Title: Quantum Theory Lecture - 091323

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Density matrix/operator

$$\rho(t_0) = |\psi(t_0)\rangle \langle \psi(t_0)|$$

$$\rho(t_0) = \sum_i W_i |\psi_i(t_0)\rangle \langle \psi_i(t_0)|$$

Time evolution

$$|\psi_i(t)\rangle = U(t,t_0)|\psi_i(t_0)\rangle$$

$$\rho(t_0) = \sum_i W_i \, U(t,t_0)\, \rho(t_0)\, U^\dagger(t,t_0)$$

Equation of motion

## Equation of motion

$$\dot{\rho}(t) = \dot{U}(t,t_0)\rho(t_0)U^\dagger(t,t_0) + U(t,t_0)\rho(t_0)\dot{U}^\dagger(t,t_0)$$

$$= \frac{1}{i\hbar} H \underbrace{U(t,t_0)\rho(t_0)U^\dagger(t,t_0)}_{\rho(t)} - \frac{1}{i\hbar}\underbrace{U(t,t_0)\rho(t_0)U^\dagger(t,t_0)}_{\rho(t)} H$$

$$\boxed{i\hbar \frac{d\rho(t)}{dt} = [H, \rho(t)]} \rightarrow \text{Liouville equation}$$

$$\frac{d}{dt}\langle\hat{A}\rangle = \frac{d}{dt}\mathrm{Tr}(\rho\hat{A}) = \mathrm{Tr}\left(\frac{d\rho}{dt}\hat{A}\right) = \frac{1}{i\hbar}\mathrm{Tr}\left(\hat{A}(H\rho - \rho H)\right)$$

$\hat{A}$ ↓ time independent

$$U^\dagger(t,t_0) + U(t,t_0)\rho(t_0)\,\dot{U}^\dagger(t,t_0)$$

$$t_0)\underbrace{U^\dagger(t,t_0)}_{\rho(t)} - \frac{1}{i\hbar}\underbrace{U(t,t_0)\rho(t_0)U^\dagger(t,t_0)}_{\rho(t)}H$$

→ Liouville equation

$$\mathrm{Tr}\left(\frac{d\rho}{dt}\hat{A}\right) = \frac{1}{i\hbar}\mathrm{Tr}\left(\hat{A}(H\rho-\rho H)\right)$$

ependent

$$\boxed{i\hbar\dot{U} = HU}$$

$$\frac{1}{i\hbar}\mathrm{Tr}\left((\hat{A}\hat{H}-H\hat{A})\rho\right)$$

$$\boxed{\frac{d\langle\hat{A}\rangle}{dt} = \frac{1}{i\hbar}\mathrm{Tr}\left([\hat{A},\hat{H}]\rho\right)}$$

## Decoherences

Construct density operator for spin-$\frac{1}{2}$ system

2 subsystems

$1^{st} \rightarrow N_a\, e^-$ $\quad |\psi_a\rangle \Rightarrow \frac{N_a}{N_a + N_b} = W_a$

$2^{nd} \rightarrow N_b\, e^-$ $\quad |\psi_b\rangle \Rightarrow \frac{N_b}{N_a + N_b} = W_b$

$$\rho = W_a |\psi_a\rangle\langle\psi_a| + W_b |\psi_b\rangle\langle\psi_b|$$

$$|\psi_a\rangle = C_0^a|0\rangle + C_1^a|1\rangle$$

$$|\psi_b\rangle = C_0^b|0\rangle + C_1^b|1\rangle$$

# Decoherences

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$$\rho = W_a |\psi_a\rangle\langle\psi_a| + W_b |\psi_b\rangle\langle\psi_b|$$

$$|\psi_a\rangle = C_0^a|0\rangle + C_1^a|1\rangle$$

$$|\psi_b\rangle = C_0^b|0\rangle + C_1^b|1\rangle$$

$$\rho = \begin{bmatrix} \underline{W_a|C_0^a|^2 + W_b|C_0^b|^2} & W_a C_0^a C_1^{a*} + W_b C_0^b C_1^{b*} \\ W_a C_0^{a*} C_1^a + W_b C_0^{b*} C_1^b & \underline{W_a|C_1^a|^2 + W_b|C_1^b|^2} \end{bmatrix}$$

