Title: Quantum Theory Lecture - 091423

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Collection: Quantum Theory 2023/24

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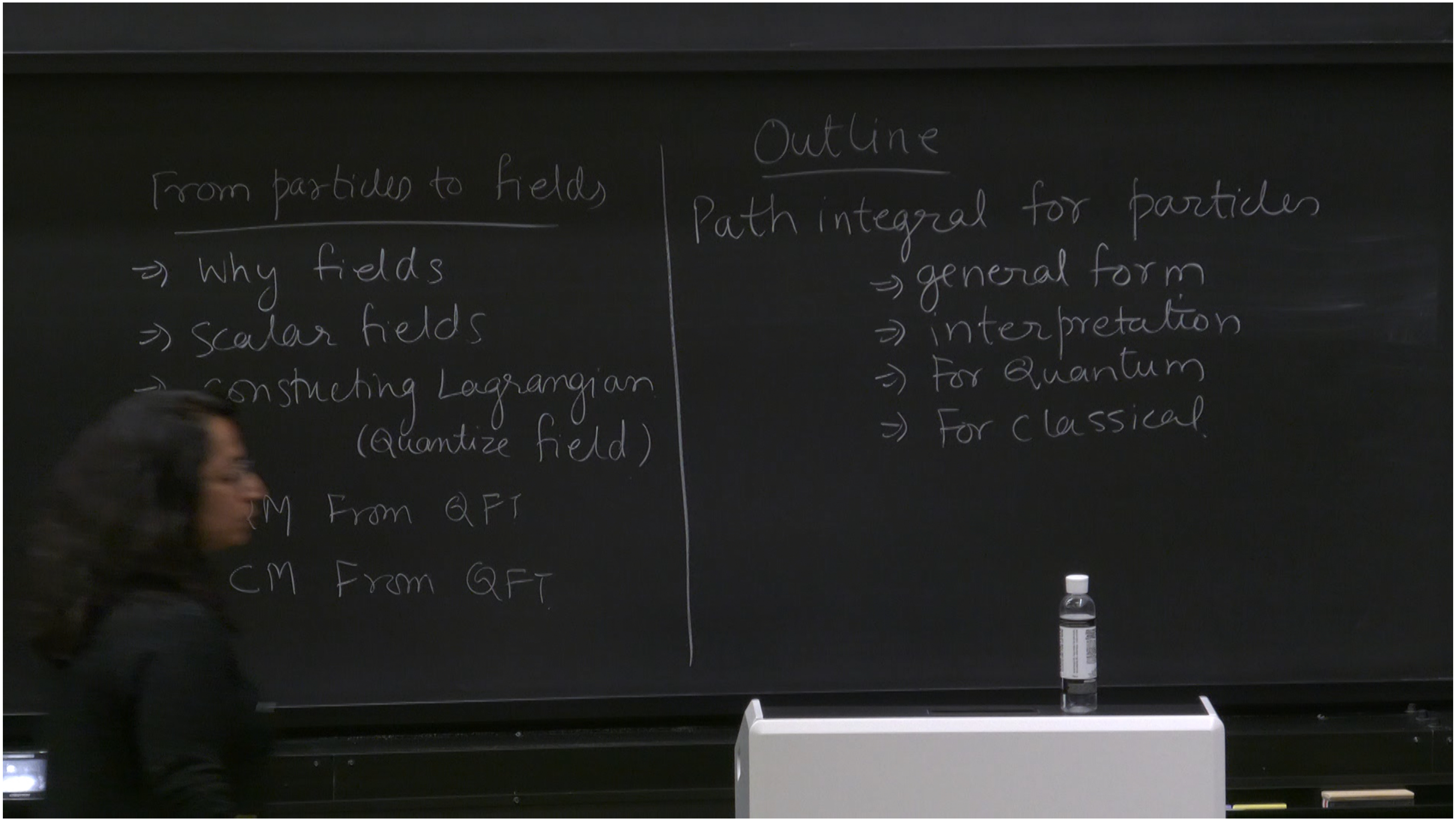
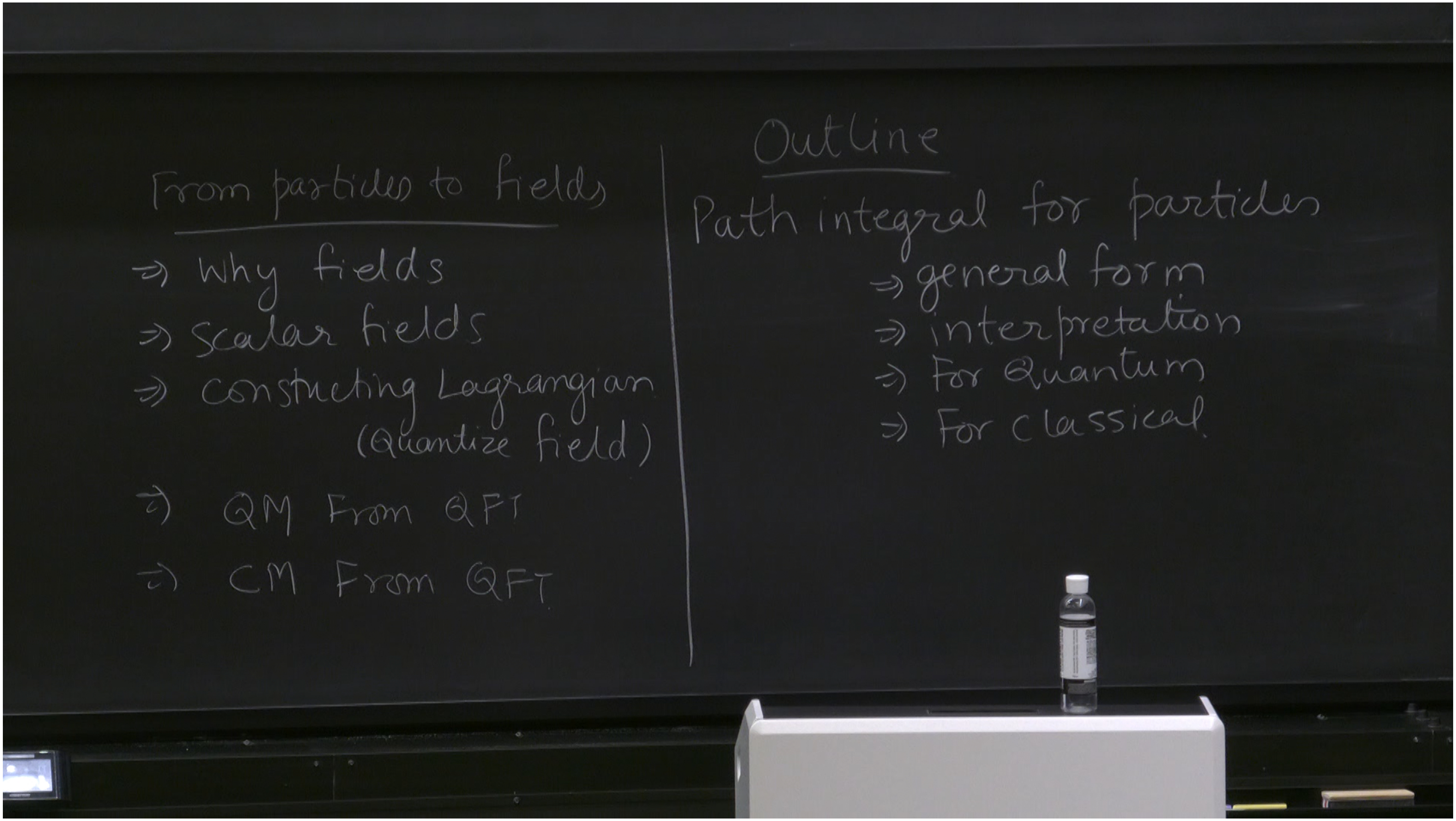

