Title: Classical Physics Lecture - 090723

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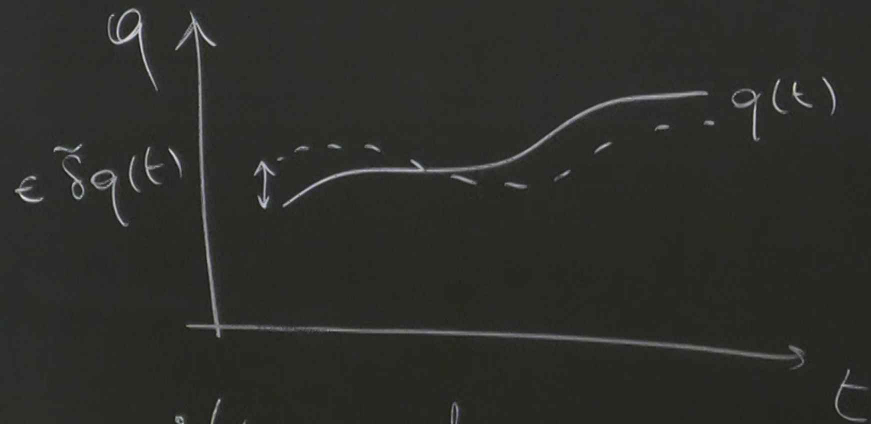
$$q'(t) = q(t) + \epsilon \underbrace{\tilde{\delta}_s q(t)}$$

can depend on $(q(t), \dot{q}(t), t)$

[this is $s(q, v, t)$]



$$\tilde{\delta}\dot{q}$$

$$\equiv$$

$$\dot{q}'(t) = \frac{d}{dt} q'(t) = \dot{q}(t) + \epsilon \boxed{\frac{d}{dt} \tilde{\delta}_s q(t)}$$

$$L(q+\epsilon\tilde{\delta}_s q,\ \dot{q}+\epsilon\tilde{\delta}_s\dot{q},\ t) = L(q,\dot{q},t) + \epsilon\frac{d}{dt}R_s(q,\dot{q},\cdots) + \mathcal{O}(\epsilon^2)$$

$\tilde{\delta}q$ is an (infinitesimal) sym

$$\Longleftrightarrow\ \tilde{\delta}L = \frac{d}{dt}R_s$$

$$L(q+\epsilon\tilde{\delta}_s q,\ \dot{q}+\epsilon\tilde{\delta}_s\dot{q},\ t) = L(q,\dot{q},t) + \epsilon\frac{d}{dt}R_s(q,\dot{q},\dots) + O(\epsilon^2)$$

$\tilde{\delta}q$ is an (infinitesimal) sym

$\Longleftrightarrow\ \tilde{\delta}L = \frac{d}{dt}R_s$

Noether 1st

If $\tilde{\delta}q$ is an infinitesimal sym., then

$$Q_s(t) := \sum_i \boxed{\frac{\partial L}{\partial \dot{q}^i}}\tilde{\delta}q^i - R_s$$

p_i the (generalized) momentum

is conserved on-shell of the e.o.m.

i.e. $\frac{d}{dt}Q_s \hat{=} 0$.

Ex 1 Cyclical Variable

q^1 cyclical iff $\frac{\partial L}{\partial q^1} = 0 \xRightarrow{\text{EoM}} p_1$ is conserved

$$\left(\frac{\partial L}{\partial q^1} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^1} \doteq 0 \;\leftrightarrow\; 0 - \frac{d}{dt} p_1 \doteq 0 \right)$$

choose

$$\tilde{\delta} q^i = \begin{cases} 1 & \text{if } i = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\Rightarrow \tilde{\delta} \dot{q}^i \equiv 0$$

$$\tilde{\delta} L = \sum_i \frac{\partial L}{\partial q^i} \underbrace{\tilde{\delta} q^i}_{-\delta^i_1} + \frac{\partial L}{\partial \dot{q}^i} \underbrace{\tilde{\delta} \dot{q}^i}_{=0}$$

$$= \frac{\partial L}{\partial q^1} \underset{\uparrow \text{cyclicity}}{=} 0$$

$\Rightarrow$ $\tilde{\delta} q$ is an infinitesimal sym for $R_S = 0$

$\Rightarrow$ $Q_S = \sum_i p_i \tilde{\delta} q^i - R_S = p_1$ is <u>conserved</u>

