Title: Classical Physics Lecture - 100323

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$n_\mu = \delta^0_\mu$

An infinitesimal variation

$$\varphi^I(x) \longmapsto \varphi^I(x) + \epsilon\, \tilde{\delta}_s \varphi^I(x)$$

is said an infinitesimal symmetry iff

$$\mathcal{L}(\varphi + \epsilon\tilde{\delta}\varphi, \partial\varphi + \epsilon\partial\delta\varphi, x) = \mathcal{L}(\varphi, \partial\varphi, x) + \epsilon \frac{d}{dx^\mu} R^\mu_s + \mathcal{O}(\epsilon^2)$$

[that is: $\tilde{\delta}_s \mathcal{L} = \frac{d}{dx^\mu} R^\mu_s$]

<u>Thm</u> (Noether) If $\tilde{\delta}_s \varphi$ is an infinitesimal sym, then

$$J^\mu_s := \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi^I)} \tilde{\delta}_s \varphi^I - R^\mu_s$$

is a conserved current on-shell

$^I(x)$

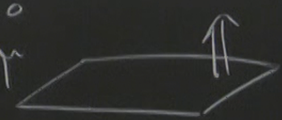
f

$) = \mathcal{L}(\varphi, \partial\varphi, x) + \epsilon \frac{d}{dx^\mu} R^\mu_s + O(\epsilon^2)$

is an infinitesimal sym, then

$- R^\mu_s$ is a conserved current on-shell

$n_\mu = \delta^0_\mu$



which means

$$\partial_\mu J^\mu_s \hat{=} 0$$

$$\Rightarrow Q_s(t) = \int_{\Sigma_t} \underbrace{n_\mu J^\mu}_{=J^0} d^3\vec{x}$$

is const. in t.

$J^\mu \equiv$ Noether's current

$Q \equiv$ — " — charge.

$d^3\vec{x}$

Rmk (Ambiguities)

The eq. $\tilde{\delta}_s \mathcal{L} = \frac{d}{dx^\mu} R^\mu_s$ does not uniquely determine R^μ_s, in fact

$$R^\mu_s \mapsto R'^\mu_s = R^\mu_s - \frac{d}{dx^\nu} \Psi^{\nu\mu}$$

with $\Psi^{\mu\nu} + \Psi^{\nu\mu} = 0$ is just as good.

However, this will affect my def of J^μ:

$$J^\mu_s \mapsto J'^\mu_s = J^\mu_s + \frac{d}{dx^\nu} \Psi^{\nu\mu}$$

What about the charge?

$$Q_S'(t) = \int_{\Sigma_t} d^3\vec{x} \left(J^0 + \frac{d}{dx^\nu} \Psi^{\nu 0} \right)$$

$$= Q_S(t) + \int_{\Sigma_t} d^3\vec{x} \, \frac{d}{dx^i} \Psi^{i0}$$

$$= Q_S(t) + \oint_{S_\infty^2 = \partial\Sigma_t} \Psi^{i0} \, d^2S_i$$

if fields at ∞ fall off fast enough

$\Psi(\varphi, \dots) \xrightarrow{\infty} 0$

$$= Q_S(t)$$

Note:

$\Psi^{\mu\nu} =$

ge?

$J^0 + \frac{d}{dx^\nu}\Psi^{\nu 0})$

$\int_{\Sigma_t} d^3\vec{x}\ \frac{d}{dx^i}\Psi^{i0}$

$+\oint_{S_\infty = \partial\Sigma_t} \Psi^{i0}\, d^2S_i$

Note:

$\Psi^{\mu\nu} = \Psi^{\mu\nu}(\varphi, \partial\varphi, \dots, x)$

"superpotential"

Translational

& stress en

$\varphi^I(x) \longmapsto \varphi$

Translational sym
& stress energy tensor

• $\varphi^I(x) \longmapsto \varphi'^I(x) = \varphi^I(x+\epsilon a)$

$= \varphi^I(x) + \epsilon \underbrace{a^\mu \partial_\mu \varphi^I(x)}_{\tilde{\delta}_s \varphi^I} + \mathcal{O}(\epsilon^2)$ for $a^\mu \in \mathbb{R}^{1,3}$ (const!)

