Title: Quantum Theory Lecture - 091123

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Collection: Quantum Theory 2023/24

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Perturbative Solution

Approximation - **Weak Perturbation**

First Order Contribution $c_m^{(1)} = \sum_n \int_0^{t_f} dt' e^{i\omega_{mn}t'} \frac{V_{mn}}{i\hbar} c_n(0)$ **at** $t = 0\ ,\ c_n(0) = \delta_{n,i}$

Transition Probability for i $\longrightarrow$ f

$$|c_f^{(1)}(t)|^2 = P_{i\to f}(t) = \frac{V_{fi}^2}{(E_f - E_i)^2} \sin^2(\frac{\omega_{fi} t}{2})$$

Fermi's Golden Rule

Energy levels are not discrete but has a finite width

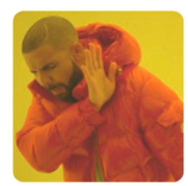
$$\sum_f P_{i\to f}(t) = \int P_{i\to f}(t)\, \overbrace{\rho(E_f)}^{\text{No. of states per unit energy around } E_f} dE_f$$

Over Entire Spectrum

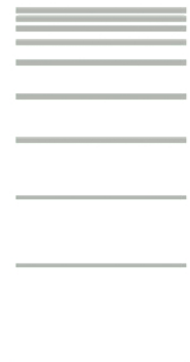
$$= \frac{2\pi |V_{fi}|^2 \rho(E_f)\, t}{\hbar}$$

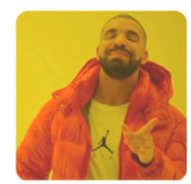
Rate of transition
Independent of time
Fermi's Golden Rule