$\tilde{\delta}\dot{q}^i$

$= 0$

sym

$R_s = p_1$ is <u>conserved</u>

<u>Detour</u>

$$L'(q,v,t) = L(q,v,t) + \frac{d}{dt}\ell(q,t)$$

equivalent Lagrangians

(see action principle with

$$\delta q(t_0) = \delta q(t_1) = 0 \;)$$

<u>Rmk</u> $\tilde{\delta}q$ is a sym of L

iff — " — L'

$\Longrightarrow R_s' = R_s + \tilde{\delta}\ell$, but Q is the same!

$$\vec{x}' = \vec{x} + \epsilon \underbrace{\vec{a} \times \vec{x}}_{\tilde{\delta}\vec{x}} \qquad R_s = 0$$

$$\vec{Q}(t) = \sum_i \frac{\partial L}{\partial \dot{x}^i} (\vec{a} \times \vec{x})^i$$

$$= \vec{p} \cdot (\vec{a} \times \vec{x}) = \vec{a} \cdot \vec{M} \quad \forall \vec{a}$$

$\Rightarrow \vec{M}$ conserved

$\times \vec{p})$

Ex Rotations (particle in central potential)

$$L = \frac{1}{2} m |\dot{\vec{x}}|^2 - V(|\vec{x}|)$$

$\downarrow$ spherical coords $\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases}$

$$= \frac{1}{2} m \left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\varphi}^2\right) - V(r)$$

$$\frac{\partial L}{\partial \varphi} = 0 \longrightarrow p_\varphi \text{ is conserved} \equiv M_z = \hat{z}\cdot(\vec{x}\times\vec{p})$$

$\vec{X}' = \vec{X} +$

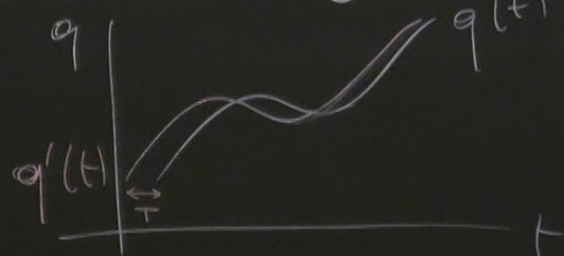
$\vec{Q}(t) = \sum_i$

$=$

EX (Time translation)

$$q'(t) = q(t+T)$$

↑ "same" history but translated in time



claim: time transl is a sym if $\frac{\partial L}{\partial t} = 0$

$$q'(t) = q(t+\epsilon) = q(t) + \epsilon\,\dot{q}(t) + O(\epsilon^2)$$

$$\tilde{\delta} q^i = \dot{q}(t)$$

n time

$$q'(t) = q(t+\epsilon)$$

$$= q(t) + \epsilon\, \dot{q}(t) + O(\epsilon^2)$$

change of config

$$\tilde{\delta} q^i = \dot{q}^i(t) \longleftarrow$$ depends on instantaneous velocity!

$$\tilde{\delta} \dot{q}^i := \frac{d}{dt} \tilde{\delta} q^i = \ddot{q}^i(t)$$

$$\tilde{\delta} L = \sum_i \frac{\partial L}{\partial q^i} \tilde{\delta} q^i + \frac{\partial L}{\partial \dot{q}^i} \tilde{\delta} \dot{q}^i = \sum_i \frac{\partial L}{\partial q^i} \dot{q}^i + \frac{\partial L}{\partial \dot{q}} \ddot{q}^i = \frac{d}{dt} L - \frac{\partial}{\partial t} L$$

...anslated in time

...m

$$q'(t) = q(t+\epsilon)$$
$$= q(t) + \epsilon\, \dot{q}(t) + O(\epsilon^2)$$

change of config depends on instantaneous velocity!

$$\tilde{\delta} q^i = \dot{q}^i(t) \longleftarrow$$

$$\tilde{\delta} \dot{q}^i := \frac{d}{dt} \tilde{\delta} q^i = \ddot{q}^i(t)$$

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$\Rightarrow$ if $\partial_t L = 0$, $\tilde{\delta} q$ is an inf. sym with $R_s = L$

$$Q_s = \sum_i p_i \tilde{\delta} q^i -$$
$$= \sum_i p_i \dot{q}^i -$$

$\ldots t) + O(\epsilon^2)$

…hange of config
…epends on
…nstantaneous velocity!

