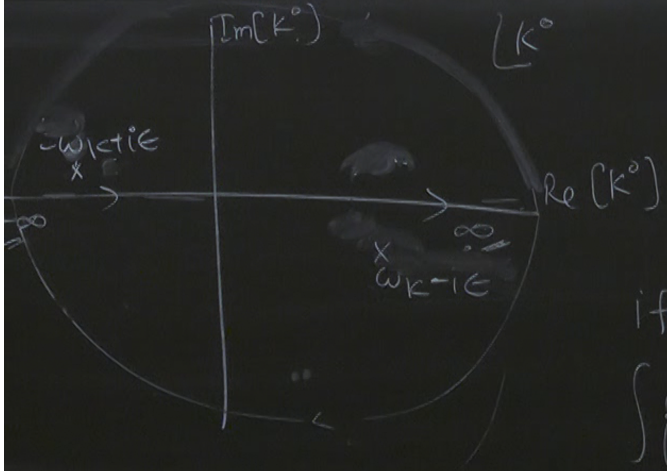
$-\omega_k + i\epsilon$



Jordan's Lemma

$$\int_{C_R} f(z)\, e^{-i\alpha z}\, dz.$$

$\alpha > 0 \Rightarrow$ lower half plane



Jordan's Lemma

$$\int_{C_R} f(z)\, e^{-i\alpha z}\, dz.$$

$t'-t<0$

$\alpha < 0 \Rightarrow$ upper half plane

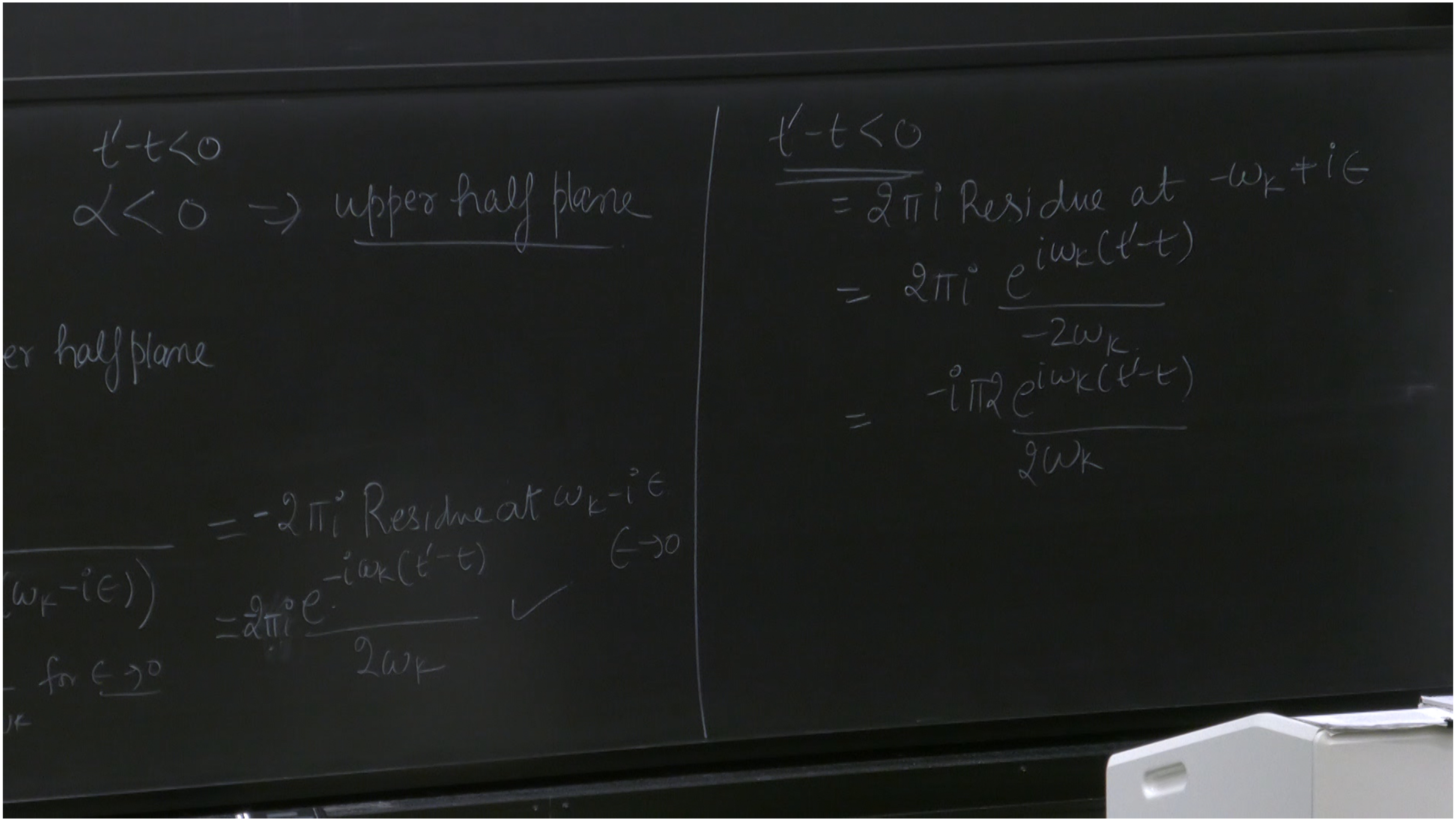
$\alpha > 0 \Rightarrow$ lower half plane