$c_1^a|1\rangle \Rightarrow N_a$

$c_1^b|1\rangle \Rightarrow N_b$

$W_a c_0^a c_1^{a*} + W_b c_0^b c_1^{b*}$

$W_a|c_1^a|^2 + W_b|c_1^b|^2$

Measurement

diagonal matrix elements $\Rightarrow$ population

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + e^{i\phi}|1\rangle)$$

Case I

$$\frac{1}{2}|0\rangle \quad \frac{1}{2}|1\rangle \Rightarrow \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

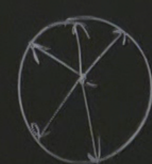
Case I $\quad |\psi\rangle = \frac{|0\rangle}{\sqrt{2}} + \frac{e^{i\phi}}{\sqrt{2}}|1\rangle$

$$\rho = \begin{bmatrix} 1 & e^{-i\phi} \\ e^{i\phi} & 1 \end{bmatrix}$$

Case II $\quad |\psi_1\rangle = \frac{|0\rangle}{\sqrt{2}} + e^{i\phi_1}\frac{|1\rangle}{\sqrt{2}}$

$$|\psi_n\rangle = \frac{|0\rangle}{\sqrt{2}} + \frac{e^{i\phi_n}|1\rangle}{\sqrt{2}}$$

$$\rho = \frac{1}{2}\begin{bmatrix} 1 & \frac{\sum_j e^{-i\phi_j}}{2n} \to 0 \\ 0 \Leftarrow \frac{\sum_j e^{i\phi_j}}{2n} & 1 \end{bmatrix}$$



1) not well prepared system

2) system are interacting with env.

$$\rho(t_0) = \begin{bmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{bmatrix} \qquad |\psi\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

$$\sigma_z \Rightarrow |\psi\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

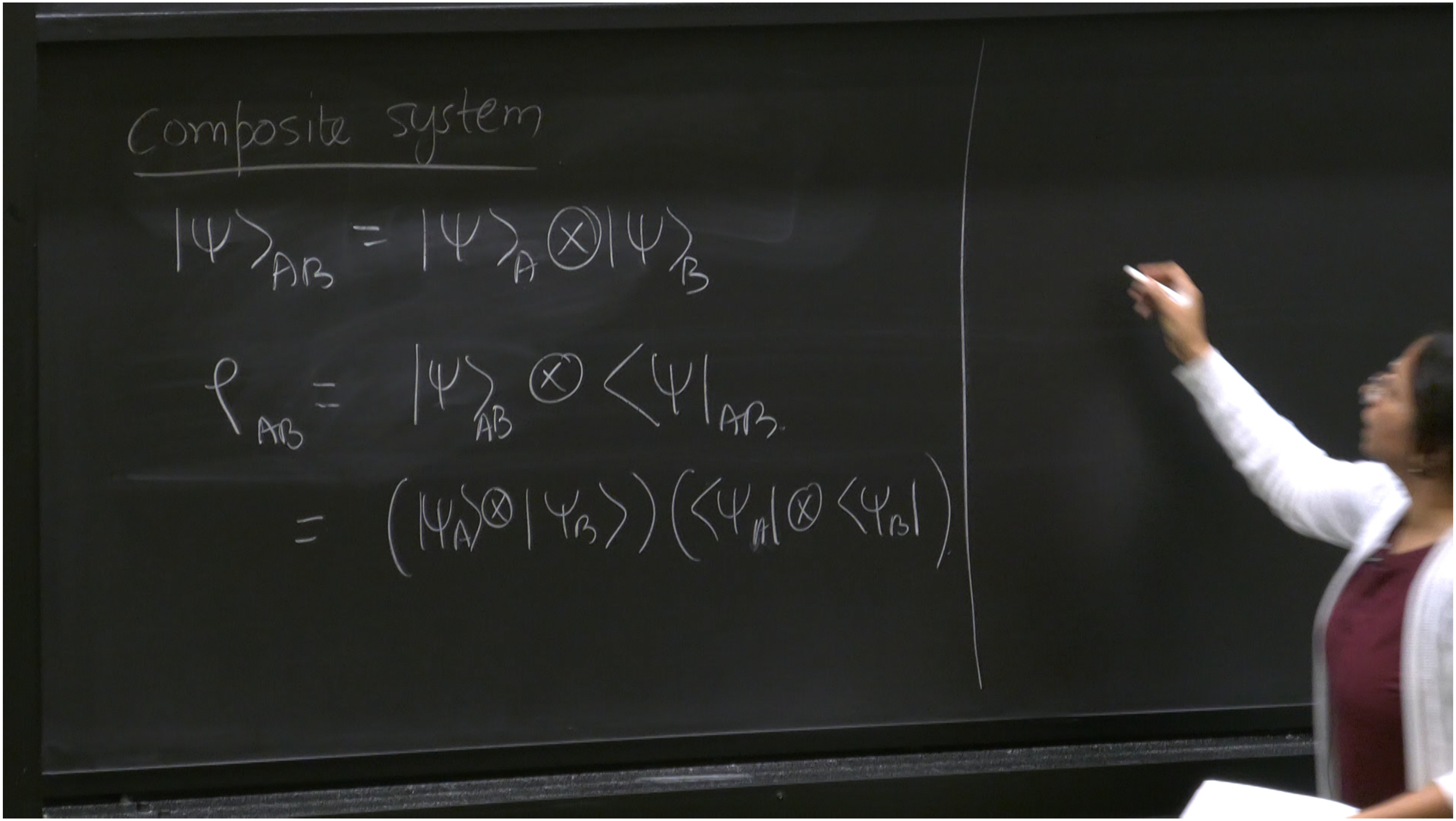
$$\rho' = \begin{bmatrix} \rho_{00} & -\rho_{01} \\ -\rho_{10} & \rho_{11} \end{bmatrix}$$