$\partial_\mu(a^\nu \partial_\nu \varphi^I) = a^\nu \partial_\nu (\partial_\mu \varphi^I)$

• symmetry?

$$\mathcal{L}(\varphi + \epsilon \tilde{\delta}_s \varphi, \dots, x) = \mathcal{L}(\varphi, \;, x) + \epsilon \left[\frac{\partial \mathcal{L}}{\partial \varphi^I} \boxed{\tilde{\delta}_s \varphi^I} + \epsilon \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi^I)} \boxed{\partial_\mu(\tilde{\delta}_s \varphi^I)} \right] + \mathcal{O}(\epsilon^2)$$

$\tilde{\delta}_s \varphi^I = a^\nu \partial_\nu \varphi^I$

$$= \mathcal{L}(\varphi, \dots) + \epsilon\, a^\nu \left(\frac{d}{dx^\nu} - \frac{\partial}{\partial x^\nu} \right) \mathcal{L}(\varphi, \partial\varphi, x) + \mathcal{O}(\epsilon^2)$$

- Translations are a symm
  iff $\partial_x \mathcal{L}(\varphi, \partial\varphi, x) = 0$
  [no <u>explicit</u> dependence on x]

$$\Rightarrow \quad R^\mu_s = a^\mu \mathcal{L} \qquad (\text{recall } a^\nu = \text{const})$$

- Noether $\Rightarrow$ $$J^\mu_a = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi^I)} \tilde{\delta}_a \varphi^I - R^\mu_a$$

$$= a^\nu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi^I)} \partial_\nu \varphi^I - \delta^\mu_\nu \mathcal{L} \right)$$

$$J^\mu_a = t^\mu{}_\nu a^\nu$$

$$\boxed{t^\mu{}_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi^I)} \partial_\nu \varphi^I - \delta^\mu_\nu \mathcal{L}}$$

canonical stress-energy tensor

Since a^μ is arbitrary and const:

$$\partial_\mu J^\mu_a \hat{=} 0 \quad \text{iff} \quad \boxed{\partial_\mu t^\mu{}_\nu \hat{=} 0}$$

$$Q = \int_{\Sigma_t} J^0_a d^3\vec{x} = a^\nu \int_{\Sigma_t} t^0{}_\nu d^3\vec{x} = a^\nu P_\nu$$

Ex

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi - V(\varphi)$$

Ex

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi - V(\varphi)$$

$$t^\mu{}_\nu = -\partial^\mu\varphi\,\partial_\nu\varphi + \delta^\mu_\nu\left(\frac{1}{2}\partial_\rho\varphi\,\partial^\rho\varphi + V(\varphi)\right)$$

$$P_\nu = \int_\Sigma d^3\vec{x}\left(\dot{\varphi}\,\partial_\nu\varphi - \delta^0_\nu\,\mathcal{L}\right)$$

$$\to P_0 = \int_\Sigma d^3\vec{x}\left(+\frac{1}{2}\dot{\varphi}^2 + \frac{1}{2}|\vec{\nabla}\varphi|^2 + V(\varphi)\right)$$

tot energy

Energy density

Ex

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi - V(\varphi)$$

$$t^\mu{}_\nu = -\partial^\mu\varphi\,\partial_\nu\varphi + \delta^\mu_\nu\left(\frac{1}{2}\partial_\rho\varphi\,\partial^\rho\varphi + V(\varphi)\right)$$

$$P_\nu = \int_\Sigma d^3\vec{x}\left(\dot{\varphi}\,\partial_\nu\varphi - \delta^0_\nu\,\mathcal{L}\right)$$

$$\rightarrow P_0 = \int_\Sigma d^3\vec{x}\left(+\frac{1}{2}\dot{\varphi}^2 + \frac{1}{2}|\vec{\nabla}\varphi|^2 + V(\varphi)\right)$$ Energy density

tot energy $\quad P_i = \int d^3\vec{x}\,\dot{\varphi}\,\partial_i\varphi$

$$\Rightarrow P_\nu = (E, \vec{p})$$ Energy-mom. 4-vector.