Discrete Width



Finite Width

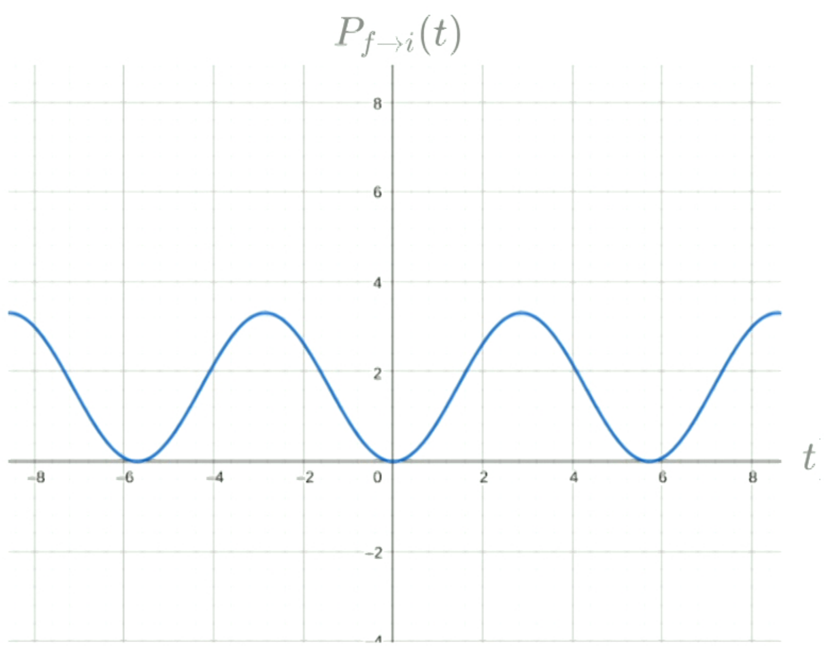


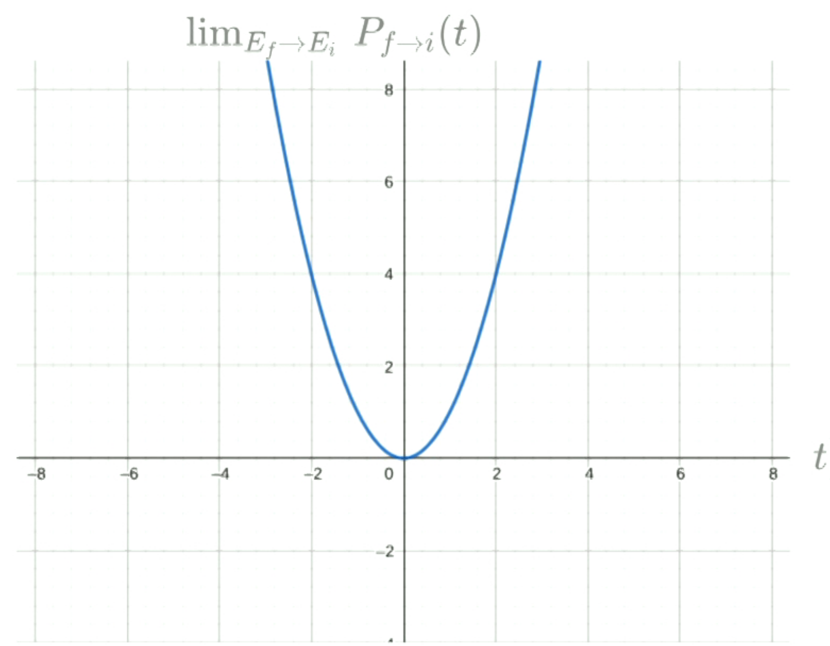


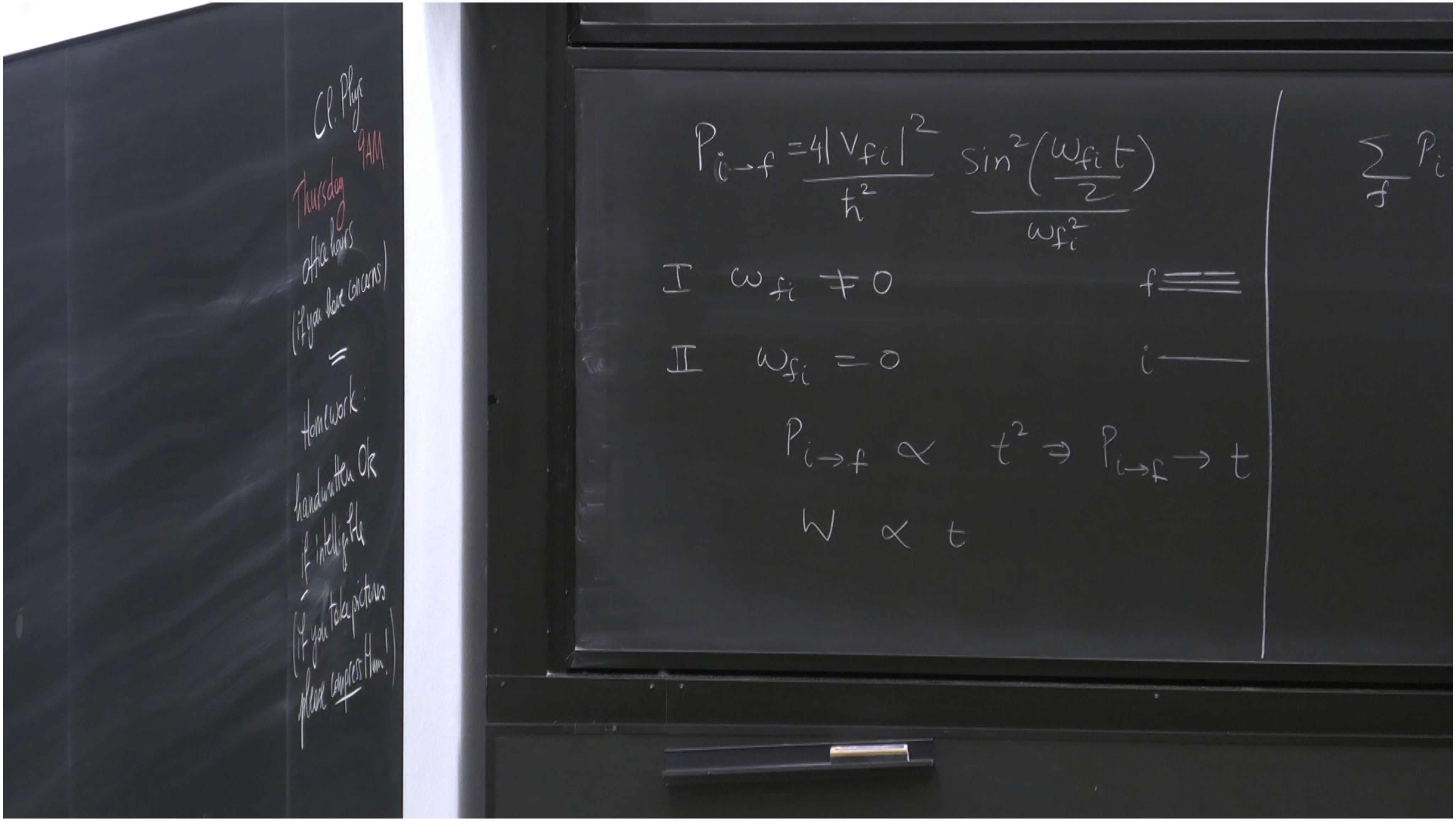
Variation Of Probability

Probability as a function of time

$$\lim_{E_f \to E_i} P_{f\to i}(t) = \frac{|V_{fi}|^2 t^2}{\hbar}$$



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Homework:
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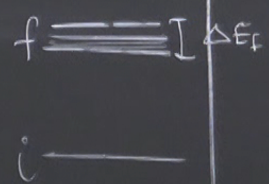
if intelligible
(if you take pictures
please compress them!)
$P_{i\to f} = \frac{4|V_{fi}|^2}{\hbar^2} \frac{\sin^2\left(\frac{\omega_{fi} t}{2}\right)}{\omega_{fi}^2}$
I $\omega_{fi} \neq 0$
II $\omega_{fi} = 0$
f
i
$P_{i\to f} \propto t^2 \Rightarrow P_{i\to f} \to t$
$W \propto t$
$\sum_f P_i$

$$|^2 \;\frac{\sin^2\left(\frac{\omega_{fi} t}{2}\right)}{\omega_{fi}^2}$$



$$\propto \; t^2 \Rightarrow P_{i\to f} \to t$$

$$\propto \; t$$

$$\sum_f P_{i\to f} = \int P_{i\to f}(t)\, \underbrace{\rho(\varepsilon_f)}_{\text{no. density}}\, d\varepsilon_f$$

no. of state per unit energy around ε_f

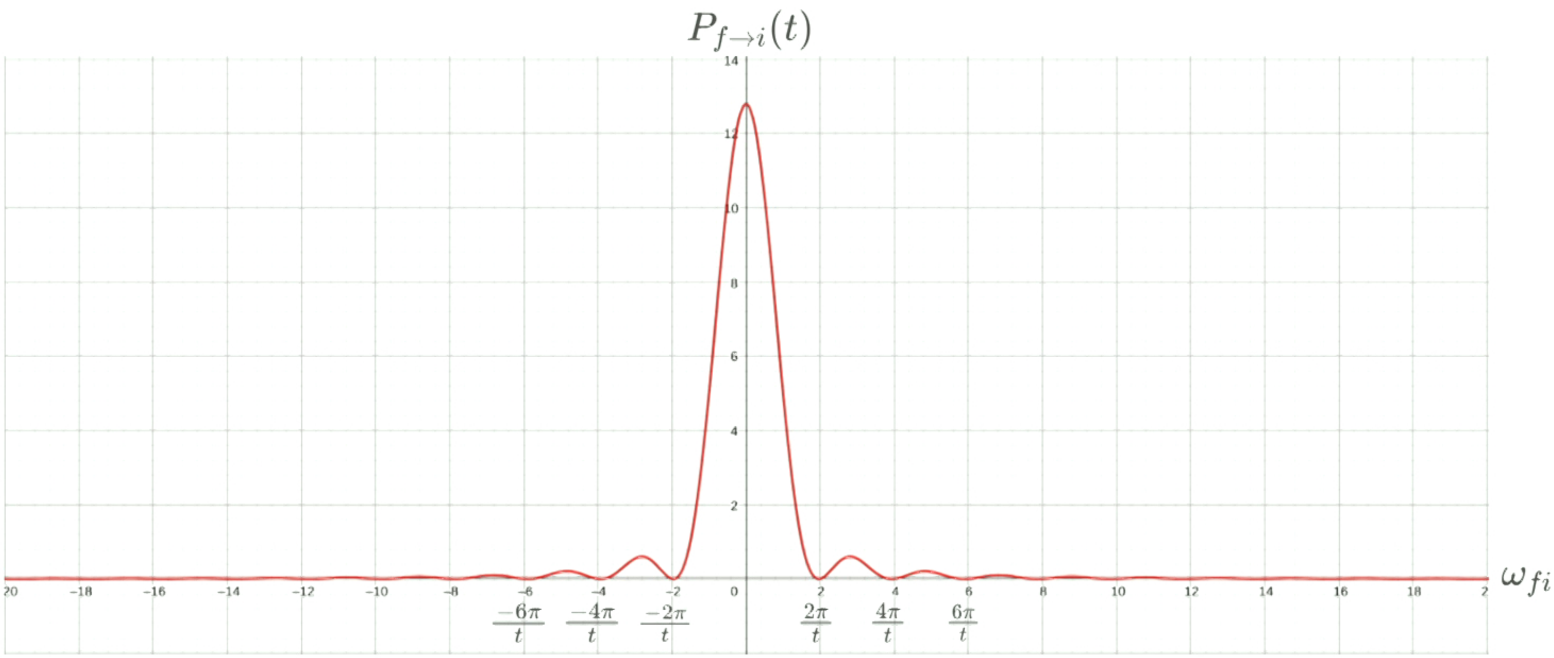
$$\sum_f P_{i\to f} = \int \frac{4|V_{fi}|^2}{\hbar^2}\, \frac{\sin^2\left(\frac{\omega_{fi} t}{2}\right)}{\omega_{fi}^2}\, \rho(\varepsilon_f)\, d\varepsilon_f$$