random unitary op. $U$

1) with prob $p$, perfor $\sigma_z$

2) with prob $(1-p)$, do nothing

$$\rho(t) = p\,\sigma_z\,\rho(t_0)\,\sigma_z^\dagger + (1-p)\,I\,\rho(t_0)\,I$$


Composite system
$|\Psi\rangle_{AB} = |\Psi\rangle_A \otimes |\Psi\rangle_B$
$\rho_{AB} = |\Psi\rangle_{AB} \otimes \langle\Psi|_{AB}$
$= (|\Psi_A\rangle \otimes |\Psi_B\rangle)(\langle\Psi_A| \otimes \langle\Psi_B|)$

## Composite system

$$|\psi\rangle_{AB} = |\psi\rangle_A \otimes |\psi\rangle_B$$

$$\rho_{AB} = |\psi\rangle_{AB} \otimes \langle\psi|_{AB}$$

$$= \left(|\psi_A\rangle \otimes |\psi_B\rangle\right)\left(\langle\psi_A| \otimes \langle\psi_B|\right)$$

$$|\psi_A\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad |\psi_B\rangle = \left[\right.$$

$|0\rangle$ $|1$

## Composite system

$$|\psi\rangle_{AB} = |\psi\rangle_A \otimes |\psi\rangle_B$$

$$\rho_{AB} = |\psi\rangle_{AB} \langle\psi|_{AB}$$

$$= \left(|\psi_A\rangle \otimes |\psi_B\rangle\right)\left(\langle\psi_A| \otimes \langle\psi_B|\right)$$

$$= \left(|\psi\rangle_A \langle\psi_A| \otimes |\psi_B\rangle\langle\psi_B|\right) = \rho_A \otimes \rho_B$$

$$|\psi_A\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad |\psi_B\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$|0\rangle \qquad\qquad |1\rangle$$

$$|\psi\rangle_{AB} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad \rho_{AB} = ($$

$$|\psi_A\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad |\psi_B\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$|0\rangle \qquad\qquad |1\rangle$$

$\psi\rangle_B$

$$|\psi\rangle_{AB} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad \rho_{AB} = \begin{bmatrix} 0 & 0 & & \\ & 1 & & \\ & & & \\ & & & \end{bmatrix}_{4\times 4}$$

$\langle\psi|_{AB}$

$\rangle)(\langle\psi_A| \otimes \langle\psi_B|)$

$$\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}$$

$A| \otimes |\psi_B\rangle\langle\psi_B| = \rho_A \otimes \rho_B$

$$|\psi_B\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$|1\rangle$

$$\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \rho_{AB} = \begin{bmatrix} 0 & 0 & & \\ & 1 & & \\ & & & \\ & & & \end{bmatrix}_{4\times 4}$$

$$\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}$$

Partial trace

Extracting information for A by tracing

$$Tr_B\left(\rho_{AB}\right) = \rho_A$$

$$Tr_B\left(|\psi_A\rangle\langle\psi_A| \otimes |\psi_B\rangle\langle\psi_B|\right)$$

$$\langle\psi_B|\psi_A\rangle\langle\psi_A| \otimes |\psi_B\rangle\langle\psi_B|\psi_B\rangle$$

$$\underbrace{|\psi_A\rangle\langle\psi_A|}_{\rho_A} \otimes \underline{\langle\psi_B|\psi_B\rangle \ \langle\psi_B|\psi_B\rangle}$$