if $t'-t > 0 \Rightarrow t' > t$

$$\int \frac{dk^0}{(2\pi)} \frac{e^{-ik(t'-t)}}{\left(k^0-(-\omega_k+i\epsilon)\right)\left(k^0-(\omega_k-i\epsilon)\right)} = -2\pi i \text{ Residue at } \omega_k - i$$

$$= \frac{e^{-i\omega_k(t'-t)}}{2\omega_k} \quad \checkmark$$

$-\frac{1}{2\omega_k}$ for $\epsilon \to 0$



$t'-t<0$

$\alpha<0 \Rightarrow$ <u>upper half plane</u>

er half plane

$(\omega_k - i\epsilon))$

for $\underline{\epsilon\to 0}$

$= -2\pi i$ Residue at $\omega_k - i\epsilon$

$\epsilon\to 0$

$= 2\pi i \dfrac{e^{-i\omega_k(t'-t)}}{2\omega_k}$ ✓

$\underline{\underline{t'-t<0}}$

$= 2\pi i$ Residue at $-\omega_k + i\epsilon$

$$= 2\pi i \frac{e^{i\omega_k(t'-t)}}{-2\omega_k}$$

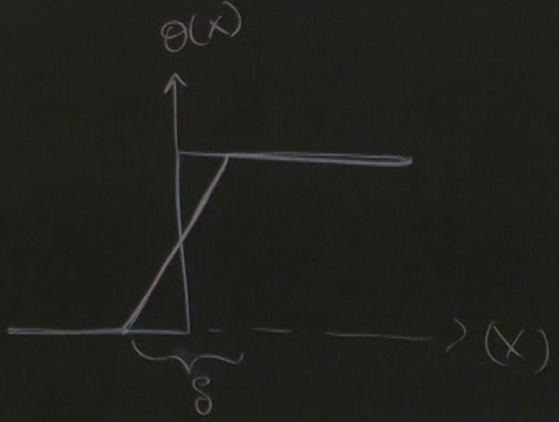
$$= \frac{-i\pi 2\, e^{i\omega_k(t'-t)}}{2\omega_k}$$

$$D(y-x) = -i\int_{-\infty}^{\infty} \frac{d^3k}{(2\pi)^{3} 2\omega_k}\Big[e^{-i\omega_k(t'-t)} \; [t'-t>0] - e^{i\omega_k} \qquad [t'-t<$$

$$\int_{-\infty}^{\infty} \frac{d^3k}{(2\pi)^3 2\omega_k} \left[e^{-i\omega_k(t'-t)} \Theta(t'-t) - e^{i\omega_k(t'-t)} \Theta(t-t') \right]$$

Heavy side step f^n.

$$\Theta(x) = \lim_{\delta \to 0} \begin{cases} 0 & \text{for } x < -\delta/2 \\ \frac{1}{\delta}(x + \delta/2) & \\ 0 & \text{for } x > \delta/2 \end{cases}$$