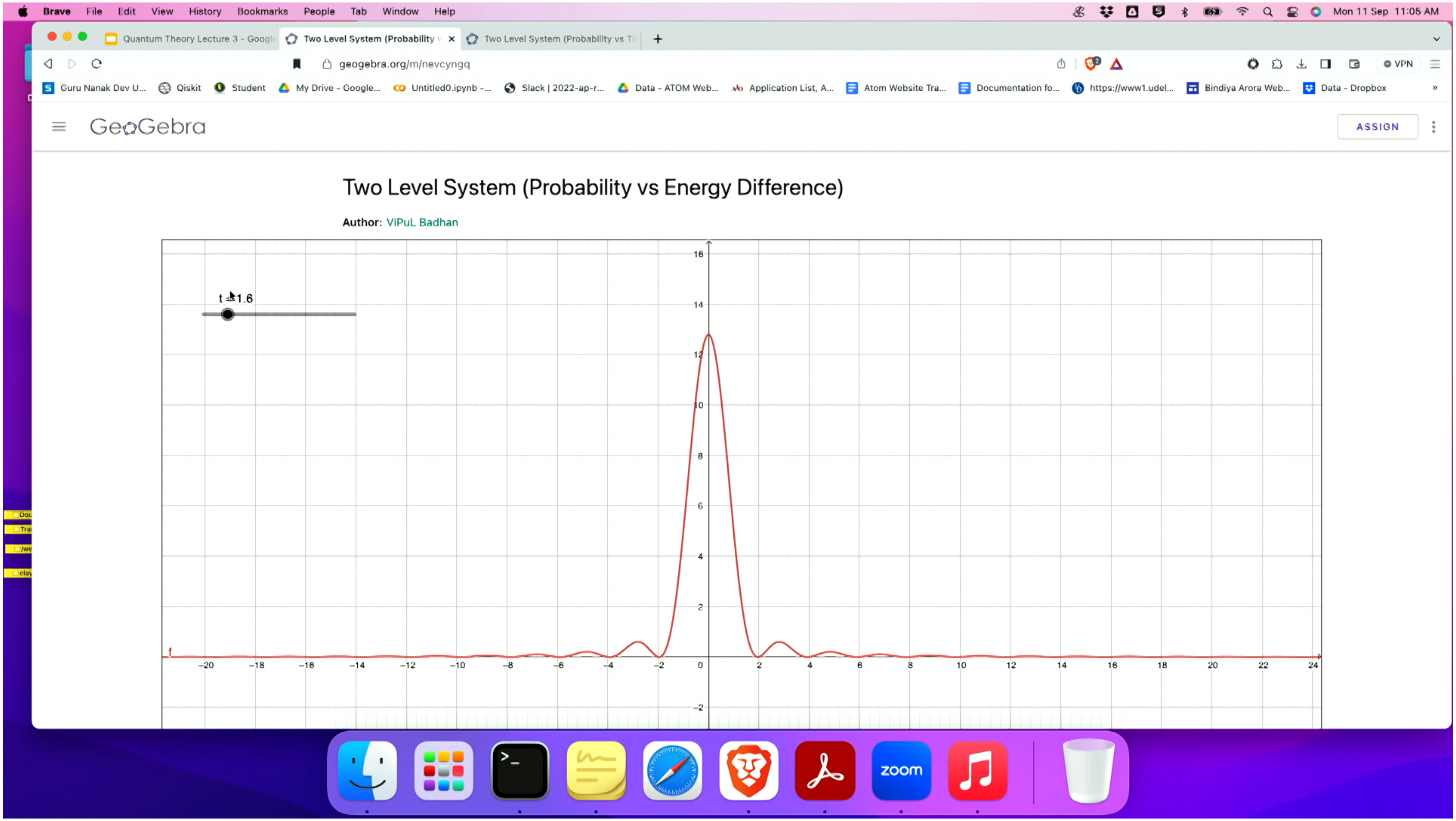
max cont $\to \omega_{fi} = 0 \quad \varepsilon_f = \varepsilon_i$

$$= \frac{4|V_{fi}|^2}{\hbar^2}\rho(\varepsilon_f) \int \frac{\sin^2\left(\frac{\omega_{fi} t}{2}\right)}{\omega_{fi}^2}\, d\varepsilon_f$$

Variation Of Probability

Probability as a function of energy difference




geogebra.org/m/nevcyngq
GeoGebra
ASSIGN
Two Level System (Probability vs Energy Difference)
Author: ViPuL Badhan
t = 1.6
f

$\hbar^2$ ω_{fi}^2

Integrand $\rightarrow \int_{-\Delta E_f/2}^{\Delta E_f/2} \frac{t^2}{4} \frac{\sin^2\left(\frac{\omega_{fi} t}{2}\right)}{\frac{\omega_{fi}^2 t^2}{4}} \, d\omega_{fi} = \frac{\pi}{2} \hbar t$

$\Delta E_f/2$

$-\Delta E_f/2$

$$\sum_f P_{i\to f} = \frac{4 |V_{fi}|^2 \rho(\varepsilon_f) \pi \hbar t}{\hbar^2}$$