From particles to fields
⇒ Why fields
⇒ Scalar fields
⇒ constructing Lagrangian
(Quantize field)
QM From QFT
CM From QFT
Outline
Path integral for particles
⇒ general form
⇒ interpretation
⇒ For Quantum
⇒ For classical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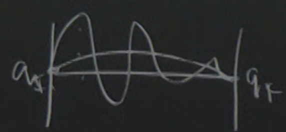
$\ldots^{iT}|q_I\rangle$



...me 'T' into N small segments $T = N\Delta t$

... $\int dq\, |q\rangle\langle q|$ at each time step.

$\ldots |q_I\rangle$

$\ldots dq_2 \ldots \int dq_{N-1}\ \langle q_F|e^{-i\hat{H}\Delta t}|q_{N-1}\rangle$

$\left(\prod \ldots dq_j\right)\quad \langle q_{N-1}|e^{-i\hat{H}\Delta t}|q_{N-2}\rangle \ldots\ldots$

$\langle q_1|e^{-i\hat{H}\Delta t}|q_I\rangle \longrightarrow \star$

...al factor

$\langle q_{j+1}|e^{-i\hat{H}\Delta t}|q_j\rangle,\quad \hat{H} = \frac{\hat{P}}{2m} + V(\hat{q})$

(II) $\langle q_{j+1}|e^{-i\left(\frac{\hat{p}^2}{2m} + V(\hat{q})\right)\Delta t}|q_j\rangle$

$$\left\{ \begin{array}{l} V(\hat{q})|q_j\rangle = V(q_j)|q_j\rangle \\ \frac{1}{2\pi}\int dp\,|p\rangle\langle p| \\ \langle p|q_j\rangle = e^{-ipq_j} \end{array} \right\}$$

$$= \frac{e^{-iV(q_j)\Delta t}}{2\pi}\int dp\, \langle q_{j+1}|e^{-i\frac{\hat{p}^2}{2m}\Delta t}|p\rangle\langle p|q_j\rangle$$

$$= \frac{e^{-iV(q_j)\Delta t}}{2\pi}\int dp\, e^{-i\frac{p^2}{2m}\Delta t}\, e^{ip(q_{j+1}-q_j)}$$

Gaussian integral

$$\int dx\, e^{-\frac{1}{2}ax^2 + Jx} = \sqrt{\frac{2\pi}{a}}\, e^{J^2/2a}$$

replace $J \to i(q_{j+1} - q_j)$

$a \to i\Delta t/m$

(III) $\langle q_{j+1}|e^{-i\hat{H}\Delta t}|q_j\rangle$

$$= \left(\frac{m}{2\pi i\Delta t}\right)^{1/2} e^{i\Delta t\left(m\frac{(q_{j+1}-q_j)^2}{2\Delta t^2} - V(q_j)\right)}$$

Substitute in $\star$

$\langle q_F|e^{-i\hat{H}T}|q_I\rangle$

$$= \left(\prod_{j=1}^{N-1}\int dq_j\right)\left[\frac{m}{2\pi i\Delta t}\right]^{N/2} e^{\overbrace{\sum_{j=0}^{N-1} \frac{im}{2}\dot{q}_j^2\Delta t - iV(q_j)}^{\int L(q,\dot{q})\,dt}}$$

Path Integral

For continuum limit $\delta t \to 0$, we get the path integral representation

$$\mathcal{A} = \langle q_F | e^{iHT} | q_I \rangle = \int Dq(t) e^{i \int_0^T dt L(\dot{q}, q)}$$ ➔ **Path Integral**

$$Z = \langle 0 | e^{iHT} | 0 \rangle = \int Dq(t) e^{i \int_0^T L(\dot{q}, q)\ dt}$$ ➔ **For I and F as ground state**

With $$\int Dq(t) = \lim_{N \to \infty} \left(\frac{im}{2\pi \Delta t} \right)^{\frac{N}{2}} \left(\Pi_{k=1}^{N-1} \int dq_k \right)$$

Example

Consider an **electron** translate from (0,0) to $q' = 1\text{cm},\ t' = 1\text{sec}$
with $m = 9.1 \times 10^{-31}\text{kg},\ \hbar = 1.05 \times 10^{-34}\text{m}^2\text{kg/sec}$