$-2\pi i$ Residue

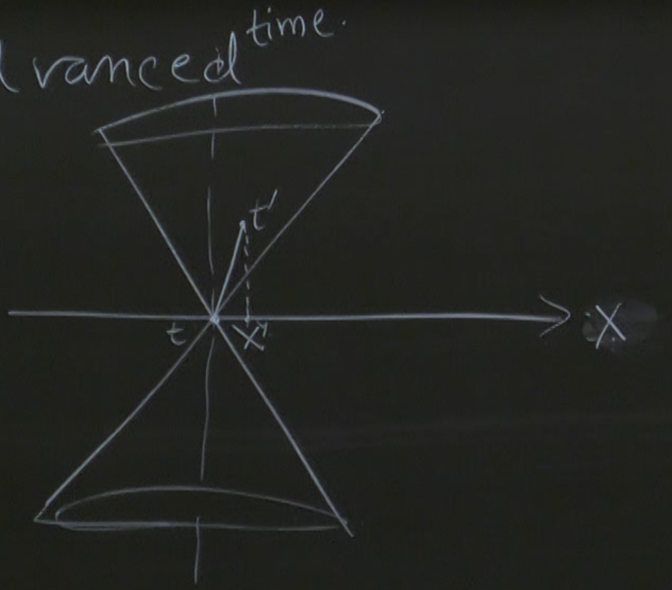
$- \omega_k + i\epsilon\ )$

$\vec{x})$

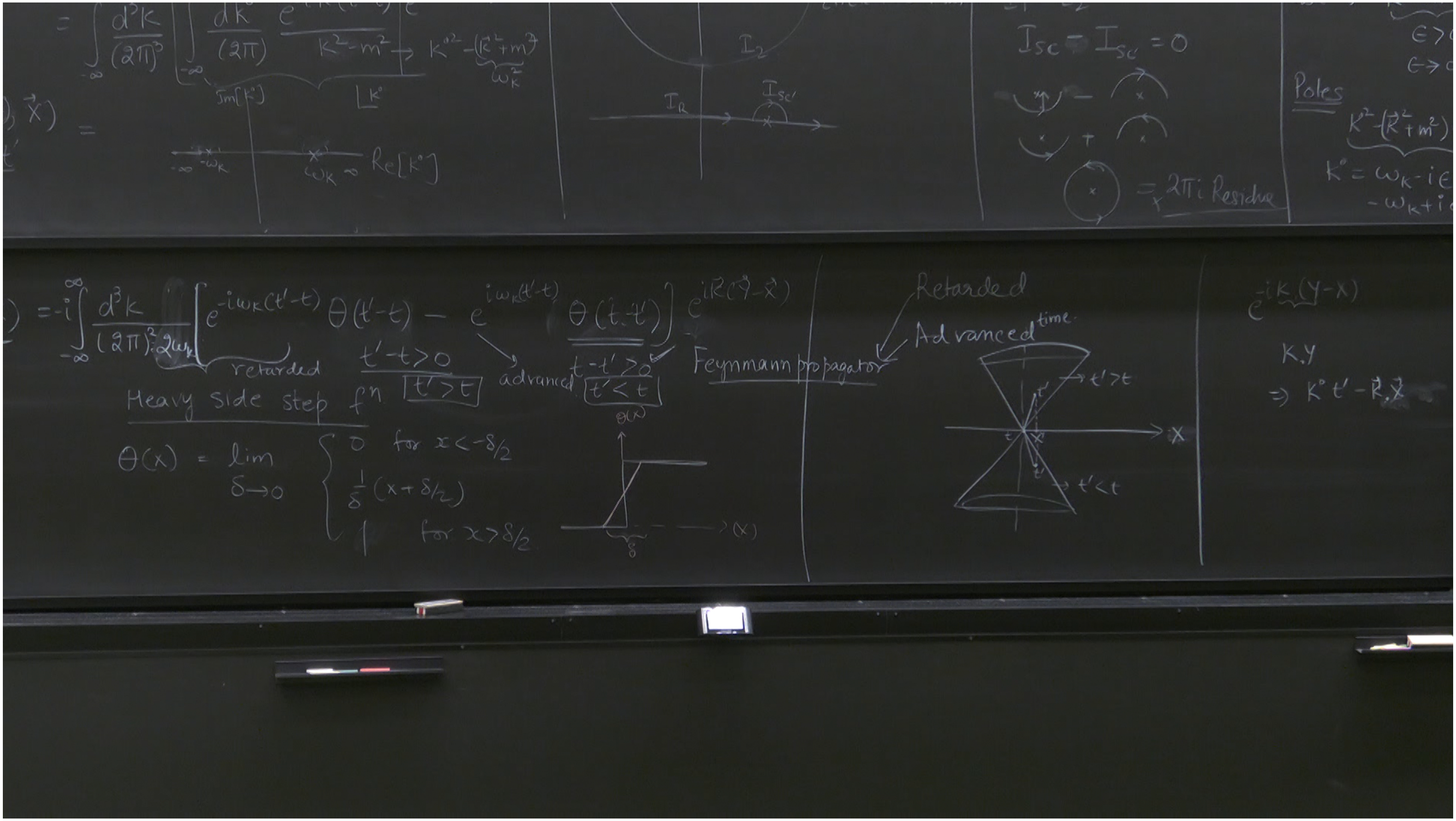
Retarded

Advanced

mann propagator



$x)$



$$= -i \int_{-\infty}^{\infty} \frac{d^3k}{(2\pi)^3\, 2\omega_k} \Big[\underbrace{e^{-i\omega_k(t'-t)}\, \theta(t'-t)}_{\text{retarded}} - e^{i\omega_k(t'-t)}\, \theta(t-t') \Big] e^{i\vec{k}(\vec{y}-\vec{x})}$$