$\ldots t)$

$$\ldots \delta \dot q^i = \sum_i \frac{\partial L}{\partial q^i}\dot q^i + \frac{\partial L}{\partial \dot q^i}\ddot q^i = \frac{d}{dt}L - \frac{\partial}{\partial t}L$$

…q is an inf. sym with $R_s = L$

$$Q_s = \sum_i p_i \,\delta q^i - R_s$$

$$= \sum_i p_i \dot q^i - L \equiv E$$ is conserved

"generalized" energy -

HAMILTONIAN FORMALISM

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} = \frac{\partial L}{\partial q^i}$$ 2nd order ODE

$$\sim \ddot{q}^i$$

$$\to \ddot{q}^i = f^i(q,\dot{q}) \to \begin{cases} \dot{q}^i = v^i \\ \dot{v}^i = f^i(q,v) \end{cases}$$

$$z = \begin{pmatrix} q \\ v \end{pmatrix} \qquad \dot{z} = \begin{pmatrix} 0 & 1 \\ f & 0 \end{pmatrix} z$$

...LISM

...nd order ODE

$$\Rightarrow \begin{cases} \dot{q}^i = v^i \\ \dot{v}^i = f^i(q,v) \end{cases}$$

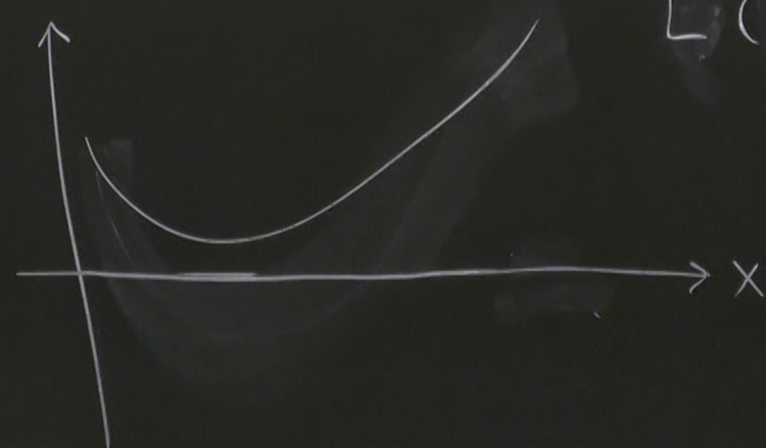
$$= \begin{pmatrix} 0 & 1 \\ f & 0 \end{pmatrix} z$$

it seems nicer to use

$p_i = \frac{\partial L}{\partial v^i}$ instead of v itself

The Legendre transform

$L(v)$ convex



$-p_i\,dq - K_s = p_1$ is conserved | $\Rightarrow K_s = K_s + \delta t$, but Q is the same!

itself

form

$L(v)$ convex $\longrightarrow H(p)$ that encodes the "same information"

where $p = \dfrac{dL}{dv}$

$\to v$

Rmk: we need $\dfrac{dL}{dv}$ to be monotonous in v $\left(\begin{array}{c}\text{to get} \\ v \leftrightarrow p \\ 1 \text{ to } 1\end{array}\right)$

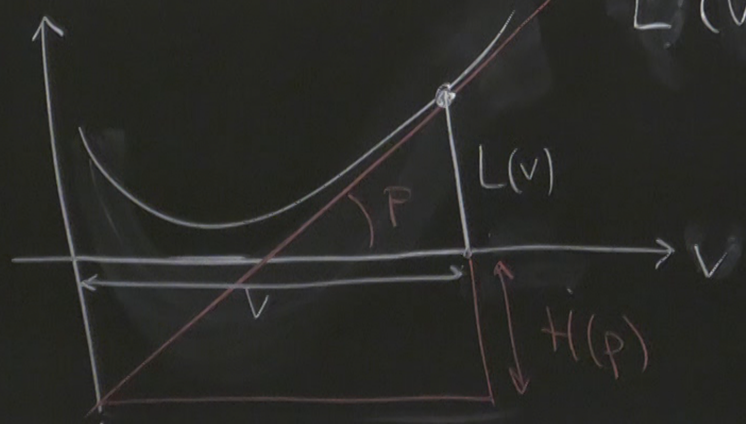
i.e. $\dfrac{d^2L}{dv^2} > 0$ i.e. L convex

$Q_j = \sum_i p_i \, \partial q - K_j = p_1$ is conserved

it seems nicer to use

$p_i = \frac{\partial L}{\partial v^i}$ instead of v itself

The Legendre transform



$L(v)$ convex $\longrightarrow$ $H(p)$ that encod

where $p =$

Rmk: we need $\frac{dL}{dV}$ to be m

i.e. $\frac{d^2 L}{d V^2} > 0$ i.e. L

$v)$

$H(p) := p\, v(p) - L(v(p))$

where $v(p)$ is the inverse of $v \to p = \frac{dL}{dv}$

Rmk $\quad pv = H(p) + L(v)$

Doing the Legendre transf twice gives back the original function

Back to physics!

