Title: Classical Physics Lecture - 092523

Speakers: Aldo Riello

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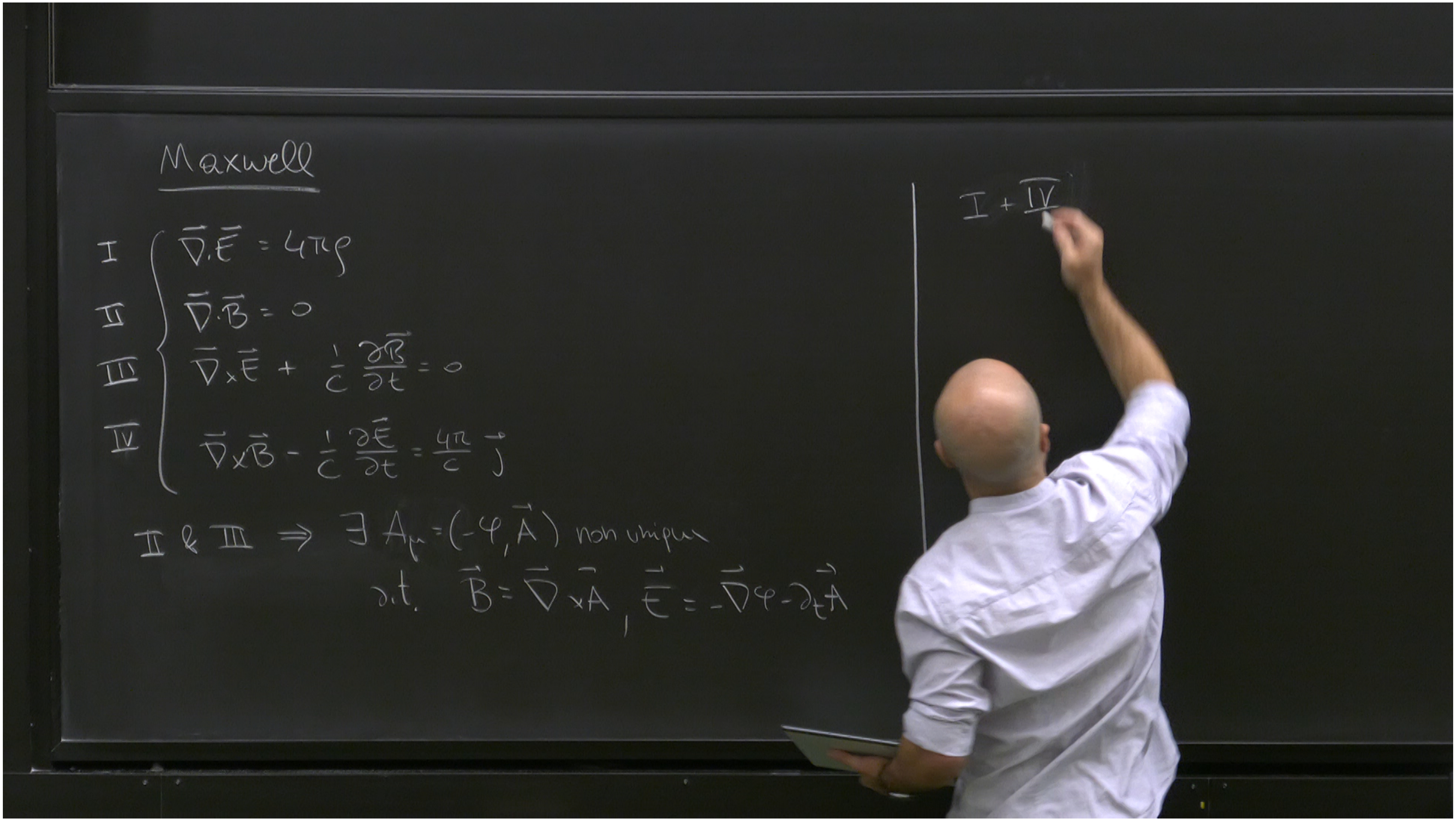


Maxwell
I $\vec{\nabla}\cdot\vec{E} = 4\pi\rho$
II $\vec{\nabla}\cdot\vec{B} = 0$
III $\vec{\nabla}\times\vec{E} + \frac{1}{c}\frac{\partial\vec{B}}{\partial t} = 0$
IV $\vec{\nabla}\times\vec{B} - \frac{1}{c}\frac{\partial\vec{E}}{\partial t} = \frac{4\pi}{c}\vec{j}$
II & III $\Rightarrow \exists\, A_\mu = (-\varphi, \vec{A})$ non unique
s.t. $\vec{B} = \vec{\nabla}\times\vec{A}$, $\vec{E} = -\vec{\nabla}\varphi - \partial_t\vec{A}$
I + IV

$$\begin{aligned} \mathrm{I} + \mathrm{IV} &\quad \left\{ \begin{aligned} &\nabla_\mu F^{\mu\nu} = 4\pi j^\nu \\ &\nabla_{[\mu} F_{\nu\rho]} = 0 \end{aligned} \right. \\ \mathrm{II} + \mathrm{III} & \end{aligned}$$

$$F_{\mu\nu} = \left(\begin{array}{c|c} 0 & -E^i \\ \hline E^i & \epsilon_{ijk} B^k \end{array} \right)$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$\begin{cases} \mathrm{I} + \mathrm{IV} & \nabla_\mu F^{\mu\nu} = 4\pi j^\nu \\ \mathrm{II} + \mathrm{III} & \nabla_{[\mu} F_{\nu\rho]} = 0 \end{cases}$$

$$F_{\mu\nu} = \left(\begin{array}{c|c} 0 & -E^i \\ \hline E^i & \epsilon_{ijk} B^k \end{array} \right)$$

$$\boxed{\mathrm{II} + \mathrm{III}} \Rightarrow \boxed{F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu} \text{ for some } A_\mu$$