$t' - t > 0$ $\boxed{t' > t}$

advanced $t - t' > 0$ $\boxed{t' < t}$

Feynmann propagator

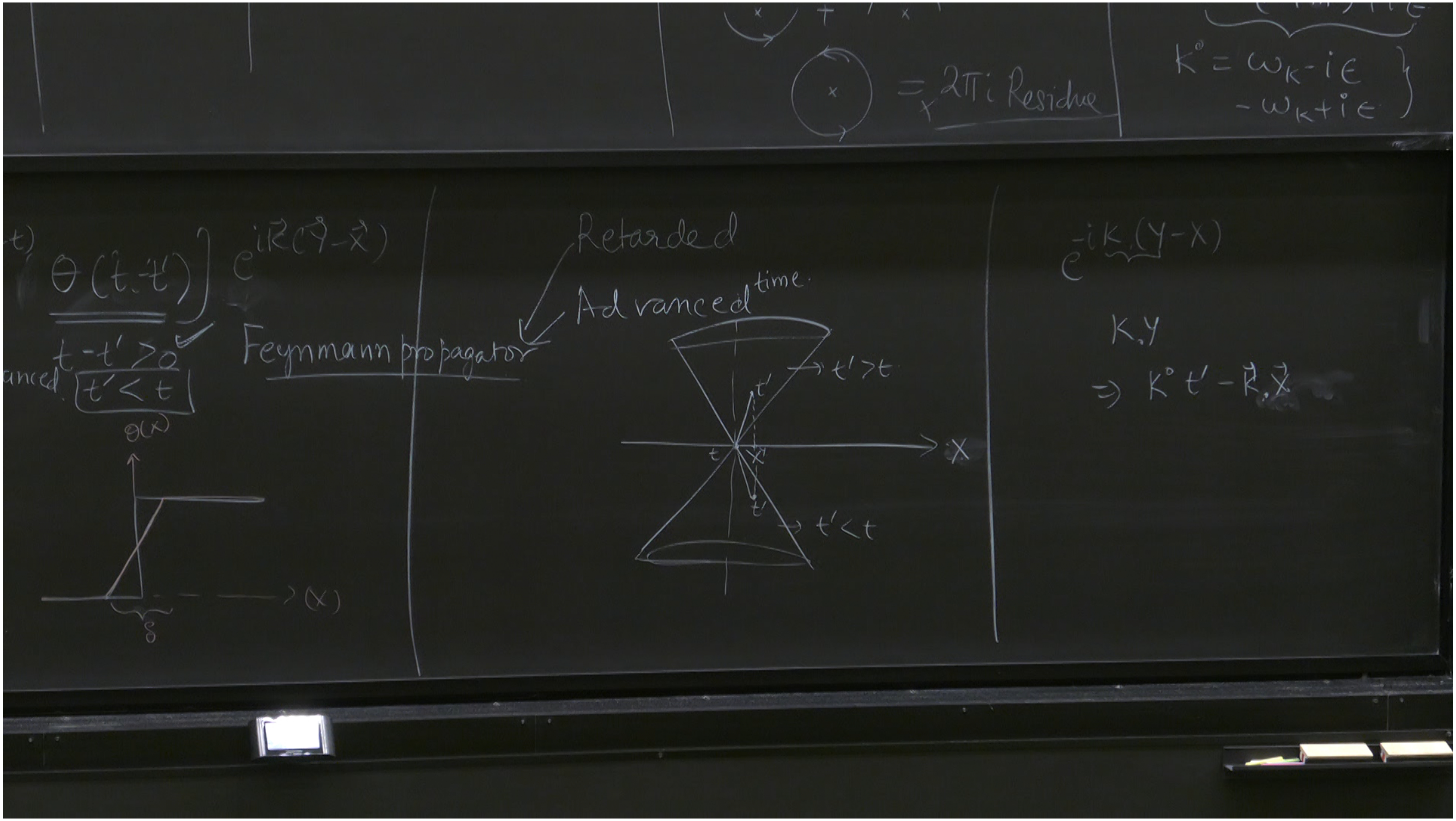
Heavy side step f^n

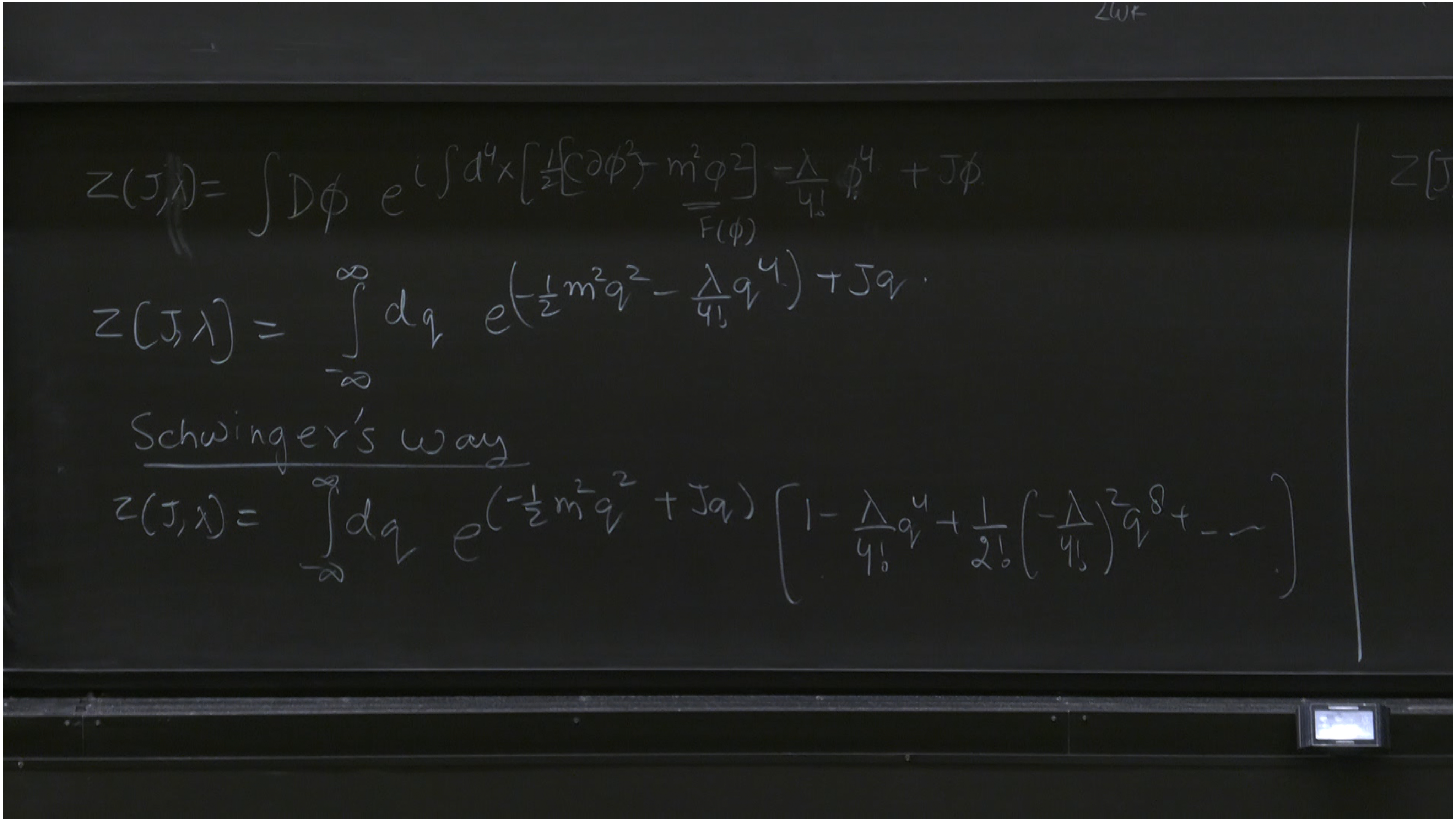
$$\theta(x) = \lim_{\delta \to 0} \begin{cases} 0 & \text{for } x < -\delta/2 \\ \frac{1}{\delta}(x + \delta/2) \\ 1 & \text{for } x > \delta/2 \end{cases}$$