$\propto t$

f

$$\boxed{W = \text{constant}}$$

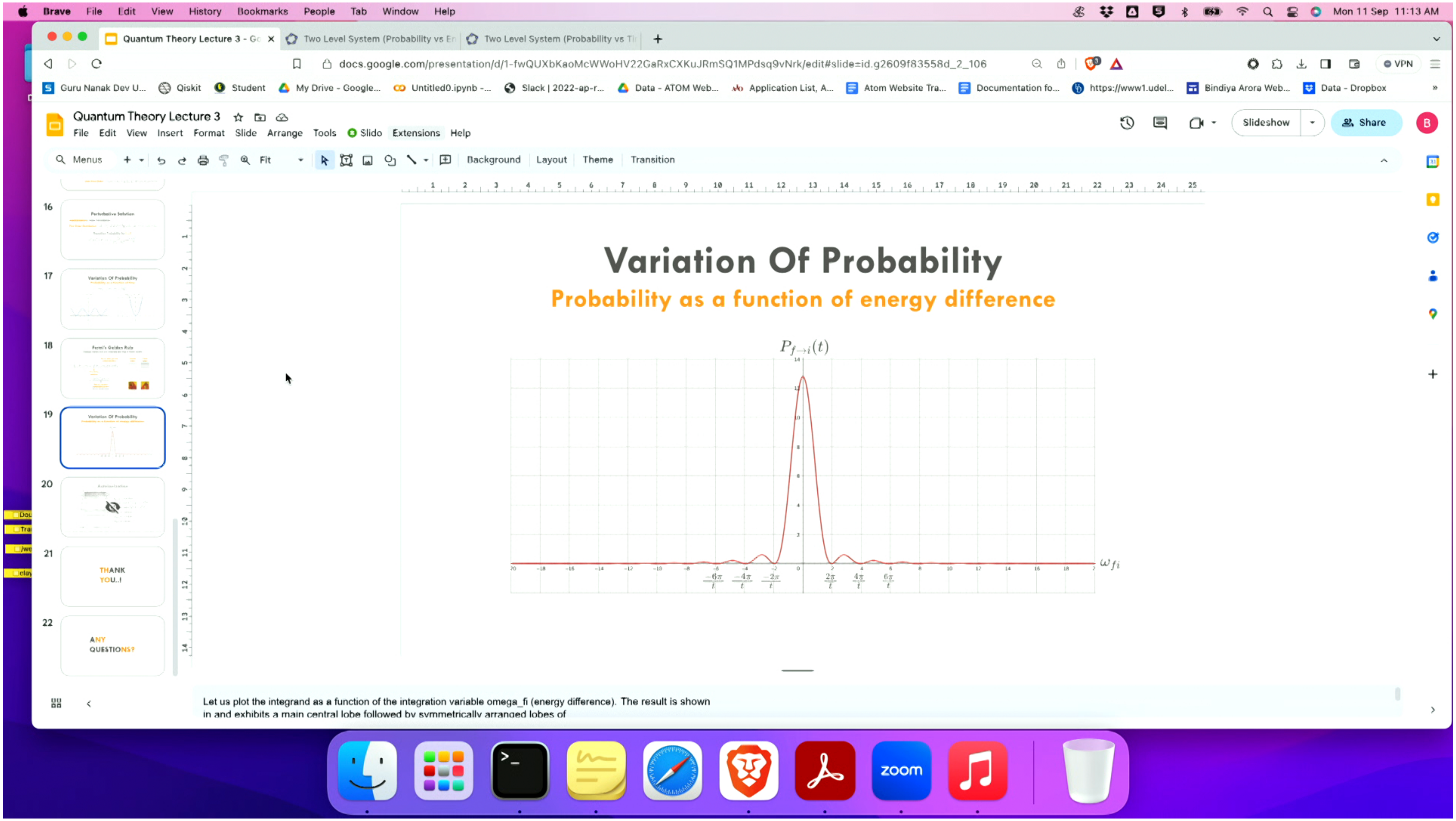
$E_f = E_i$

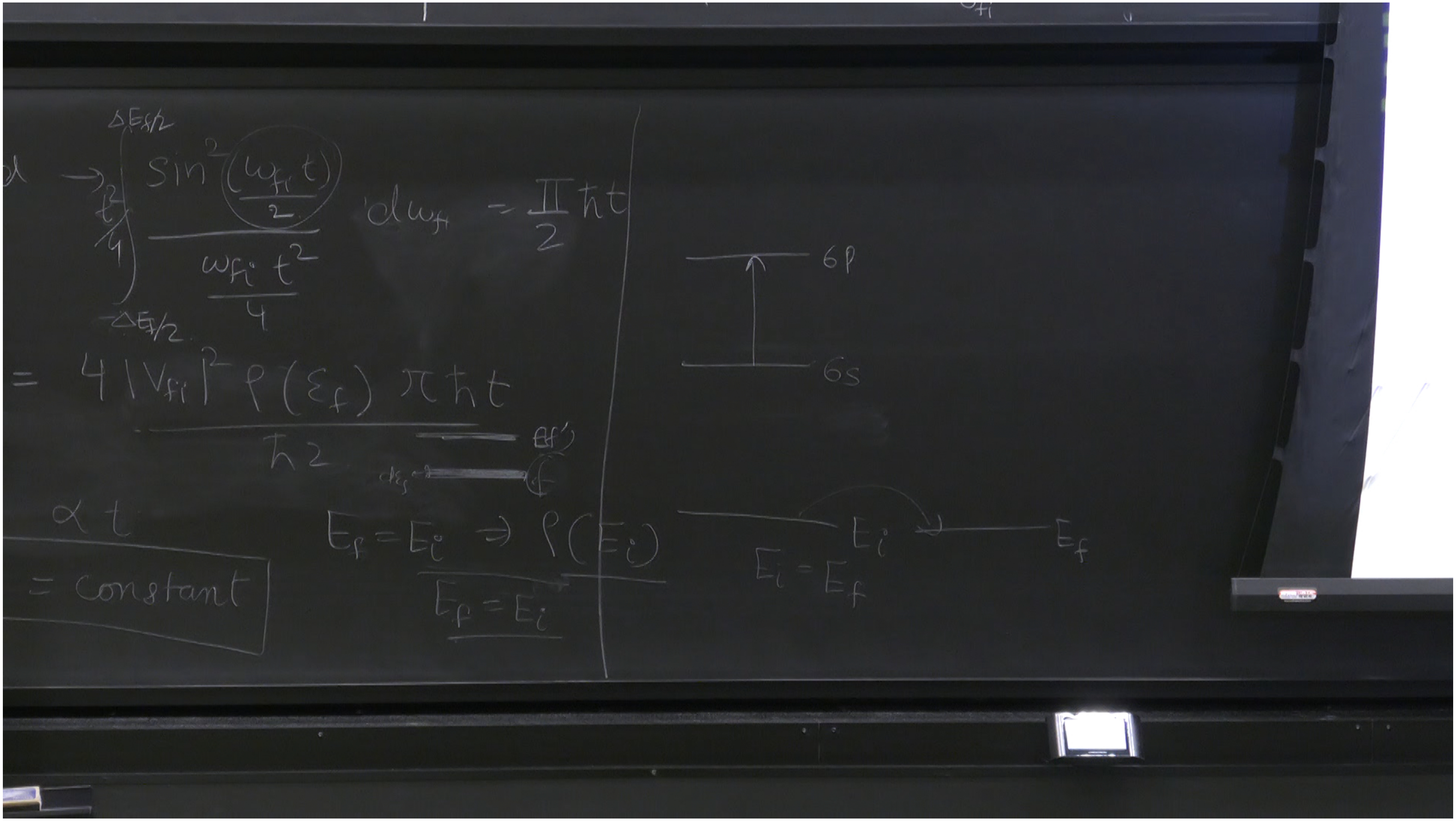
6p

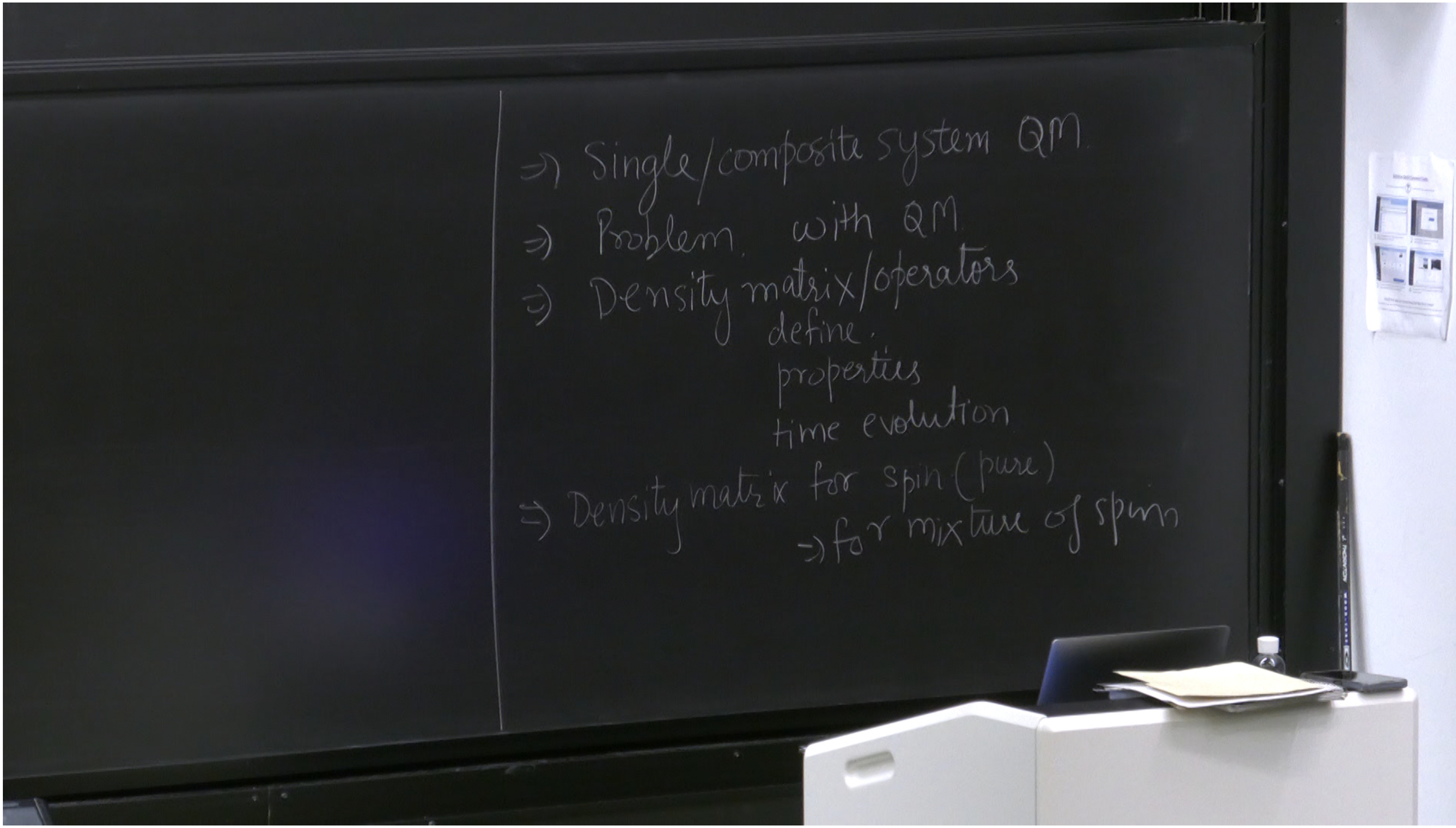
6s

E_i E_f

$E_i = E_f$


Quantum Theory Lecture 3
Variation Of Probability
Probability as a function of energy difference
$P_{f\to i}(t)$
ω_{fi}
$-\frac{6\pi}{t}$ $-\frac{4\pi}{t}$ $-\frac{2\pi}{t}$ $\frac{2\pi}{t}$ $\frac{4\pi}{t}$ $\frac{6\pi}{t}$
Let us plot the integrand as a function of the integration variable omega_fi (energy difference). The result is shown
in and exhibits a main central lobe followed by symmetrically arranged lobes of


ΔE_f/2
sin²(ω_fi t/2) / (ω_fi t²/4) dω_fi = π/2 ħt
-ΔE_f/2
= 4|V_fi|² ρ(E_f) πħt / ħ2
∝ t
= constant
E_f = E_i ⇒ ρ(E_i)
E_f = E_i
6p
6s
E_i
E_f
E_i = E_f


⇒ Single/composite system QM.
⇒ Problem with QM
⇒ Density matrix/operators
define.
properties
time evolution
⇒ Density matrix for spin (pure)
⇒ for mixture of spin

Single $\Rightarrow |\psi\rangle$, A, S.E., $|\langle n|\psi\rangle|^2$

$$= \sum_i c_n |n\rangle$$

Composite (2 particles)

$$|\psi\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$$

$$|\psi_A\rangle = |0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \checkmark$$

$$|\psi_B\rangle = |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \checkmark$$

$$|\psi\rangle_{AB} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Operators (local)

$$\hat{O} = \hat{A} \otimes \hat{I}_B + I_A \otimes \hat{B}$$

$$\hat{A} = |1\rangle\langle 0|$$

$$\hat{O} = \hat{A} \otimes \mathbb{I}_B$$

$$= |1\rangle\langle 0| \otimes \mathbb{I}_B$$

$$(|\psi_A\rangle \otimes |\psi_B\rangle)$$

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Homework:
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if intelligible
(if you take pictures
please compress them!)

$$H = H_A \otimes I_B + I_A \otimes H_B$$

$$U = e^{-iHt/\hbar} = e^{-iH_A t/\hbar} \otimes e^{-iH_B t/\hbar}$$