Bell State $\Rightarrow \dfrac{|0\rangle^A|0\rangle^B + |1\rangle^A|1\rangle^B}{\sqrt{2}}$

$$\rho_{AB} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

$\rho_{AB} =$

$$\Rightarrow \frac{|0\rangle^A|0\rangle^B + |1\rangle^A|1\rangle^B}{\sqrt{2}}$$

$$\rho_{AB} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

$$\rho_{AB} = \frac{|0\rangle^A|0\rangle^B\langle 0|^A\langle 0|^B + |0\rangle^A|1\rangle^B\langle 0|^A\langle 1|^B + |1\rangle^A|0\rangle^B\langle 1|^A\langle 0|^B + |1\rangle^A|1\rangle^B\langle 1|^A\langle 1|^B}{2} \Rightarrow Tr_B(\rho^{AB})$$

$$= \langle 0|^B \rho^{AB} |0$$

$$\rho_A = Tr_B(\rho_{AB}) = \frac{1}{2}\left[|0\rangle^A\langle 0|^A + |1\rangle^A\langle 1|^A\right] = \frac{I}{2} \Rightarrow$$

$$|1\rangle^A |1\rangle^B$$

$$\begin{matrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{matrix}\Bigg]$$

$$\frac{\cdots^B + |0\rangle^A |1\rangle^B \langle 0|^A \langle 1|^B + |1\rangle^A |0\rangle^B \langle 1|^A \langle 0|^B + |1\rangle^A |1\rangle^B \langle 1|^A \langle 1|^B}{2} \Rightarrow Tr_B(\rho^{AB})$$

$$= \langle 0|^B \rho^{AB} |0\rangle^B + {}^B\langle 1|\rho^{AB}|1\rangle^B$$

$$= \frac{1}{2}\left[ |0\rangle^A \langle 0|^A + |1\rangle^A \langle 1|^A \right] = \frac{I}{2} \Rightarrow$$

$$\rho_B = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\rho_A \otimes \rho_B = \rho_{AB}$$

$$|1\rangle^B \langle 0|^A \langle 1|^B + |1\rangle^A |0\rangle^B \langle 1|^A \langle 0|^B + |1\rangle^A |1\rangle^B \langle 1|^A \langle 1|^B \Rightarrow \mathrm{Tr}_B(\rho^{AB})$$

$$= \langle 0|^B \rho^{AB} |0\rangle^B + {}^B\langle 1| \rho^{AB} |1\rangle^B$$

$$\frac{1}{2}\left[ |0\rangle^A {}^A\langle 0| + |1\rangle^A {}^A\langle 1| \right] = \frac{I}{2} \Rightarrow$$

*"Everything that can happen, does happen. But this does not mean everything happens. The rest of physics is about describing what can happen and what can't"*

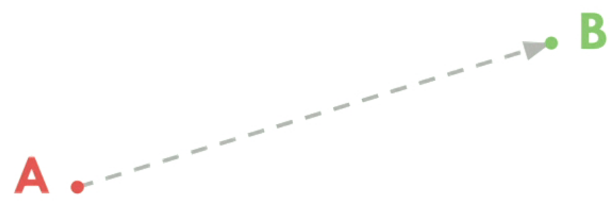
*Antony Garrett Lisi*

Image Source : The New Yorker, Surfing the Universe, www.newyorker.com/magazine/2008/07/21/surfing-the-universe