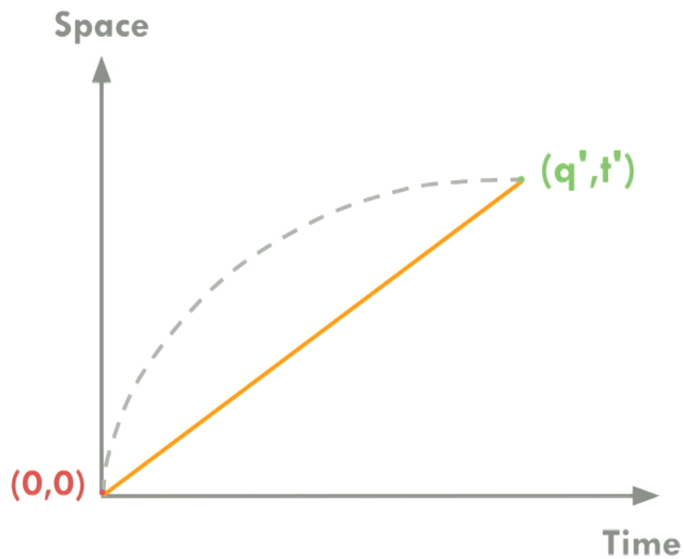
Straight Path $\frac{q}{t} = \frac{q'}{t'} = v$

$S_{st} =$ **?**

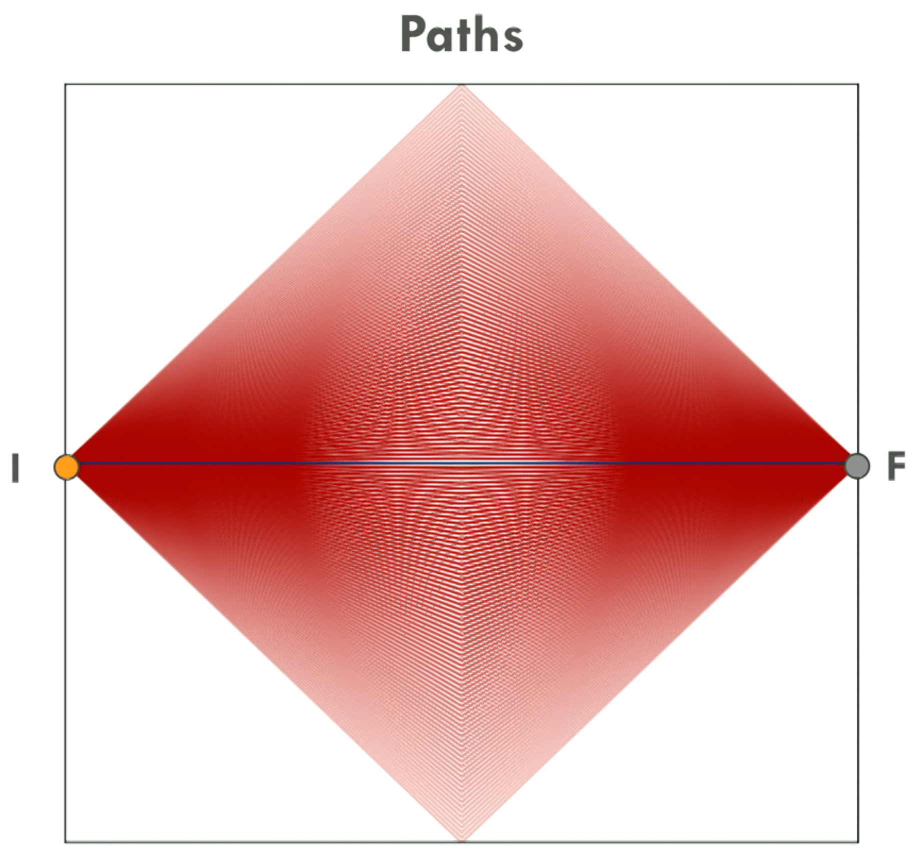
Parabolic Path $\frac{q}{t^2} = \frac{q'}{t'^2}$

$S_{pr} =$ **?**

Difference $\Delta S =$ **?**



For 100 distinct paths from S to O

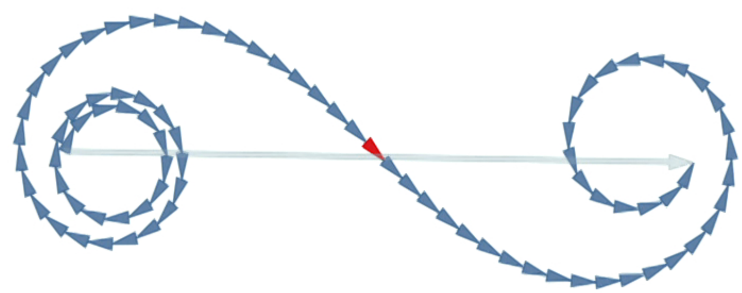


Summing Up Phases

Add all phases, connecting head to tail

- Phases of paths that **deviate minimally** from the classical path **contribute maximum** to the final amplitude

- Phases of paths **deviating significantly** from the classical path swirl around **canceling each other**



Example

Consider a **classical particle** translate from (0,0) to $q' = 1\text{cm}, \ t' = 1\text{sec}$
with $m = 1\text{g}, \ \hbar = 1.05 \times 10^{-34}\text{m}^2\text{kg/sec}$