$$|\psi\rangle = |\psi\rangle_A \otimes |\psi_B\rangle$$

$$= \frac{|0\rangle + |1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

$$= |0\rangle_A |0\rangle_B \Rightarrow \text{Joint measurement}$$

Partial measur

Partial measurement

Singlet state

$$|\psi\rangle = \frac{|+\rangle^A \otimes |-\rangle^B - |-\rangle^A \otimes |+\rangle^B}{\sqrt{2}}$$

A

B

Classical mix of $|+\rangle$ $|-\rangle$

Alice's outcome	Joint state	Bob's state
$\Pr[|+\rangle_x] = \frac{1}{2}$	$|+\rangle_x^A \otimes |-\rangle_x^B$	$|-\rangle_x$
$\Pr[|-\rangle_x] = \frac{1}{2}$	$|-\rangle_x^A \otimes |+\rangle_x^B$	$|+\rangle_x$

$$|\psi\rangle^B \neq \frac{|+\rangle - |-\rangle}{\sqrt{2}}$$

Ensemble

$$\left\{\frac{1}{2}, |+\rangle\right\}, \left\{\frac{1}{2}, |-\rangle\right\}$$

$|1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ ✓

Operators (local)

$$\hat{O} = \hat{A} \otimes \hat{I}_B + I_A \otimes \hat{B}$$

Alice's outcome	Joint state	Bob's state
$Pr[\|+\rangle_x] = \frac{1}{2}$	$\|+\rangle_x^A \otimes \|-\rangle_x^B$	$\|-\rangle_x$
$Pr[\|-\rangle_x] = \frac{1}{2}$	$\|-\rangle_x^A \otimes \|+\rangle_x^B$	$\|+\rangle_x$

$$|\psi\rangle^B \neq \frac{|+\rangle - |-\rangle}{\sqrt{2}}$$

ensemble

$\left\{\frac{1}{2}, |+\rangle\right\}, \left\{\frac{1}{2}, |-\rangle\right\}$

$|+\rangle_z, |-\rangle_z \Rightarrow z$

$|+\rangle_x, |-\rangle_x \Rightarrow x$

$|+\rangle_y, |-\rangle_y \Rightarrow y$

$\Rightarrow \left\{\frac{1}{2}, |+\rangle_x\right\} \left\{\frac{1}{2}, |-\rangle_x\right\}$

Bob's $\xrightarrow{\text{State}}$ $\left\{\frac{1}{4}\, |+\rangle_x\right\}$ $\left\{\frac{1}{4}\, |-\rangle_x\right\}$ $\left\{\frac{1}{4}\, |+\rangle\right\}$ $\left\{\frac{1}{4}\, |-\rangle\right\}$

(2) $$|\psi\rangle = \frac{1}{\sqrt{2}}|+\rangle + \frac{e^{i\phi_1}}{\sqrt{2}}|-\rangle$$

$\phi_1 \quad \phi_2 \; - - - \qquad \phi_{100} \Rightarrow$ not good coherence.

$\{\frac{1}{4}\ |-\rangle\}$

$\Rightarrow$ not good coherence.