# Which path does a particle traveling from A to B take?

## Path of least distance

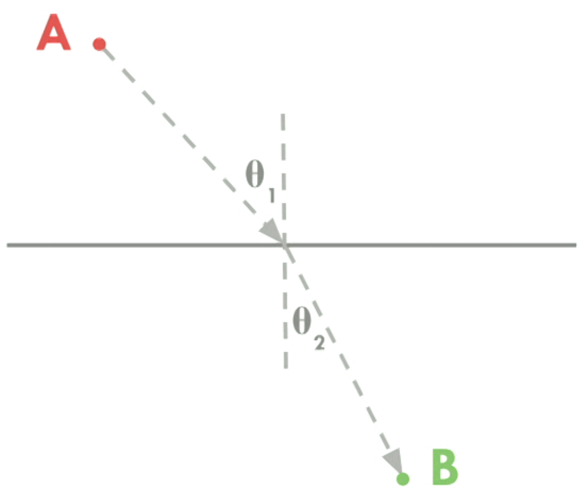
- The concept of "**least distance**" refers to the shortest path between two points
- This inherently implies motion in a **straight line**




## Path of least time (Fermat's Principle)

- The concept of "**least time**" denotes the route that minimizes the overall duration of travel
- Snell's law directly arises from the principle of least time

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{v_1}{v_2}$$



# Path of Least Action

## Nature operates in a manner that minimizes a fundamental quantity known as action

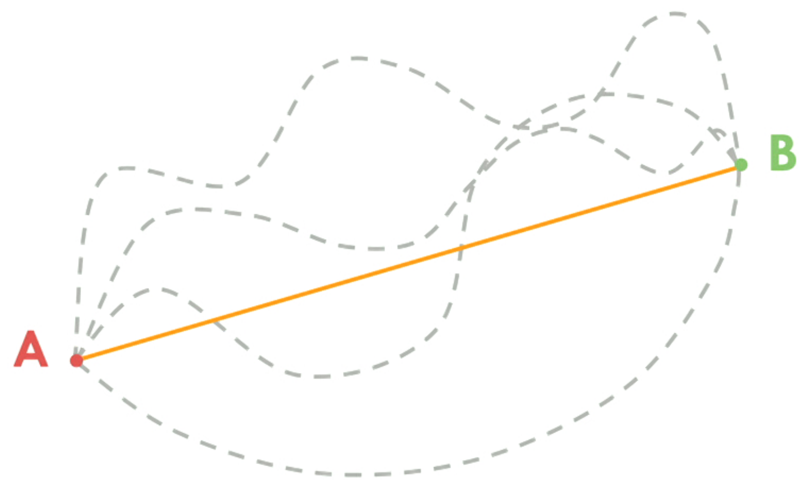
Particles follows the path where the action is stationary

$\delta S = 0$ ➔ **Stationary Action**

Where

$S = \int_{t_1}^{t_2} L dt$ ➔ **Action**

$L = T - V$ ➔ **Lagrangian**



# Why are some paths special ?

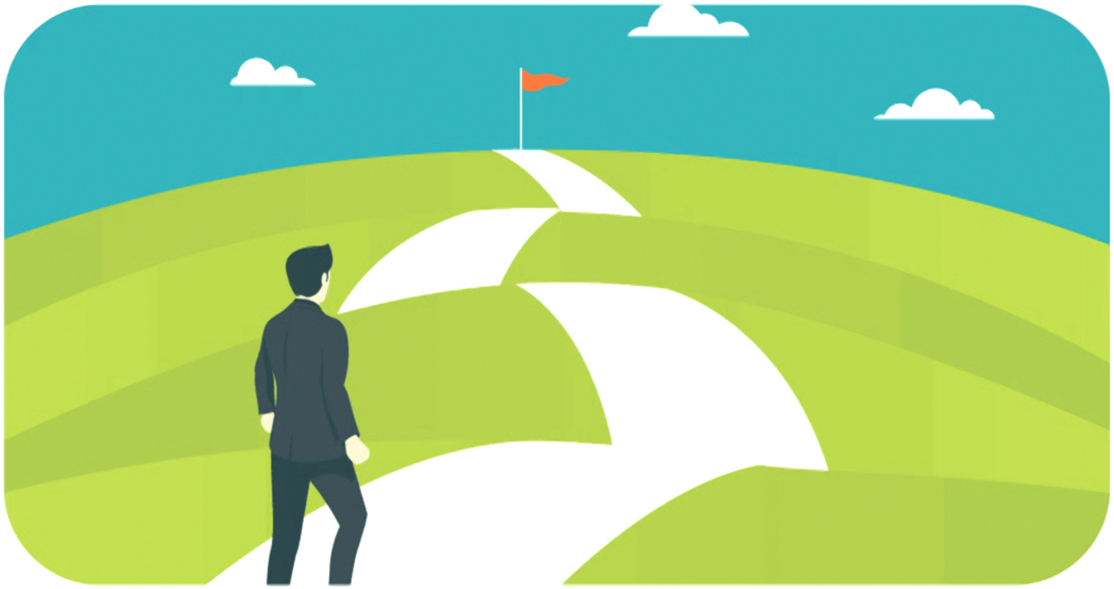


Image Source : Insperity, Engaging today's workforce, www.insperity.com/blog/employee-career-path/