$$J^\mu_a \equiv t^\mu{}_\nu a^\nu$$

$$\boxed{t^\mu{}_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi^I)} \partial_\nu \varphi^I - \delta^\mu_\nu \mathcal{L}}$$

canonical stress-energy tensor

Since a^μ is arbitrary and const:

$$\partial_\mu J^\mu_a \doteq 0 \quad \text{iff} \quad \boxed{\partial_\mu t^\mu{}_\nu \doteq 0}$$

$$Q = \int_{\Sigma_t} J^0_a d^3\vec{x} = a^\nu \int_{\Sigma_t} t^0{}_\nu \, d^3\vec{x} = a^\nu \boxed{P_\nu}$$

Energy-mom 4 vector

Ex

$$\mathcal{L} = -\frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi - \ldots$$

$$t^\mu{}_\nu = -\partial^\mu \varphi \partial_\nu \varphi \ldots$$

$$P_\nu = \int_\Sigma d^3\vec{x} \left(\ldots \dot{\varphi} \partial_\nu \ldots \right.$$

$$\rightarrow P_0 = \int_\Sigma d^3\vec{x} \left(+ \ldots \right.$$

tot energy $\quad P_i = \int \ldots$

$$\Rightarrow P_\nu = (E, \vec{P})$$

$$\ldots^\mu_a \qquad - \delta^\mu_\nu \mathcal{L})$$

Ex

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi - V(\varphi)$$

$$t^\mu{}_\nu = -\partial^\mu\varphi\,\partial_\nu\varphi + \delta^\mu_\nu\left(\frac{1}{2}\partial_\rho\varphi\,\partial^\rho\varphi + V(\varphi)\right)$$

$$P_\nu = \int_\Sigma d^3\vec{x}\left(\dot{\varphi}\,\partial_\nu\varphi - \delta^0_\nu\,\mathcal{L}\right)$$

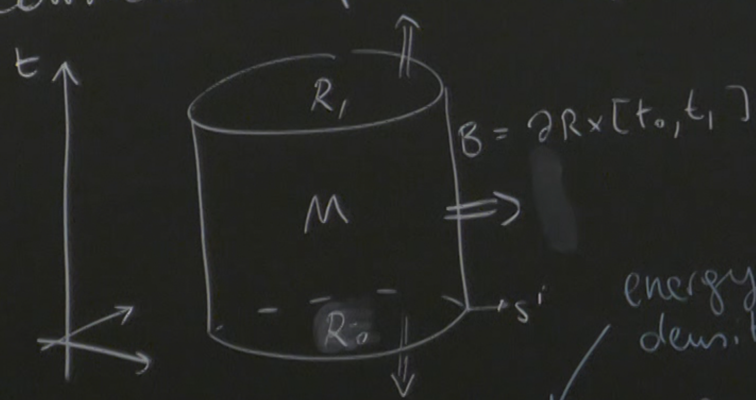
$$\rightarrow\ P_0 = \int_\Sigma d^3\vec{x}\left(+\frac{1}{2}\dot{\varphi}^2 + \frac{1}{2}|\vec{\nabla}\varphi|^2 + V(\varphi)\right)$$ Energy density

tot energy $\quad P_i = \int d^3\vec{x}\ \dot{\varphi}\,\partial_i\varphi$

$\Rightarrow\ P_\nu = (E, \vec{P})$ Energy-mom. 4-vector by Lorentz cov.

om 4 vector

Consider a spacetime region



energy-mom. density

energy momentum fluxes

$$0 \hat{=} \int_M \partial_\mu t^\mu{}_\nu = \int_{R_1} t^0{}_\nu - \int_{R_0} t^0{}_\nu + \int_{t_0}^{t_1} \oint_{\partial R} s_i \boxed{t^i{}_\nu}$$

$$= P_\nu(R_1) - P(R_0) + \text{Flux}$$

on

$[t_0, t_1]$

energy-mom. density

energy momentum fluxes

$$\int_{R_0} t^0{}_\nu + \int_{t_0}^{t_1} \oint_{\partial R} s_i \boxed{t^i{}_\nu}$$