$e^{-ik.(y-x)}$

k.y

$\Rightarrow k^0 t' - \vec{k}.\vec{x}$


$= 2\pi i$ Residue
$k^0 = \omega_k - i\epsilon$
$-\omega_k + i\epsilon$
$-t)$
$\theta(t-t')\big] e^{i\vec{k}(\vec{y}-\vec{x})}$
$t - t' > 0$
anced. $\boxed{t' < t}$
$\theta(x)$
(x)
δ
Retarded
Advanced
Feynmann propagator
time.
$t' > t$
t'
t
x
X
t'
$t' < t$
$e^{-ik.(y-x)}$
$k.y$
$\Rightarrow k^0 t' - \vec{k}.\vec{x}$



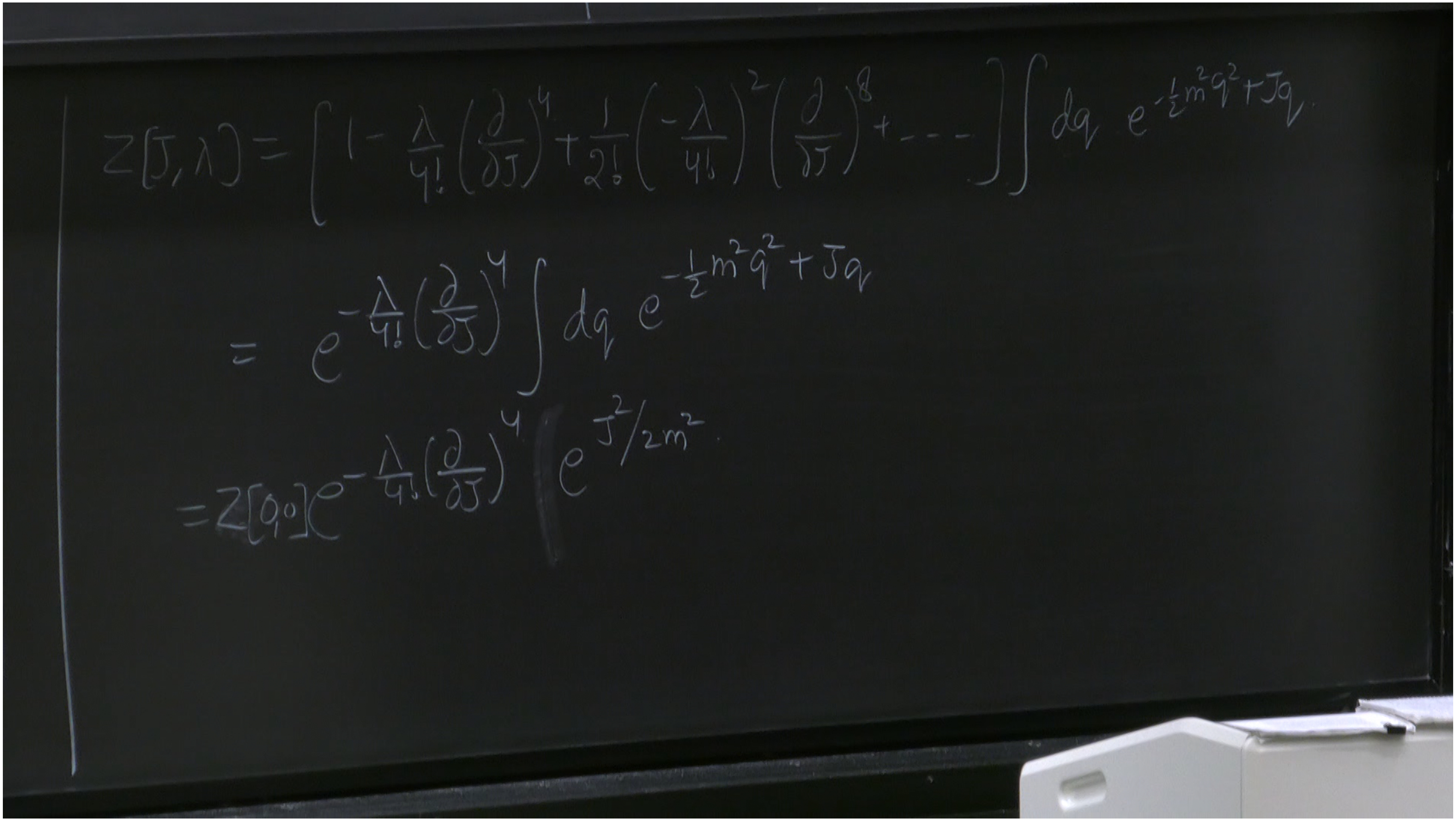
$$Z(J,\lambda) = \int D\phi \; e^{i\int d^4x \left[\frac{1}{2}[(\partial\phi)^2 - \underset{F(\phi)}{\underline{\underline{m^2\phi^2}}}] - \frac{\lambda}{4!}\phi^4 + J\phi\right]}$$