# Quantum Perspective

## Double Slit to Multiple Slits

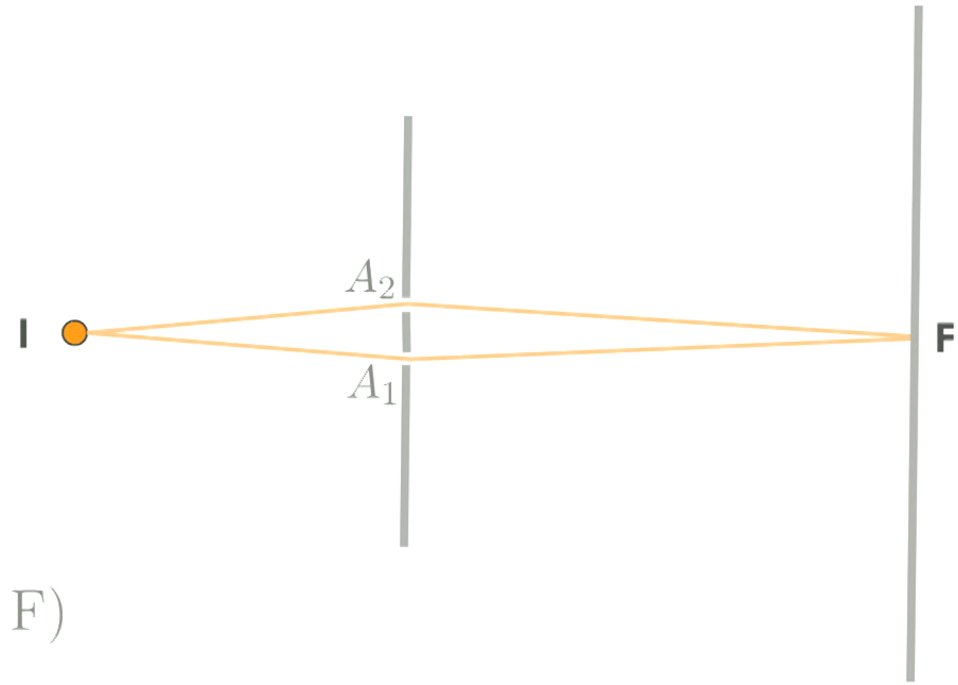
### In Quantum Mechanics

- No well-defined trajectory
- Only probabilities of finding the particle at locations

Probability Transition Amplitude from I to F is given by

**With two slits**

$$\mathcal{A}(\text{detected at F}) = \sum_{i=1}^{2} \mathcal{A}(\text{I} \to \text{A}_i \to \text{F})$$