③ pure states $\Rightarrow$ $|+\rangle$, $|-\rangle$

mixutires $\frac{|+\rangle + |-\rangle}{\sqrt{2}}$

maximally mixed states

50% $|\uparrow\rangle$ 50% $|\downarrow\rangle$

Density matrix formulation

$|\psi\rangle\langle\psi| \Rightarrow$ for pure

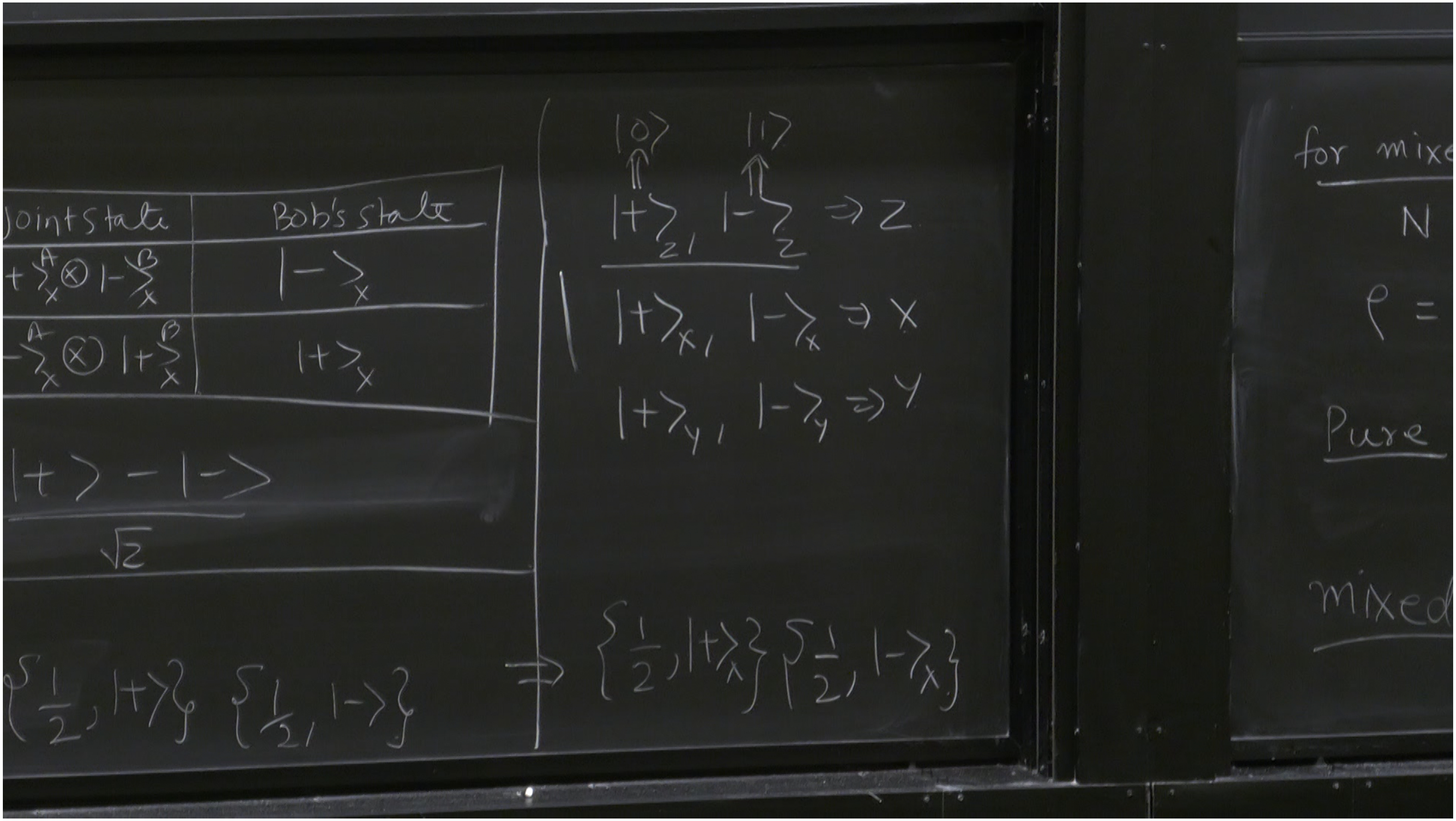
all of same kind

$\Rightarrow$ Single/com

$\Rightarrow$ Problem.

$\Rightarrow$ Density m

$\Rightarrow$ Density matrix


Joint state
Bob's state
$|+\rangle_x^A \otimes |-\rangle_x^B$
$|-\rangle_x$
$|-\rangle_x^A \otimes |+\rangle_x^B$
$|+\rangle_x$
$\frac{|+\rangle - |-\rangle}{\sqrt{2}}$
$\{\frac{1}{2}, |+\rangle\}, \{\frac{1}{2}, |-\rangle\}$
$|0\rangle$ $|1\rangle$
$|+\rangle_z, |-\rangle_z \Rightarrow Z$
$|+\rangle_x, |-\rangle_x \Rightarrow X$
$|+\rangle_y, |-\rangle_y \Rightarrow Y$
$\Rightarrow \{\frac{1}{2}, |+\rangle_x\} \{\frac{1}{2}, |-\rangle_x\}$
for mixe
N
$\rho =$
Pure
mixed

for mixed states

$N \to$ subsystems $|\psi_i\rangle$ $W_i \Rightarrow$ classical prob.

$$\rho = \sum_i W_i |\psi_i\rangle\langle\psi_i|$$

Pure $\to$ $|\psi\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$ $\rho = \frac{(|0\rangle + |1\rangle)(\langle 1| + \langle 0|)}{2}$

mixed $\to \left\{ \frac{1}{2} |0\rangle, \frac{1}{2} |1\rangle \right\}$ $\rho = \frac{1}{2} |0\rangle\langle 0| + \frac{1}{2} |1\rangle\langle 1|$

$$|\psi_i\rangle = \sum_n C_n^i |n\rangle$$

Observables

$$\langle \hat{A} \rangle_i = \langle \psi_i | \hat{A} | \psi_i \rangle$$

$$\langle A \rangle = \mathrm{Tr}(\rho \underline{A})$$

$$\langle A \rangle = \sum_i W_i \langle \psi_i | \hat{A} | \psi_i \rangle$$

$$= \sum_i W_i \sum_n \sum_{n'} C_n^{i*} C_{n'}^i \langle n' | A | n \rangle$$

$$= \sum_n \sum_{n'} \langle n | \rho | n' \rangle \langle n' | A | n \rangle = \mathrm{Tr}(\rho A)$$

$$= \sum_n \langle n | \rho A | n \rangle$$

$$\rho(\,) = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\rho(1) = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$|\psi_i\rangle \quad W_i \Rightarrow$ classical prob

$\langle\psi_i| \quad O$

$\frac{|0\rangle+|1\rangle}{\sqrt{2}}$

$, \frac{1}{2}|1\rangle\}$

$|\psi_i\rangle = \quad i=1,2$

$50\,|0\rangle \qquad 50\%\,|1\rangle$

$|\psi_1\rangle \qquad |\psi_2\rangle$

$$\rho = \frac{(|0\rangle+|1\rangle)(\langle 1|+\langle 0|)}{2} = \frac{1}{2}\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}$$

$$\rho = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1| = \frac{1}{2}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