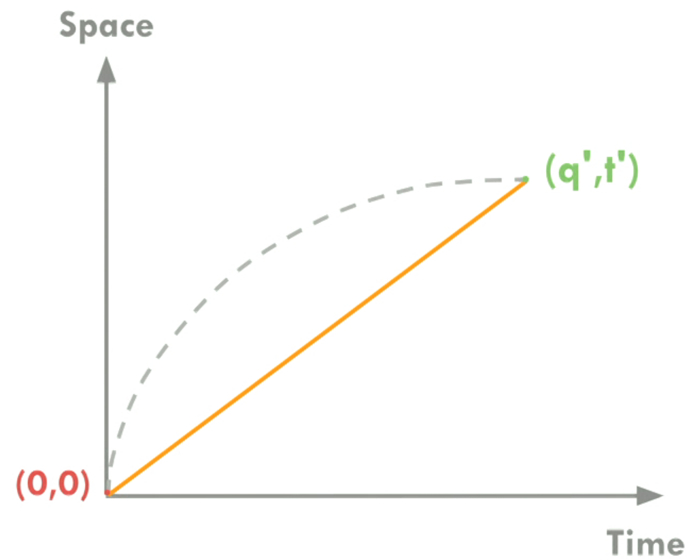
Straight Path $\frac{q}{t} = \frac{q'}{t'} = v$

$$S_{st} = \frac{1}{2}\frac{mq'^2}{t'}$$

Parabolic Path $\frac{q}{t^2} = \frac{q'}{t'^2}$

$$S_{pr} = \frac{2}{3}\frac{mq'^2}{t'}$$

Difference $\Delta S =$?



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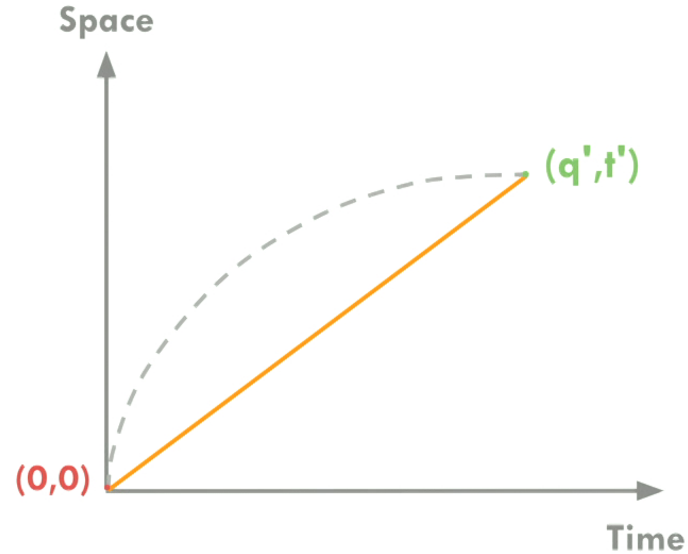
Difference $\Delta S = \frac{1}{6}\frac{mq'^2}{t'} \sim 10^{27}$

Random Phase



Classical Limit

If Quantum mechanics applies to all particles, why does a macroscopic particle appear to follow a specific path?

Classical Limit $S >> \hbar$

As S increases, phases of even slightly deviating paths from the classical trajectory swirl around and cancel out

Infinite paths reduces to one determined by Euler-Lagrange equation

?

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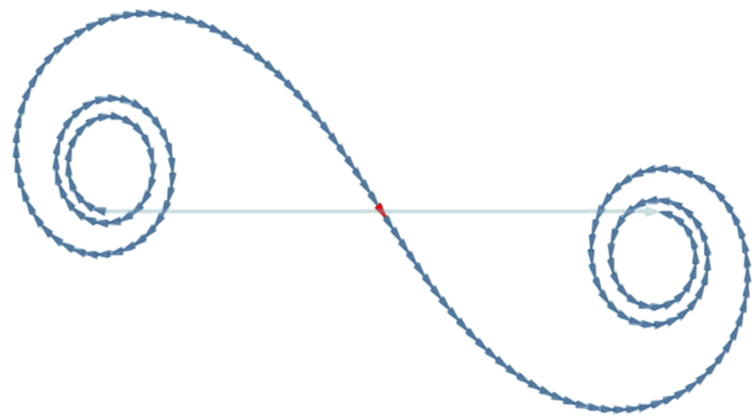

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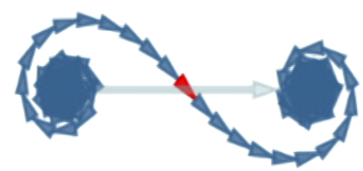


"A theory with mathematical beauty is more likely to be correct than an ugly one that fits some experimental data"

P.A.M. Dirac

Image Source : Wikipedia, Paul Dirac, en.wikipedia.org/wiki/Paul_Dirac

Problems with QM

1. **Particles can live and die**
   Particles never appear or disappear in QM

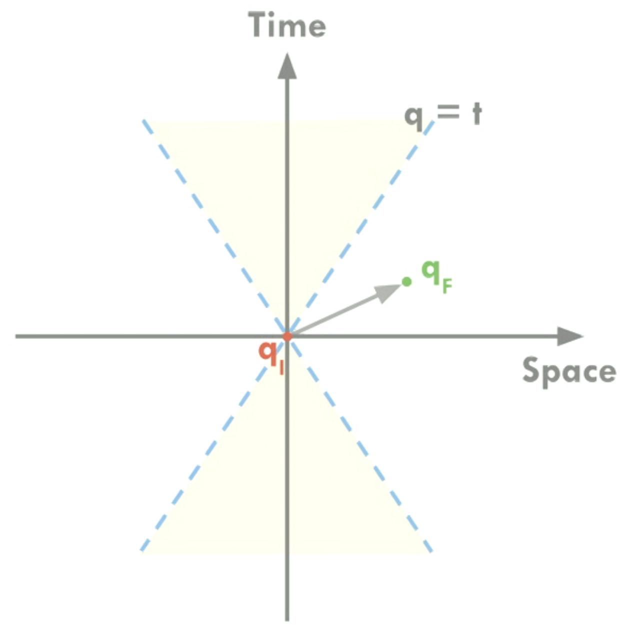
2. **Asymmetric treatment of space and time**
   Space is treated as observable and time as parameter

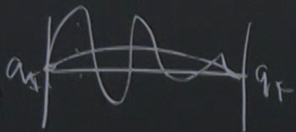
3. **Atom-Light Interaction**
   Light is treated as field and electron as particle

Faster Than Light

Consider a particle moves from q_I (at origin) to q_F (in region q>>t)