# Quantum Perspective

## Double Slit to Multiple Slits

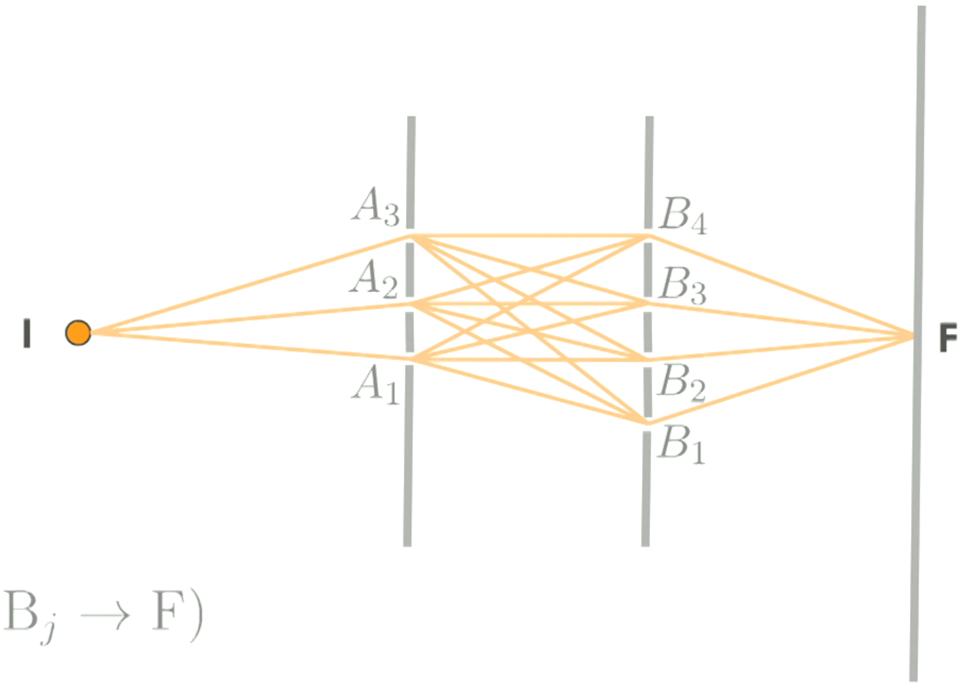
### In Quantum Mechanics

- No well-defined trajectory
- Only probabilities of finding the particle at locations

Probability Transition Amplitude from I to F is given by

**With seven slits and two screens**

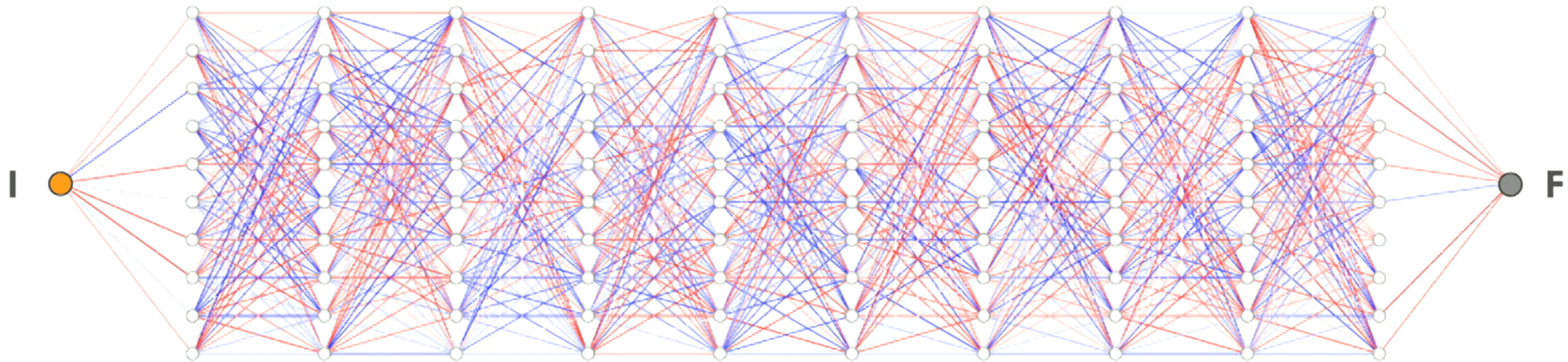
$$\mathcal{A}(\text{detected at F}) = \sum_{i,j} \mathcal{A}(\text{I} \to \text{A}_i \to \text{B}_j \to \text{F})$$



# Feynman's Idea

## Infinite slits and infinite screens

$$\mathcal{A}(\text{detected at F}) = \sum_{\text{infinite paths}} \mathcal{A}(\text{I} \to \text{F in time } T \text{ following a particular path})$$