Observables

$$\langle\hat{A}\rangle_i = \langle\psi_i|\hat{A}|\psi_i\rangle$$

$$\langle A\rangle = \sum_i W_i \langle\psi_i|\hat{A}$$

$$= \sum_i W_i \sum_n \sum_{n'} C_n^{i*}$$

$$= \sum_n \sum_{n'} \langle n|\rho|n'\rangle$$

$$= \sum_n \langle n|\rho A|n\rangle$$

$$|\psi_1\rangle = \sqrt{\tfrac{2}{3}}|+\rangle + \sqrt{\tfrac{1}{3}}|-\rangle, \quad |\psi_2\rangle = \sqrt{\tfrac{4}{5}}|+\rangle + \sqrt{\tfrac{1}{5}}|-\rangle$$

$$|\psi_i\rangle = i = 1,2$$

systems $|\psi_i\rangle$ $W_i \Rightarrow$ classical prob

$W_i \; |\psi_i\rangle\langle\psi_i|$

50 $|0\rangle$ 50% $|1\rangle$

$|\psi_1\rangle$ $|\psi_2\rangle$

$$|\psi\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

$$\rho = \frac{(|0\rangle + |1\rangle)(\langle 1| + \langle 0|)}{2} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\left\{\frac{1}{2}|0\rangle, \frac{1}{2}|1\rangle\right\}$$

$$\rho = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1| = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Observable

$$\langle \hat{A}\rangle_i =$$

$$\langle A\rangle = $$

$$= \sum_i$$

$$= \sum_n$$

$$=$$

Properties

1) $\rho = \left(\sum w_i |\psi_i\rangle\langle\psi_i| \right)^\dagger = \rho$

2) $\mathrm{Tr}(\rho) = 1$

$$= \sum_n \langle n | \rho | n \rangle$$

$$= \sum_{n,i} w_i \langle n | \psi_i \rangle \langle \psi_i | n \rangle$$

$$= \sum_i w_i \langle \psi_i | \psi_i \rangle = \sum_i w_i = 1$$

3) $\mathrm{Tr}(\rho^2)$

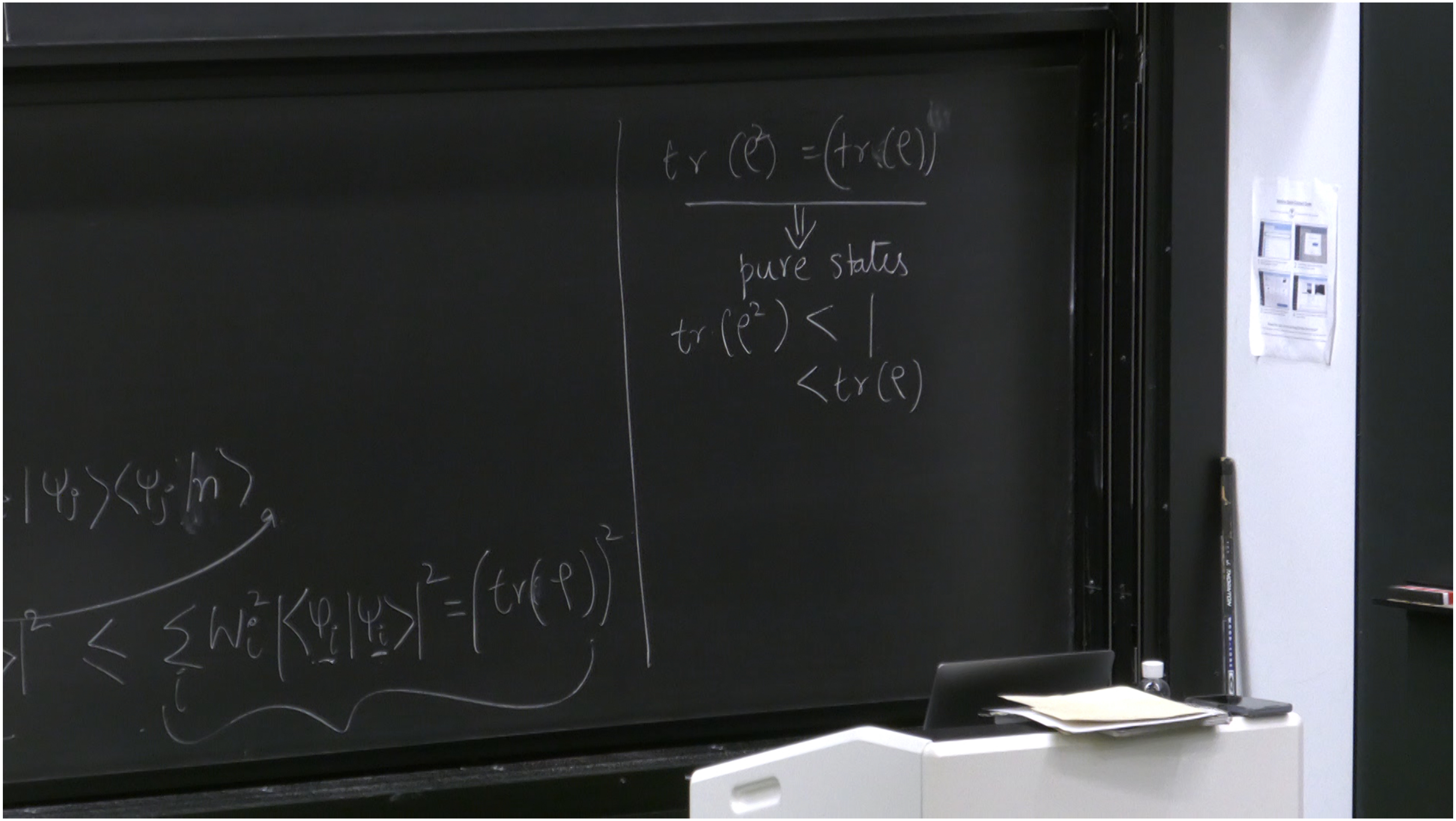
3) $\mathrm{Tr}(\rho^2) \leq \left(\mathrm{Tr}(\rho)\right)^2$

$$\mathrm{Tr}(\rho^2) = \sum_n \langle n | \rho^2 | n \rangle$$

$$= \sum_n \langle n | \rho \ \rho | n \rangle$$

$$= \sum_{n,i,j} w_i w_j \langle n | \psi_i \rangle \langle \psi_i | \psi_j \rangle \langle \psi_j | n \rangle$$

$$= \sum_{i,j} w_i w_j \left| \langle \psi_i | \psi_j \rangle \right|^2 \leq \sum_i w_i^2 \left| \langle \psi_i | \psi_i \rangle \right|^2 = \left(\mathrm{tr}(\rho)\right)^2$$


$tr(\rho^2) = (tr(\rho))^2$
$\Downarrow$
pure states
$tr(\rho^2) < 1$
$< tr(\rho)$

I^{st}

$|\psi_1\rangle = |0\rangle \rightarrow \left(\frac{2}{3}\right)$ Prob

$|\psi_2\rangle = |1\rangle \rightarrow \frac{1}{2}$

II^{nd}

$|\psi_1\rangle = \sqrt{\frac{2}{3}}|+\rangle + \sqrt{\frac{1}{3}}|-\rangle$ Prob $\frac{1}{2}$