$) - P(R_0) + \text{Flux}$

$t^i{}_0$ = flux along x^i of energy

$t^i{}_j$ = flux — " — of j-momentum

$$t^i{}_j = \left(\begin{array}{c|c} \boxed{\varepsilon} & \boxed{\text{energy flux}} \\ \hline \boxed{\text{mom. density}} & \boxed{\begin{array}{c}\text{momentum}\\ \text{fluxes}\\ \text{(stress)}\end{array}} \end{array}\right)$$

(thermalized) perfect fluid $\left(\begin{array}{c|ccc} \varepsilon & & 0 & \\ \hline & p & & \\ 0 & & p & \\ & & & p \end{array}\right)$ p = pressure

$$t^i{}_0 = \text{flux along } x^i \text{ of energy}$$

$$t^i{}_j = \text{flux} \text{ — '' — of } j\text{-momentum}$$

energy
momentum
fluxes

$\downarrow$

$s_i \boxed{t^i{}_\nu}$

flux

$$t^i{}_j = \left(\begin{array}{c|c} \boxed{\varepsilon} & \boxed{\text{energy flux}} \\ \hline \boxed{\text{mom. density}} & \boxed{\text{momentum fluxes (stresses)}} \end{array}\right)$$

(Thermalized) perfect fluid $\left(\begin{array}{c|ccc} \varepsilon & & 0 & \\ \hline & p & & \\ 0 & & p & \\ & & & p \end{array}\right)$ p: pressure

P_μ is the linear momentum of the field and is conserved.

not to be confused with

$$\Phi_{\vec{x}}(t) = \varphi(t,\vec{x})$$

$$\Pi_{\vec{x}}(t) = \partial_t \varphi(t,\vec{x}) \leftarrow \text{canonical mom.}$$

$$\{\Phi_{\vec{x}}, \Pi_{\vec{y}}\} = \delta(\vec{x},\vec{y})$$

$$P_i = \int d^3\vec{x}\; \Pi_{\vec{x}}\, \partial_i \Phi_{\vec{x}} \quad , \quad \{a^i P_i, \Phi_{\vec{x}}\} = a^i \partial_i \Phi_{\vec{x}}$$

$$\{a^i P_i, \Pi_{\vec{x}}\} = a^i \partial_i \Pi_{\vec{x}}$$

ssure

Maxwell theory

$$S[A] = \int d^4x \, -\frac{1}{4} F^{\mu\nu} F_{\mu\nu}$$

$$F_{\mu\nu} \equiv F_{\mu\nu}(A) = \partial_\mu A_\nu - \partial_\nu A_\mu$$

Eq of motion:

$$\delta S = \int d^4x \, -\frac{1}{2} F^{\mu\nu} \left(\partial_\mu \delta A_\nu - \partial_\nu \delta A_\mu \right)$$

$$= \int d^4x \, - F^{\mu\nu} \partial_\mu \delta A_\nu$$

$$= \int d^4x \, \underbrace{\left(\partial_\mu F^{\mu\nu} \right)}_{\text{Maxwell EOM}} \delta A_\nu + \oint n_\mu F^{\mu\nu} \underbrace{\boxed{\delta A_\nu}}_{} $$

$\boxed{\delta A_\nu} = 0$ boundary cond. for action principle

oundary
ond. for
ction
principle

translation invariance $\partial_x \mathcal{L}_{Max} = 0$ ✓

$$\hookrightarrow \quad t^\mu{}_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \boxed{A_\rho})} \partial_\nu \boxed{A_\rho} - \delta^\mu_\nu \mathcal{L}$$

φ^I

$$\boxed{t^\mu{}_\nu = -F^{\mu\rho} \boxed{\partial_\nu A_\rho} + \frac{1}{4} \delta^\mu_\nu F^{\alpha\beta} F_{\alpha\beta}}$$