QM : $\langle q_F | e^{-i\hat{H}t} | q_I \rangle \propto e^{\frac{i\Delta q^2 m}{2t}} \neq 0$



$\langle q_F | \hat{A}(T)$ 

into N small segments $T = N\Delta t$

$|q\rangle\langle q|$ at each time step.

$$\int dq_{N-1} \langle q_F | e^{-i\hat{H}\Delta t} | q_{N-1}\rangle$$

$$\langle q_{N-1} | e^{-i\hat{H}\Delta t} | q_{N-2}\rangle \cdots\cdots$$

$$\langle q_1 | e^{-i\hat{H}\Delta t} | q_I\rangle \text{ — } \star$$

or

$$e^{-i\hat{H}\Delta t} | q_j\rangle, \quad \hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{q})$$

(II) $$\langle q_{j+1} | e^{-i\left(\frac{\hat{p}^2}{2m} + V(\hat{q})\right)\Delta t} | q_j\rangle$$

$$\left\{ \begin{aligned} & V(\hat{q})|q_j\rangle = V(q_j)|q_j\rangle \\ & \frac{1}{2\pi}\int dp\, |p\rangle\langle p| \\ & \langle p | q_j\rangle = e^{-ipq_j} \end{aligned} \right\}$$

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Gaussian integral

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Substitute in $\star$

$$\langle q_F | e^{-i\hat{H}T} | q_I\rangle = \left(\prod_{j=1}^{N-1} \int dq_j\right) \left[\frac{m}{2\pi i\Delta t}\right]^{N/2} e^{\overbrace{\sum_{j=0}^{N-1} i\frac{m}{2}\dot{q}_j^2\Delta t - iV(q_j)}}$$

$$\underbrace{\qquad\qquad}_{\int L(q,\dot{q})\,dt}$$

Why Coulomb potential in 1/r ?

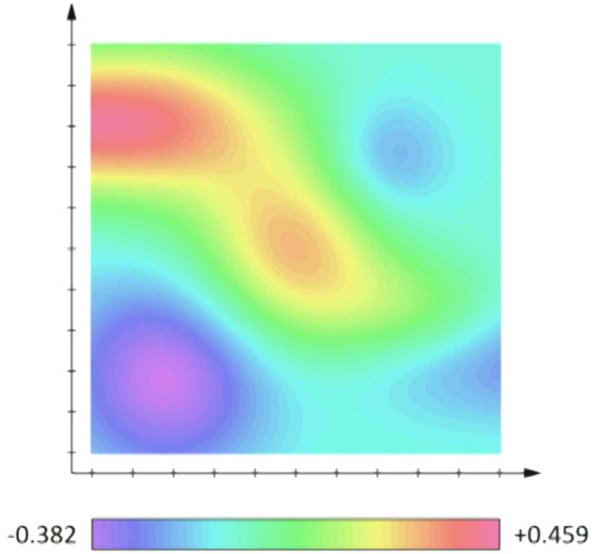
Why are all electron same ?

Why same charges repel ?

Field

Assign Scalar, Vector, Tensor, Spinor,or Corn, to each point in space

Corn Field



Temperature Field

A General Scalar Field

Assign scalar number to each point in space

Scalar Field $\phi(x) \in \mathbb{R}$

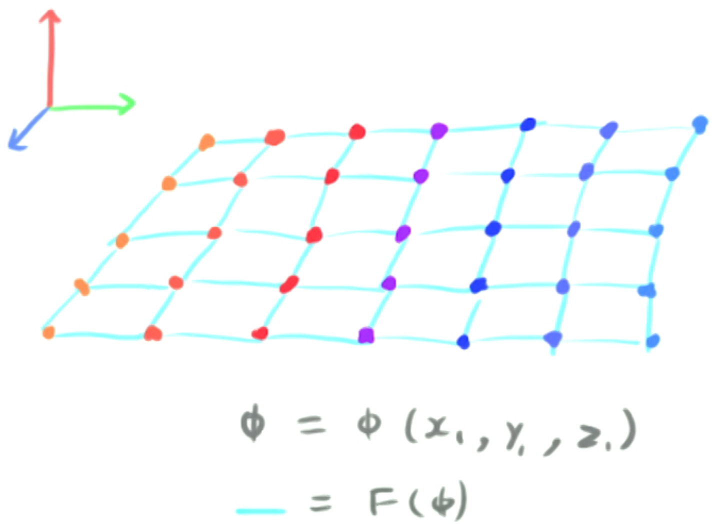
Potential Within the Field

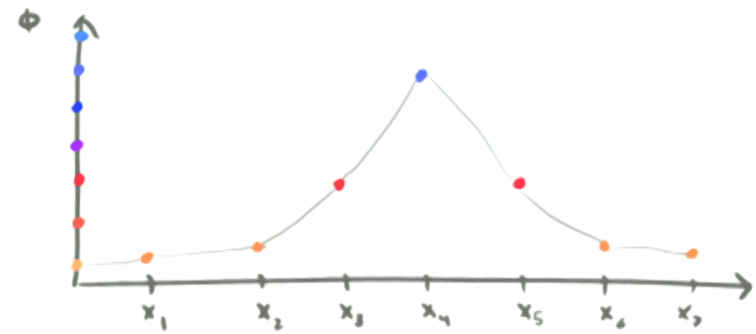
$$F(\phi) = \cancel{e} + \cancel{f\phi} + \frac{1}{2}m^2\phi^2 + \frac{g}{3!}\phi^3 + \frac{\lambda}{4!}\phi^4 + \cdots$$

Scale Factor (e)

Produces No Effect ($f\phi$)

- Imagine discretized space
- To each point is assigned a number represented by field value $\phi(x_i, y_i, z_i, t)$
- Each $\phi(x_i, y_i, z_i, t)$ is connected through a potential within the field
  (when value of $\phi(x_i, y_i, z_i, t)$ at one point in space changes it affects nearby)