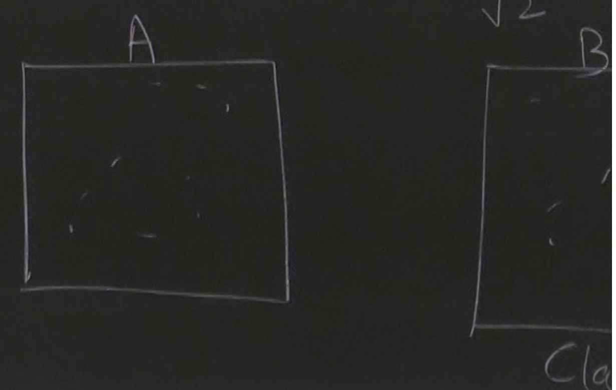
$|\psi_2\rangle = \sqrt{\frac{2}{3}}|+\rangle - \sqrt{\frac{1}{3}}|-\rangle$ Prob $\frac{1}{2}$

$$\rho = \begin{pmatrix} \frac{2}{3} & 0 \\ 0 & \frac{1}{3} \end{pmatrix}$$

Partial measurement

(1) Singlet state

$$|\psi\rangle = \frac{|+\rangle^A \otimes |-\rangle^B - |\ldots}{\sqrt{2}}$$



2^{nd}

$|\psi_1\rangle = \sqrt{\frac{2}{3}}|+\rangle + \sqrt{\frac{1}{3}}|-\rangle$

Prob $\frac{1}{2}$

$|\psi_2\rangle = \sqrt{\frac{2}{3}}|+\rangle - \sqrt{\frac{1}{3}}|-\rangle$

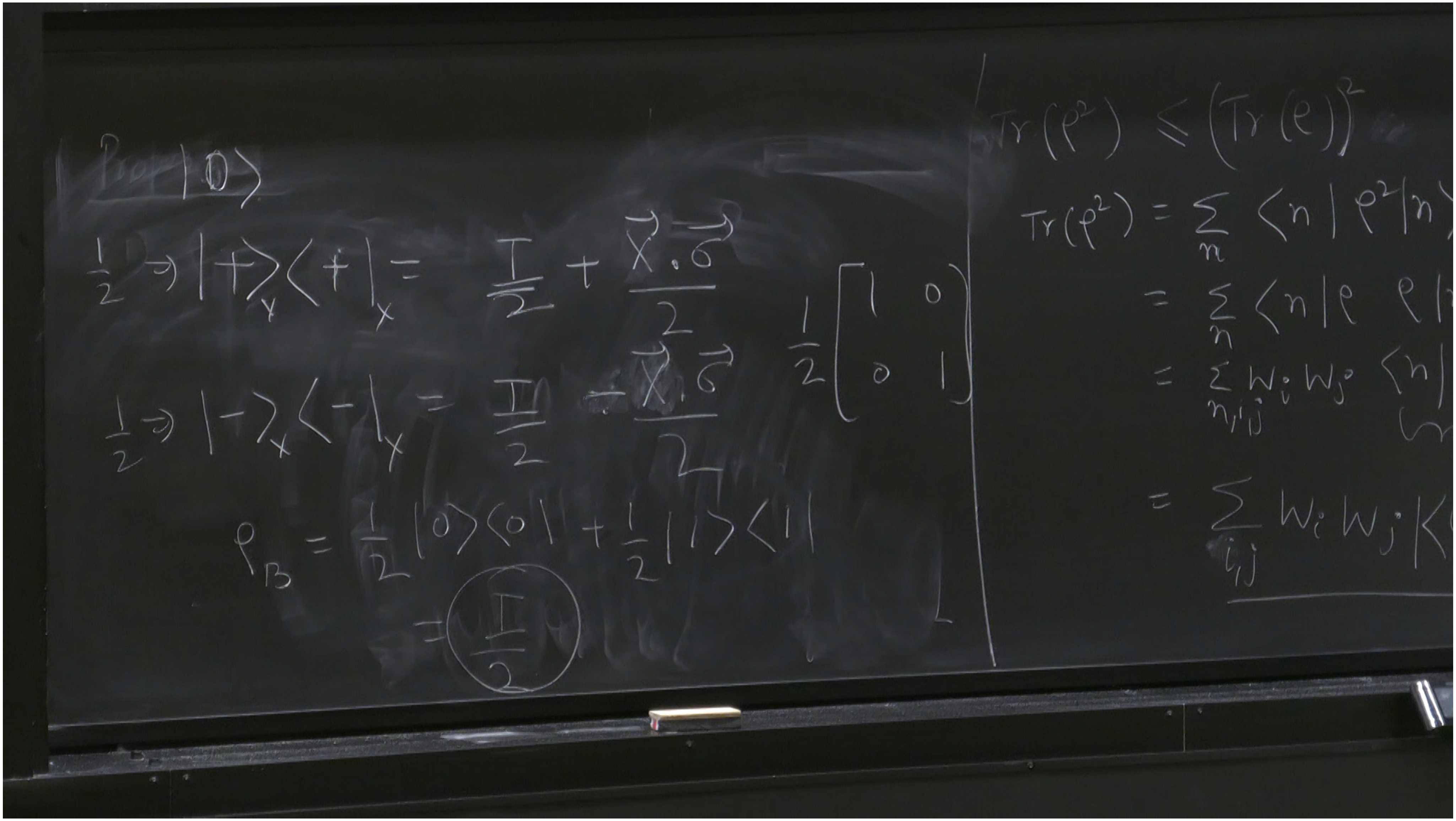
Prob $\frac{1}{2}$

Density for pure spin $\frac{1}{2}$

$$|\psi\rangle = \cos\frac{\theta}{2}|+\rangle + \sin\frac{\theta}{2}e^{i\phi}|-\rangle$$

$$|\psi\rangle\langle\psi| = \begin{bmatrix} \cos^2\theta/2 & \cos\frac{\theta}{2}\sin\frac{\theta}{2}e^{-i\phi} \\ \cos\frac{\theta}{2}\sin\frac{\theta}{2}e^{i\phi} & \sin^2\theta/2 \end{bmatrix}$$

$|+\rangle\ |-\rangle$


Prep $|0\rangle$
$\frac{1}{2} \to |+\rangle_x \langle +|_x = \frac{I}{2} + \frac{\vec{X}\cdot\vec{\sigma}}{2}$
$\frac{1}{2} \to |-\rangle_x \langle -|_x = \frac{I}{2} - \frac{\vec{X}\cdot\vec{\sigma}}{2}$
$\frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
$\rho_B = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|$
$= \frac{I}{2}$
$\mathrm{Tr}(\rho^2) \leq (\mathrm{Tr}(\rho))^2$
$\mathrm{Tr}(\rho^2) = \sum_n \langle n|\rho^2|n\rangle$
$= \sum_n \langle n|\rho\ \rho|$
$= \sum_{n,i,j} w_i w_j \langle n|$
$= \sum_{i,j} w_i w_j |\langle$

$$\frac{1}{2}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

Tr

$\frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1| \Rightarrow$ Bob mix in Z-basis

$\frac{1}{2}\Big[\,|+\rangle_x\,{}_x\langle x| + \frac{1}{2}|+\rangle_-\,{}_-\langle +| \Rightarrow$ " " " X-basis

$$\frac{|0\rangle+|1\rangle}{\sqrt{2}} \qquad \frac{|0\rangle-|1\rangle}{\sqrt{2}}$$

$$|\psi_i\rangle|^2 = \left(tr(\cdot\,\rho)\right)^2$$

tr

tr