$L(q, v, t) \xrightarrow[\text{on } v]{\text{L. transf.}} H(q, p, t)$

$p = \frac{\partial L}{\partial v} \qquad v = \frac{\partial H}{\partial p}$

Back to physics!

$$L(q,v,t) \xrightarrow[\text{on } v]{\text{L. transf.}} H(q,p,t)$$

$$p = \frac{\partial L}{\partial v}$$

(L convex b/c quadratic in v)

$$v = \frac{\partial H}{\partial p} \quad (\heartsuit)$$

$$\frac{\partial L}{\partial q} = -\frac{\partial H}{\partial q}$$

$$\Rightarrow \quad \frac{d}{dt}\frac{\partial L}{\partial v} = \frac{\partial L}{\partial q} \quad (*)$$

$$\rightsquigarrow \begin{cases} \dot{q} \overset{(\heartsuit)}{=} \dfrac{\partial H}{\partial p} \\ \dot{p} \overset{(*)}{=} -\dfrac{\partial H}{\partial q} \end{cases}$$

Hamilton's eom

$H(q,p,t)$ is the Hamiltonian function

Rmk. $H(q,p,t)$ is a function on $T^*Q \times \mathbb{R}$ — cotangent space

$((q,p), t)$

Given a history $q(t), p(t)$

$$H(q,p) = p\dot{q} - L(q,\dot{q},t) \equiv E$$

1st order action principle

T^*Q = "phase space"

$$H : T^*Q \times \mathbb{R} \longrightarrow \mathbb{R}$$
$$(q, p, t) \longmapsto H(q, p, t)$$

$$\gamma : \mathbb{R} \longrightarrow T^*Q$$
$$t \longmapsto (q, p) = \left(\gamma_q(t), \gamma_p(t)\right)$$

$$S[\gamma, t_0, t_1] = \int_{t_0}^{t_1} \sum_i \gamma_p^i(t)\,\dot{\gamma}_q^i(t) - H\left(\gamma_q(t), \gamma_p(t), t\right)$$

or, more intelligibly:

$$S = \int_{t_0}^{t_1} dt \left(\sum_i p_i \dot{q}^i - H(q,p,t) \right)$$

I have a history in phase space.

$$\delta S = \int_{t_0}^{t_1} dt \left(\sum_i \delta p_i \, \dot{q}^i + p_i \underbrace{\delta \dot{q}^i}_{\equiv \frac{d}{dt}\delta q^i} - \frac{\partial H}{\partial q^i}\delta q^i - \frac{\partial H}{\partial p_i}\delta p_i \right)$$

history variation

$$= \int_{t_0}^{t_1} dt \left(\sum_i \delta p_i \left(\dot{q}^i - \frac{\partial H}{\partial p_i} \right) + \delta q^i \left(-\dot{p}_i - \frac{\partial H}{\partial q^i} \right) \right) + \left[\sum_i p_i \delta q^i \right]_{t_0}^{t_1}$$

$\dot{q}^i - H(q, p, t) \Big)$

I have a history in phase space

$\dot{q}^i + p_i \underbrace{\delta \dot{q}^i}_{\equiv \frac{d}{dt}\delta q^i} - \frac{\partial H}{\partial q^i}\delta q^i - \frac{\partial H}{\partial p_i}\delta p_i \Big)$

need to fix only the q's at t_0 & t_1

$$\sum_i \delta p_i \left(\dot{q}^i - \frac{\partial H}{\partial p_i}\right) + \delta q^i \left(-\dot{p}_i - \frac{\partial H}{\partial q^i}\right)\Bigg) + \Big[\sum_i p_i \delta q^i\Big]_{t_0}^{t_1}$$

1st order action principle

- fix $\delta q = 0$ at t_0, t_1
- no conditions on p!

"same" as in Lagrangian action principle (no conditions on v)

- $\delta S\big|_{\delta q(t_0) = \delta q(t_1) = 0} = 0 \iff$ Hamilton's eqs (which are 1st order)