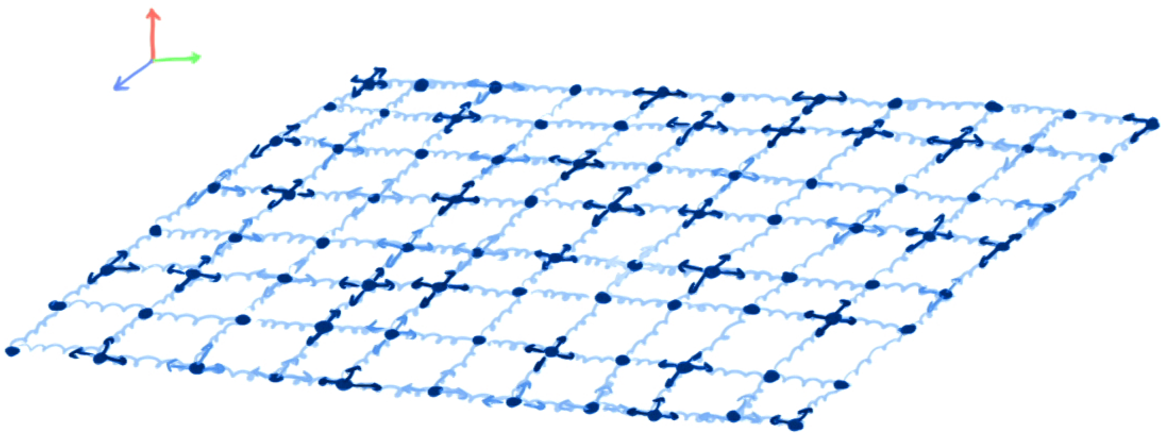
Path Integral

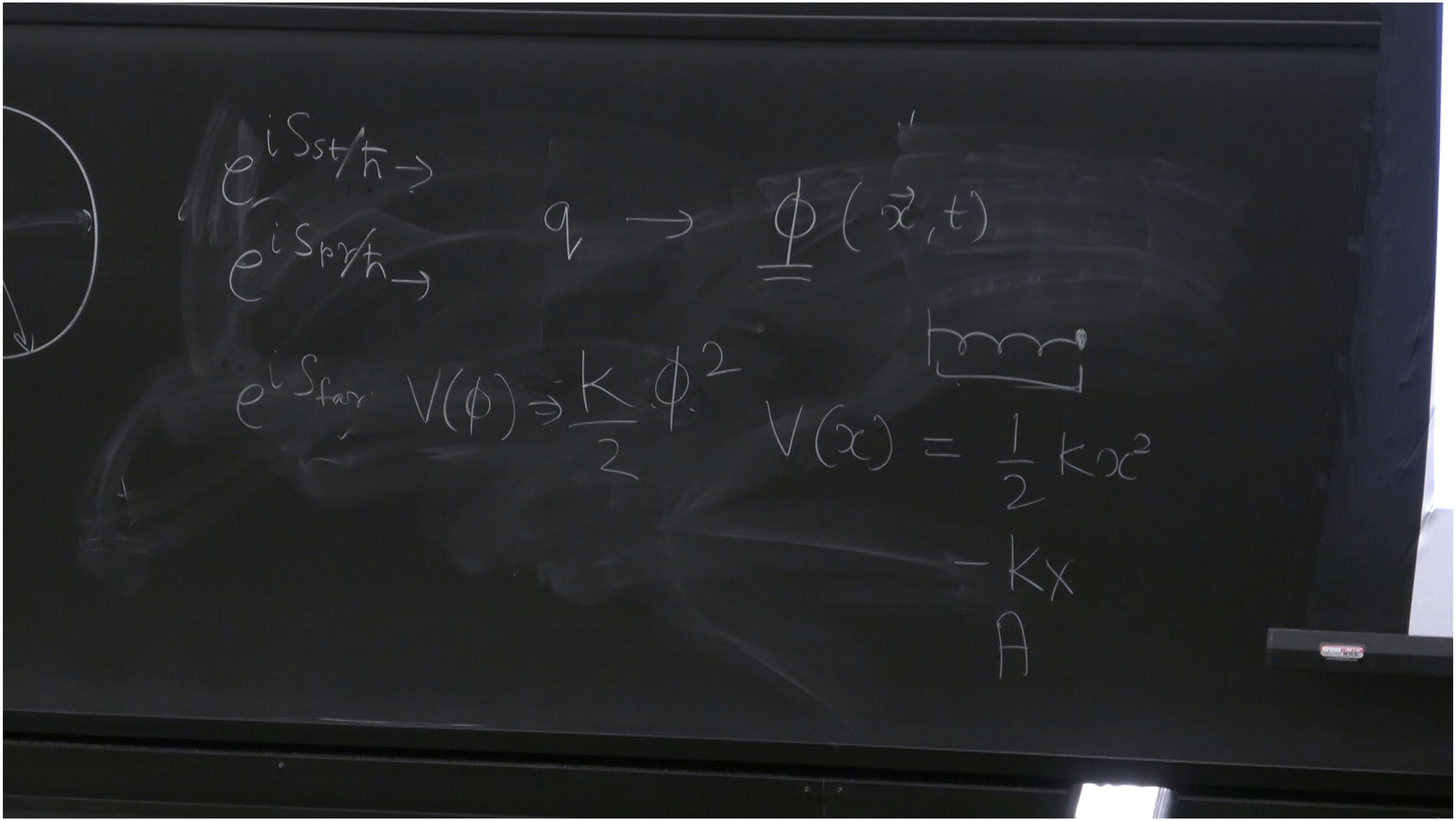
$$Z = \langle 0|e^{iHT}|0\rangle = \int Dq(t)e^{i\int_0^T L(\dot{q},q)dt}$$ → **For Particles**