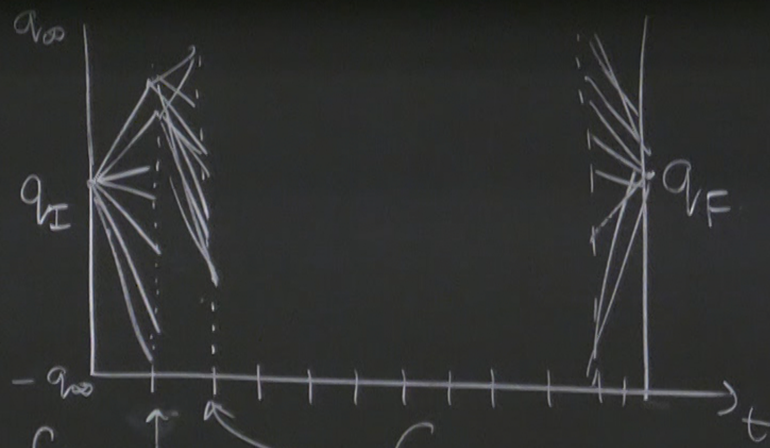
## Discretize Time $( \hbar = 1 )$

**Divide time T into N segments of duration T/N**

$$\mathcal{A} = \langle q_F | e^{-iHT} | q_I \rangle = \langle q_F | e^{-iH\Delta t} e^{-iH\Delta t} \cdots e^{-iH\Delta t} | q_I \rangle$$

**Inserting** $\int dq |q\rangle\langle q|$ **between exponentials**

$$\mathcal{A} = \langle q_F | e^{-iHT} | q_I \rangle = \left( \prod_{j=1}^{N-1} \int dq_j \right) \langle q_F | e^{-iH\Delta t} | q_{N-1} \rangle \langle q_{N-1} | e^{-iH\Delta t} | q_{N-2} \rangle \cdots \langle q_1 | e^{-iH\Delta t} | q_I \rangle$$




$$\int dq_1 |q_1\rangle\langle q_1| \quad \int dq_2 |q_2\rangle\langle q_2|$$

$$= \mathbb{1}$$

$$A = \langle q_F | e^{-iHT} | q_I \rangle$$

$$\underline{\underline{\hbar = 1}} \quad = \langle q_F | e^{-iH\Delta t} e^{-iH\Delta t} \dots\dots |q_I\rangle$$

$$= \langle q_F | e^{-iH\Delta t} \int dq_1 |q_1\rangle \langle q_1| e^{-iH\Delta} \int dq_2 |q_2\rangle$$

$$A = \langle q_F | e^{-iHT} | q_I \rangle$$

$\hbar = 1$
$$= \langle q_F | e^{-iH\Delta t} e^{-iH\Delta t} \cdots | q_I \rangle$$

$$= \langle q_F | e^{-iH\Delta t} \int dq_{N-1} |q_{N-1}\rangle\langle q_{N-1}| e^{-iH\Delta t} \int dq_{N-2} |q_{N-2}\rangle\langle q_{N-2}| \cdots \int dq_1 |q_1\rangle\langle q_1| e$$

$$= \left(\prod_{j=1}^{N-1} \int dq_j\right) \langle q_F | e^{-iH\Delta t} | q_{N-1}\rangle \langle q_{N-1} | e^{-i\Delta t} | q_{N-2}\rangle \cdots$$

$$-\frac{i\hat{p}}{2m}\Delta t\Big)\Big(1 - iV(\hat{q})\Delta t\Big) + O(t^2)$$

$$\underbrace{e^{-i\hat{p}^2/2m\,\Delta t}\; e^{-iV(\hat{q})\Delta t}}\ (\text{upto order in } \Delta t)$$

$$\langle q_{j+1}| e^{-i\hat{H}\Delta t}|q_j\rangle = e^{-iV(q)\Delta t}\int \langle q_{j+1}|e^{-i\hat{p}^2\Delta t/2m}|p\rangle\underbrace{\langle p|q_j\rangle}_{e^{-ipq_j}} dp$$

$$e^{ipq_{j+1}}$$

$$= e^{-iV(q)\Delta t}\int e^{ip(q_{j+1}-q_j)}\, e^{-ip^2\Delta t/2m}\, dp.$$

$$= \underbrace{q\left(\frac{-im}{2\pi\Delta t}\right)}_{B} e^{-iV(q_i)\Delta t}\, e^{+i\frac{m}{2}\Delta t\underbrace{\frac{(q_{i+1}-q_i)^2}{\Delta t^2}}_{\dot{q}_i}}$$

$$= B\; e^{i\underbrace{\left(\frac{m}{2}\dot{q}_i^2 - V(q_i)\right)}_{L(q_i,\dot{q}_i)}\Delta t}$$

$$\langle q_F | e^{-i\hat{H}t} | q_I \rangle = B \int Dq(t)\, e^{i \underbrace{\int L\, dt}_{S}}$$

$$\int Dq(t) = B \left( \prod_{j=1}^{N-1} \int dq_j \right)$$