$$Z(J,\lambda) = \int_{-\infty}^{\infty} dq \; e^{\left(-\frac{1}{2}m^2q^2 - \frac{\lambda}{4!}q^4\right) + Jq}$$

Schwinger's way

$$Z(J,\lambda) = \int_{-\infty}^{\infty} dq \; e^{\left(-\frac{1}{2}m^2q^2 + Jq\right)} \left[1 - \frac{\lambda}{4!}q^4 + \frac{1}{2!}\left(-\frac{\lambda}{4!}\right)^2 q^8 + \dots\right]$$

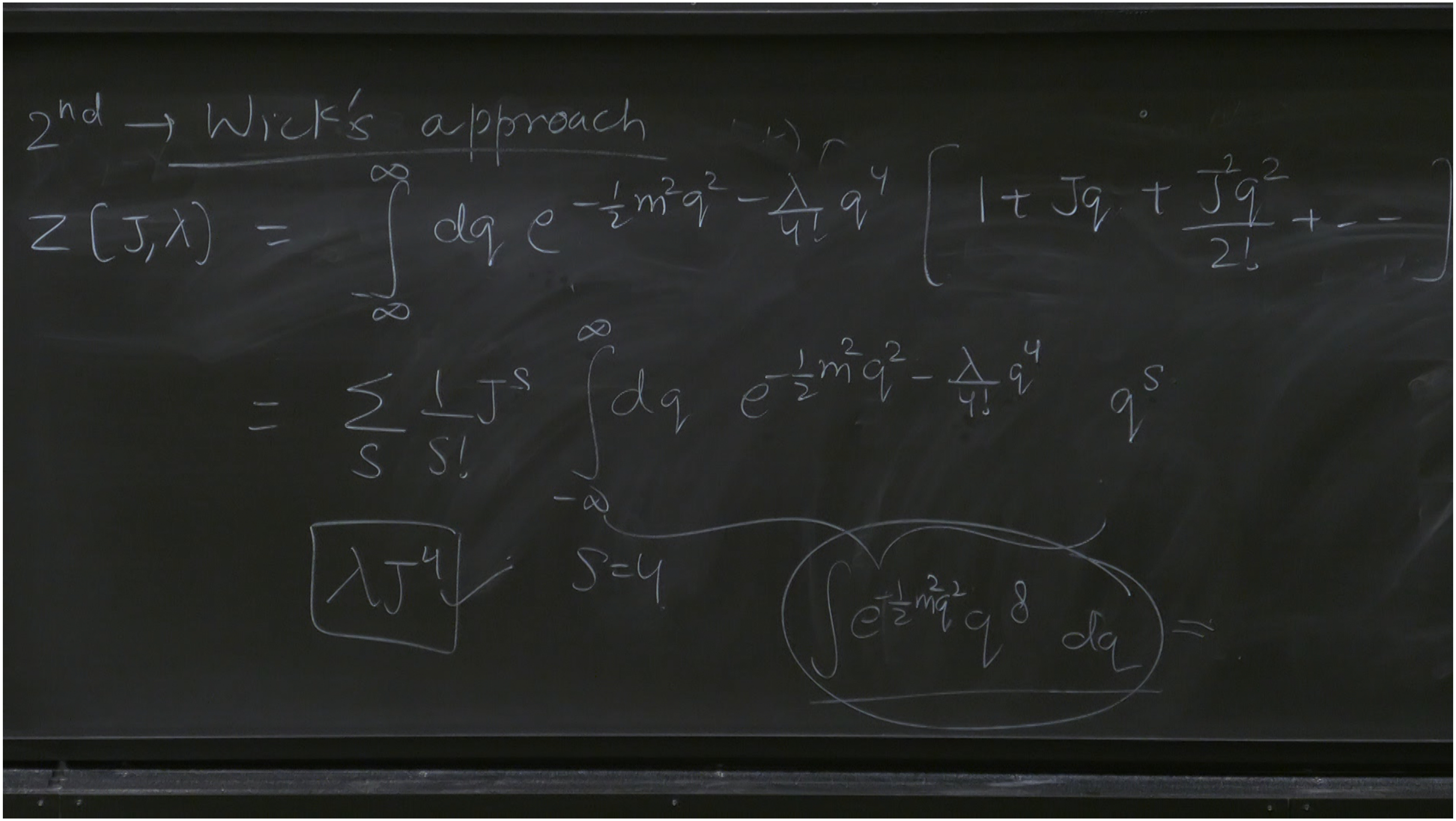
$Z[J$



$$Z[J,\lambda] = \left[1 - \frac{\lambda}{4!}\left(\frac{\partial}{\partial J}\right)^4 + \frac{1}{2!}\left(-\frac{\lambda}{4!}\right)^2\left(\frac{\partial}{\partial J}\right)^8 + \cdots\right]\int dq\; e^{-\frac{1}{2}m^2q^2 + Jq}$$

$$= e^{-\frac{\lambda}{4!}\left(\frac{\partial}{\partial J}\right)^4}\int dq\; e^{-\frac{1}{2}m^2q^2 + Jq}$$

$$= Z[0,0]\, e^{-\frac{\lambda}{4!}\left(\frac{\partial}{\partial J}\right)^4}\, e^{J^2/2m^2}$$



2nd → Wick's approach

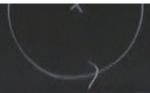
$$Z(J,\lambda) = \int_{-\infty}^{\infty} dq\, e^{-\frac{1}{2}m^2q^2 - \frac{\lambda}{4!}q^4}\left[1 + Jq + \frac{J^2q^2}{2!} + \cdots\right]$$

$$= \sum_{s} \frac{1}{s!} J^s \int_{-\infty}^{\infty} dq\, e^{-\frac{1}{2}m^2q^2 - \frac{\lambda}{4!}q^4}\, q^s$$

$\boxed{\lambda J^4}$: $s=4$

$$\int e^{-\frac{1}{2}m^2q^2}\, q^8\, dq =$$

$$-2\frac{d}{da}\left(\int_{-\infty}^{\infty} dx\, e^{-\frac{1}{2}ax^2}\right) = -2\frac{d}{da}\left(\left(\frac{2\pi}{a}\right)^{1/2}\right)$$



Residue

$-\omega_k + i\epsilon$

$$Z(J,\lambda) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} dq_1 \cdots dq_N \, e^{-\frac{1}{2}\sum_{ij} q_i A_{ij} q_j - \frac{\lambda}{4!}\sum_i q_i^4 + \sum_i J_i q_i}$$

$$= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} dq_1 \cdots dq_N \, e^{-\frac{1}{2}\sum_{ij} q_i A_{ij} q_j + \sum_i J_i q_i} \left(1 - \frac{\lambda}{4!}\sum_i q_i^4 + \left(-\frac{\lambda}{4!}\right)^2 \left(\sum_i q_i^4\right)^2 + \cdots\right.$$

$$= \left[1 - \frac{\lambda}{4!}\sum_i \left(\frac{\partial}{\partial J_i}\right)^4 + \left(-\frac{\lambda}{4!}\right)^2 \left(\sum_i \left(\frac{\partial}{\partial J_i}\right)^4\right)^2 + \cdots \right. \iiint \cdots \int e^{-\frac{1}{2}\sum_{ij} q_i A_{ij} q_j + \sum_i J_i q_i}$$

$$e^{-\lambda/4! \sum_i \left(\frac{\partial}{\partial J_i}\right)^4}$$

$$Z(J,\lambda) = \int D\phi \; e^{i\int d^4x \left[\underbrace{\frac{1}{2}[(\partial\phi)^2 - m^2\phi^2]}_{F(\phi)} - \frac{\lambda}{4!}\phi^4 + J\phi\right]}$$

$$Z^{\text{Teenage}}(J,\lambda) = e^{-\frac{\lambda}{4!}\sum_i \left(\frac{\partial}{\partial J_i}\right)^4} \; e^{\frac{1}{2} J A^{-1} J} \; Z[0,0]$$