<u>not</u> gauge invariant!

translation invariance $\partial_x \mathcal{L}_{Max} = 0$ ✓

$$\hookrightarrow t^\mu{}_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \boxed{A_\rho})} \partial_\nu \boxed{A_\rho} - \delta^\mu_\nu \mathcal{L}$$

φ^I

$$\boxed{t^\mu{}_\nu = -F^{\mu\rho}\boxed{\partial_\nu A_\rho} + \frac{1}{4}\delta^\mu_\nu F^{\alpha\beta}F_{\alpha\beta}}$$

<u>not</u> gauge invariant!

Idea, use ambiguity to fix this issue:

$$J^\mu \mapsto J^\mu + \frac{d}{dx^\rho}\Psi^{\rho\mu}$$

$$\boxed{t^\mu{}_\nu \mapsto t^\mu{}_\nu + \frac{d}{dx^\rho}\Psi^{\rho\mu}{}_\nu = T^\mu{}_\nu}$$

Is there a $\Psi^{\rho\mu}{}_\nu = -\Psi^{\mu\rho}{}_\nu$ that makes $T^\mu{}_\nu$ gauge invariant?

$$\Psi^{\rho\mu}{}_{\nu} = +F^{\mu\rho} A_\nu$$

$$\hookrightarrow T^{\mu}{}_{\nu} = \partial_\rho\left(+F^{\mu\rho}A_\nu\right) - F^{\mu\rho}\partial_\nu A_\rho + \frac{1}{4}\delta^\mu_\nu F^{\alpha\beta}F_{\alpha\beta}$$

$$= +\underbrace{\left(\partial_\rho F^{\mu\rho}\right)}_{\hookrightarrow\ \text{vanishes on shell}\ \checkmark} A_\nu + F^{\mu\rho}\left(\partial_\rho A_\nu - \partial_\nu A_\rho\right) + \frac{1}{4}\delta^\mu_\nu F^{\alpha\beta}F_{\alpha\beta}$$

$$\hat{=} \boxed{F^{\mu\rho}F_{\rho\nu} + \frac{1}{4}\delta^\mu_\nu F^{\alpha\beta}F_{\alpha\beta} = T^{\mu}{}_{\nu}} \checkmark$$

- $T_{\mu\nu}$ is gauge inv.
  - symmetric
  - equiv. to $t^{\mu}{}_{\nu}$ up to a superpot. (on shell)

"Improved", Belinfante stress-energy tensor

$\eta_\mu F^{\mu\nu} \boxed{\delta A_\nu}$ action principle

$$J^\mu \mapsto J^\mu + \frac{d}{dx^\rho} \Psi^{\rho\mu}$$

$$\boxed{t^\mu{}_\nu \mapsto t^\mu{}_\nu + \frac{d}{dx^\rho} \Psi^{\rho\mu}{}_\nu = \overline{T}^\mu{}_\nu}$$

Is there a $\Psi^{\rho\mu}{}_\nu$ = ... that makes $\overline{T}^\mu{}_\nu$ ge...

$\frac{1}{4} \delta^\mu_\nu F^{\alpha\beta} F_{\alpha\beta}$

"Improved", Belinfante stress-energy tensor

Components

$$T^0{}_0 = F^{0\rho} F_{\rho 0} + \frac{1}{4} F^{\alpha\beta} F_{\alpha\beta} = (F_{i0})^2 - \frac{1}{2}(F_{i0})^2 + \frac{1}{4} F^{ij} F_{ij}$$

$$= \frac{1}{2}(E^2 + B^2)$$

$$T^0{}_i = F^{0\rho} F_{\rho i} = F^{0j} F_{ji} = E^j \epsilon_{jik} B^k = -(\vec{E} \times \vec{B}) \quad \text{Poynting}$$

$$T^i{}_j = F^{i\rho} F_{\rho j} + \frac{1}{4} \delta^i_j F^{\alpha\beta} F_{\alpha\beta} = \dots$$