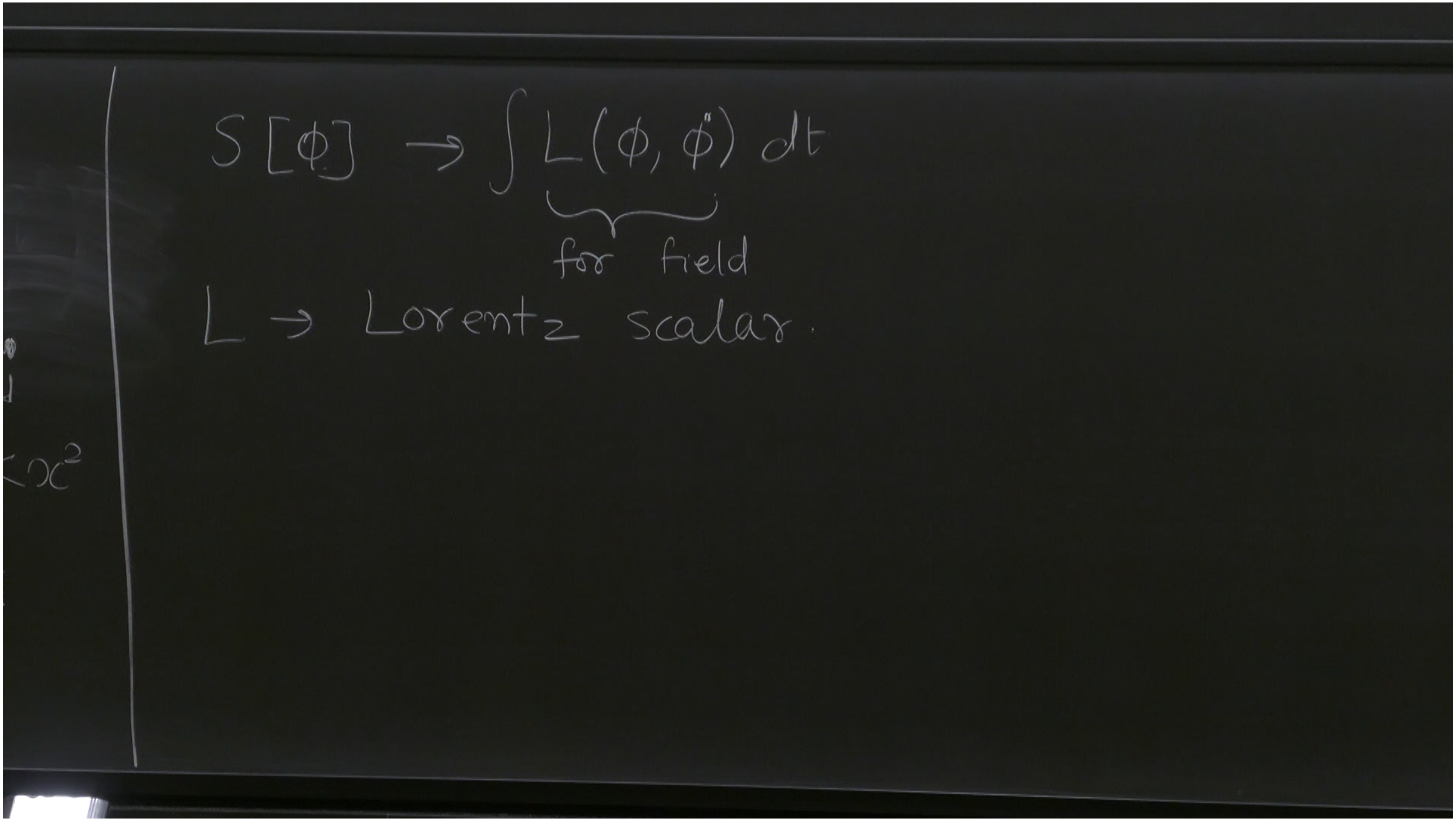
$$= \int D\phi e^{i\int_0^T L(\partial_t\phi,\phi)dt}$$ → **For Scalar Field**

↓ **Field Lagrangian**

A Bunch Of Oscillators




e^{iS_st/ħ} →
e^{iS_pr/ħ} →
q → φ(x⃗, t)
e^{iS_far} V(φ) = k/2 φ²
V(x) = ½ kx²
−kx


$S[\phi] \rightarrow \int L(\phi, \dot{\phi})\, dt$
for field
$L \rightarrow$ Lorentz scalar.

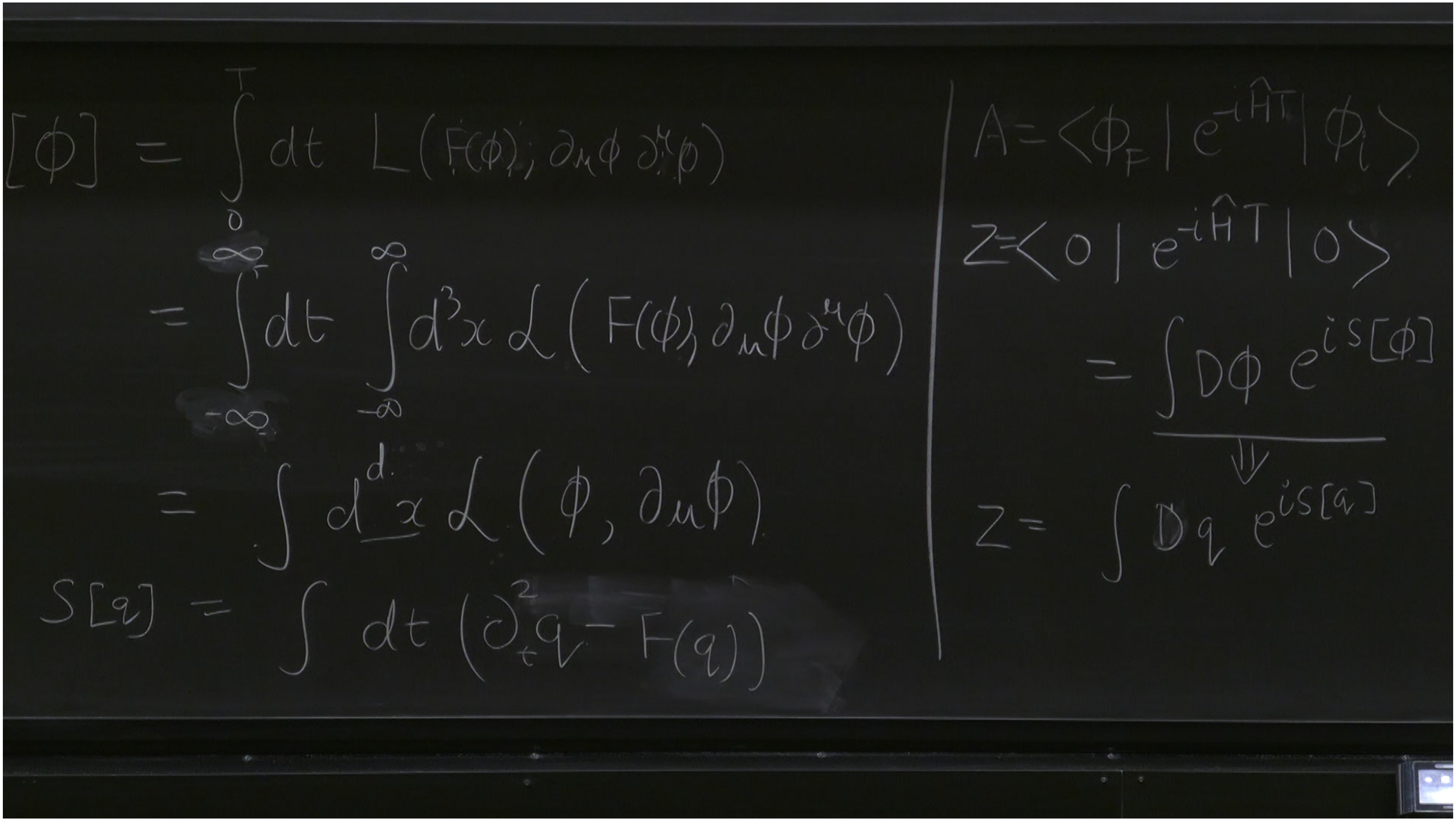
$$S[\phi] \rightarrow \int L(\phi, \dot{\phi})\, dt \longleftarrow L(q, \dot{q})$$