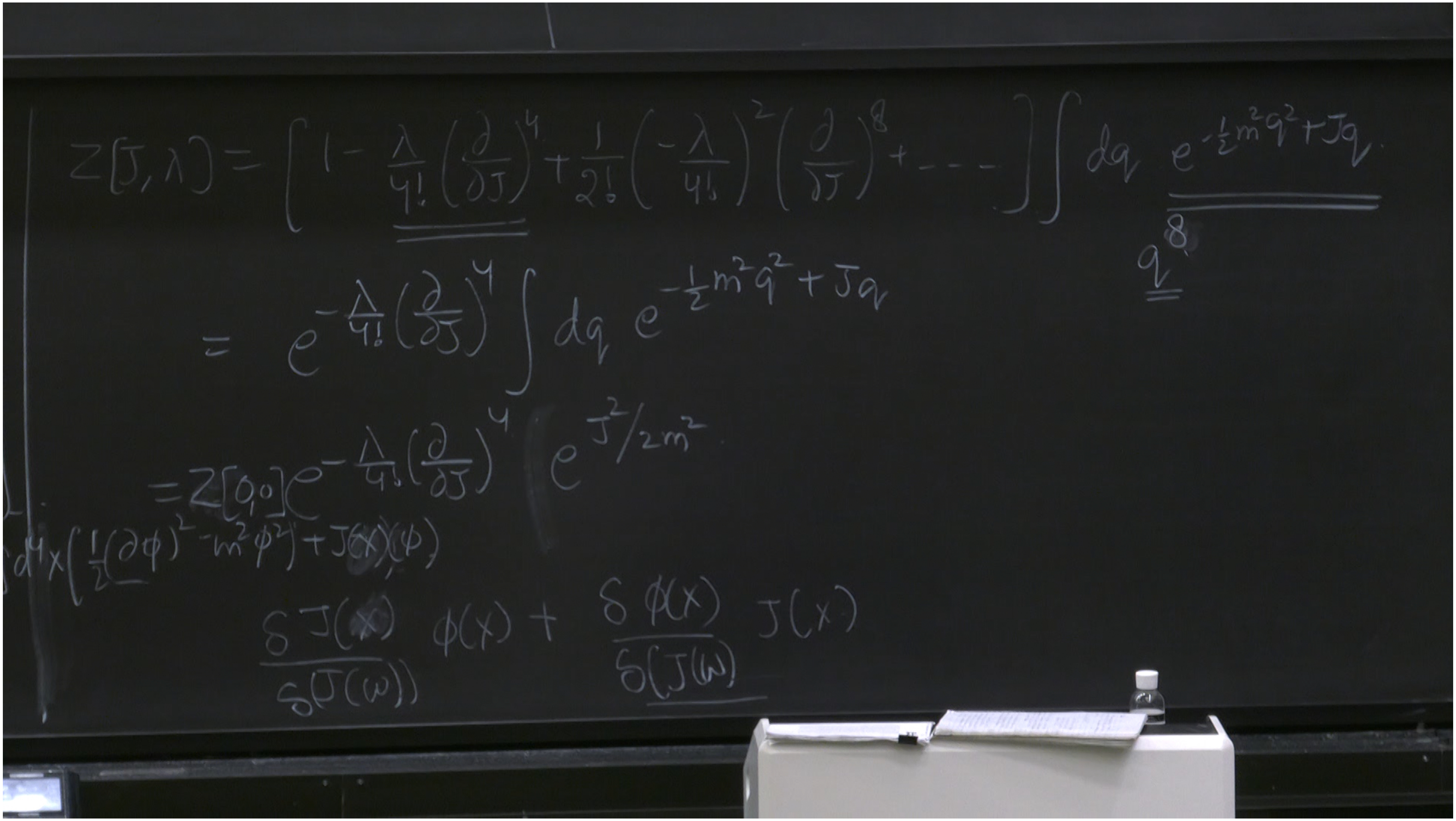
$$Z(J,\lambda) = \int D\phi \; e^{i\int d^4x \left[\frac{1}{2}(\partial\phi)^2 - m^2\phi^2\right) + J\phi} \left[1 - \frac{\lambda}{4!}\int d^4x\, \phi^4(x) + \left(-\frac{\lambda}{4!}\right)^2 \left[\int d^4x (\phi^4(x))\right]^2 + \right]$$

$$= \left[1 - \frac{i\lambda}{4!}\int d^4w \left(\frac{\delta}{i\delta(J(w))}\right)^4 + \left(\frac{-i\lambda}{4!}\right)^2 \left[\int d^4w \left(\frac{\delta}{i\delta(J(w))}\right)^4\right]^2 + \dots\right] \int D\phi \; e^{i\int d^4x \left(\frac{1}{2}(\partial\phi)^2 - m^2\phi^2\right)}$$

$$Z[J,\lambda] = ($$

$$= e$$

$$= Z[0,0]($$



$$Z[J,\lambda] = \left[1 - \underline{\underline{\frac{\lambda}{4!}\left(\frac{\partial}{\partial J}\right)^4}} + \frac{1}{2!}\left(-\frac{\lambda}{4!}\right)^2\left(\frac{\partial}{\partial J}\right)^8 + \cdots\right]\int dq\, \frac{e^{-\frac{1}{2}m^2q^2+Jq}}{\underline{\underline{q^8}}}$$

$$= e^{-\frac{\lambda}{4!}\left(\frac{\partial}{\partial J}\right)^4}\int dq\, e^{-\frac{1}{2}m^2q^2+Jq}$$

$$= Z[0,0]\, e^{-\frac{\lambda}{4!}\left(\frac{\partial}{\partial J}\right)^4}\, e^{J^2/2m^2}$$

$$\int d^4x\left(\frac{1}{2}(\partial\phi)^2 - m^2\phi^2\right) + J(x)\phi)$$

$$\frac{\delta J(x)}{\delta(J(\omega))}\phi(x) + \frac{\delta\phi(x)}{\delta(J(\omega)}J(x)$$