$$\underbrace{L(\phi, \dot{\phi})}_{\text{for field}} \qquad \dot{\phi} \rightarrow \partial_t \phi \rightarrow \partial_\mu \phi \rightarrow \underline{\partial_\mu \phi\, \partial^\mu \phi}$$

$L \rightarrow$ Lorentz scalar.

$$L(\phi, \dot{\phi}) \rightarrow \underbrace{\int d^3x\, \mathcal{L}\left(F(\phi), \partial_\mu \phi\, \partial^\mu \phi\right)}_{L}$$

$$\mathcal{L}\left(F(\phi), (\partial_\mu \phi)(\partial^\mu \phi)\right) = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - F(\phi)$$



$[\phi] = \int_0^T dt \; L(F(\phi), \partial_\mu\phi \, \partial^\mu\phi)$
$= \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} d^3x \, \mathcal{L}(F(\phi), \partial_\mu\phi \, \partial^\mu\phi)$
$= \int d^d\underline{x} \, \mathcal{L}(\phi, \partial_\mu\phi)$
$S[q] = \int dt \, (\partial_t^2 q - F(q))$
$A = \langle \phi_F | e^{-i\hat{H}T} | \phi_i \rangle$
$Z = \langle 0 | e^{-i\hat{H}T} | 0 \rangle$
$= \int D\phi \, e^{iS[\phi]}$
$\Downarrow$
$Z = \int Dq \, e^{iS[q]}$

$\hbar$

$= 0$

ϕ_c

$$S[\phi_c + \delta\phi] - S[\phi_c] = 0$$

$$\delta S \Rightarrow \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi} \delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\mu(\delta\phi) \right] = 0$$

$$\Rightarrow \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi} \delta\phi \right] + \cancelto{0}{\delta\phi \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Bigg|_{-\infty}^{\infty}} - \int d^4x \, \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi$$

$$\frac{\quad}{(\partial_\mu \phi)} \delta\phi$$

$$S[\phi] = \int d^4x \, \delta\phi \left[\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right] = 0$$

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}$$

$$\mathcal{L} = \frac{1}{2} \left((\partial_\mu \phi)(\partial^\mu \phi) - F(\phi) \right)$$

$$\mathcal{L} = \frac{1}{2}\left((\partial_\mu \phi)(\partial^\mu \phi) - \frac{m^2}{2}\phi^2 - \frac{\gamma}{3!}\phi^3 - \cdots\right]$$

$$\partial_\mu$$

$$\frac{\partial \mathcal{L}}{\partial \phi} = - m^2 \phi - \frac{\gamma}{2!}\phi^3 - \cdots$$

$$\partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right) = \partial_\mu\left[\frac{\partial}{\partial(\partial_\mu \phi)}(\partial_\alpha \phi)(\partial^\alpha \phi)\right)$$

$$g^{\alpha\beta}\partial_\beta \phi = \partial^\alpha \phi$$

$$\partial_\mu\left(\frac{\quad}{\partial(\partial_\mu\phi)}\right) \Rightarrow \frac{1}{2}\left[\partial_\mu\left(\partial^\mu\phi + \partial^\mu\phi\right)\right] = \partial_\mu\partial^\mu\phi = \partial^2\phi$$

$$\boxed{\partial^2\phi + m^2\phi + \frac{\gamma}{2!}\phi^2 + \cdots = 0}$$

Free Field Theory

The QFT integral defies resolution except for harmonic approximation

$$\mathcal{L}(\phi) = \frac{1}{2}\left[(\partial\phi)^2 - m^2\phi^2\right]$$ **Free Field Lagrangian**

Equation of motion for this lagrangian is

$$\left(\partial^2 + m^2\right)\phi(x) = 0$$ **Klein Gordon Equation**

Source

We add source to disturb the field

$J(t, \vec{x}) \longrightarrow$ **Source Function**

Free Field Integral With Added Source

$$Z = \int D\phi e^{i \int d^d x \left((\partial\phi)^2 - F(\phi) + J(x)\phi(x) \right)}$$

Image Source : Doodlewash, Skipping Stones, doodlewash.com/skipping-stones/