Not unique $A_\mu \to A_\mu + \partial_\mu \xi$ (gauge freedom)

Choosing "Lorentz gauge", $\nabla^\mu A_\mu^{(L)} = 0$, then

$\mathrm{I} + \mathrm{III}$ read $\boxed{\Box A_\mu^{(L)} = -4\pi j_\mu}$

Maxwell

Gauss

$$
\begin{array}{ll}
\mathrm{I} & \vec{\nabla}\cdot\vec{E} = 4\pi\rho \leftarrow \text{Gauss} \\
\mathrm{II} & \vec{\nabla}\cdot\vec{B} = 0 \\
\mathrm{III} & \vec{\nabla}\times\vec{E} + \frac{1}{c}\frac{\partial\vec{B}}{\partial t} = 0 \\
\mathrm{IV} & \vec{\nabla}\times\vec{B} - \frac{1}{c}\frac{\partial\vec{E}}{\partial t} = \frac{4\pi}{c}\vec{j}
\end{array}
$$

II & III $\Rightarrow \exists\, A_\mu = (-\varphi, \vec{A})$ non unique

s.t. $\vec{B} = \vec{\nabla}\times\vec{A}\,,\ \vec{E} = -\vec{\nabla}\varphi - \partial_t\vec{A}$

$\mathrm{I} + \mathrm{IV}$

$\mathrm{II} + \mathrm{III}$

$F_{\mu\nu} =$

$\mathrm{II} + \mathrm{III} \Rightarrow \boxed{F_{\mu\nu} = \partial}$

Not uniqu

Choosing "

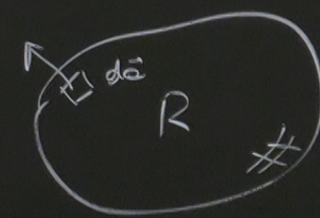
$\mathrm{II} + $

Gauss law

$$Q_R(\vec{x},t) = \int_R \rho(\vec{x},t)\, d^3x \qquad \text{charge in } R \subset \mathbb{R}^3$$

$$= \frac{1}{4\pi}\int_R \vec{\nabla}\cdot\vec{E}\, d^3x \underset{\uparrow}{=} \frac{1}{4\pi}\oint_{\partial R} \vec{E}\cdot d\vec{a}_{\partial R}$$

divergence thm

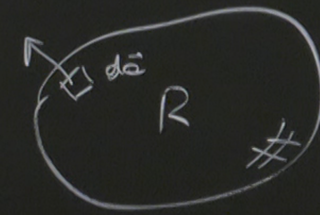


$$\to \boxed{Q_R(t) = \frac{1}{4\pi}\Phi_{\partial R}(\vec{E})}$$

Gauss law

$Q_R(\vec{x},t) = \int_R \rho(\vec{x},t)\, d^3x$ charge in $R \subset \mathbb{R}^3$

$$= \frac{1}{4\pi}\int_R \vec{\nabla}\cdot\vec{E}\, d^3x \underset{\substack{\uparrow \\ \text{divergence} \\ \text{thm}}}{=} \frac{1}{4\pi}\oint_{\partial R} \vec{E}\cdot d\vec{a}_{\partial R}$$



$$\rightarrow \boxed{Q_R(t) = \frac{1}{4\pi}\Phi_{\partial R}(\vec{E})}$$

Now: how to solve for $\vec{E}$ in R given ρ

(in the electrostatic regime)

Gauss law

$$Q_R(\vec{x},t) = \int_R \rho(\vec{x},t)\, d^3x \qquad \text{charge in } R \subset \mathbb{R}^3$$

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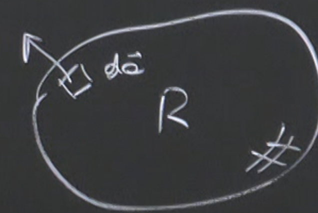


$$\rightarrow \boxed{Q_R(t) = \frac{1}{4\pi}\,\Phi_{\partial R}(\vec{E})}$$

Now: how to solve for $\vec{E}$ in R given ρ

(in the electrostatic regime: $\vec{A}=0,\ \partial_t = 0 \Rightarrow \vec{E} = -\vec{\nabla}\varphi$)

$\downarrow$

$\vec{B} = 0$

In electrostatic regime:

$$\vec{\nabla}\cdot\vec{E} = 4\pi\rho \rightsquigarrow \boxed{\Delta\varphi = -4\pi\rho}$$

Laplace-Poisson eq.

$$\Delta = \vec{\nabla}\cdot\vec{\nabla} = \partial_x^2 + \partial_y^2 + \partial_z^2$$

Neuman

The L.P. eq. admits unique solutions for φ, when we include appropriate boundary conditions:

Dirichlet
$$\begin{cases} \Delta\varphi(x) = -4\pi\rho(x), & x\in R \\ \varphi(x) = V(x), & x\in\partial R \end{cases}$$

III read $\boxed{\Box A_\mu = -4\pi j_\mu}$ $\beta = 0$

Neumann
$$\begin{cases} \Delta\varphi(x) = -4\pi\rho(x), & x \in R \\ \vec{S}\cdot\vec{\nabla}\varphi(x) = -f(x), & x \in \partial R \end{cases}$$

$(\vec{S} \perp \partial R)$

III read $\boxed{\Box A_\mu = -4\pi j_\mu}$

$B=0$

Neumann

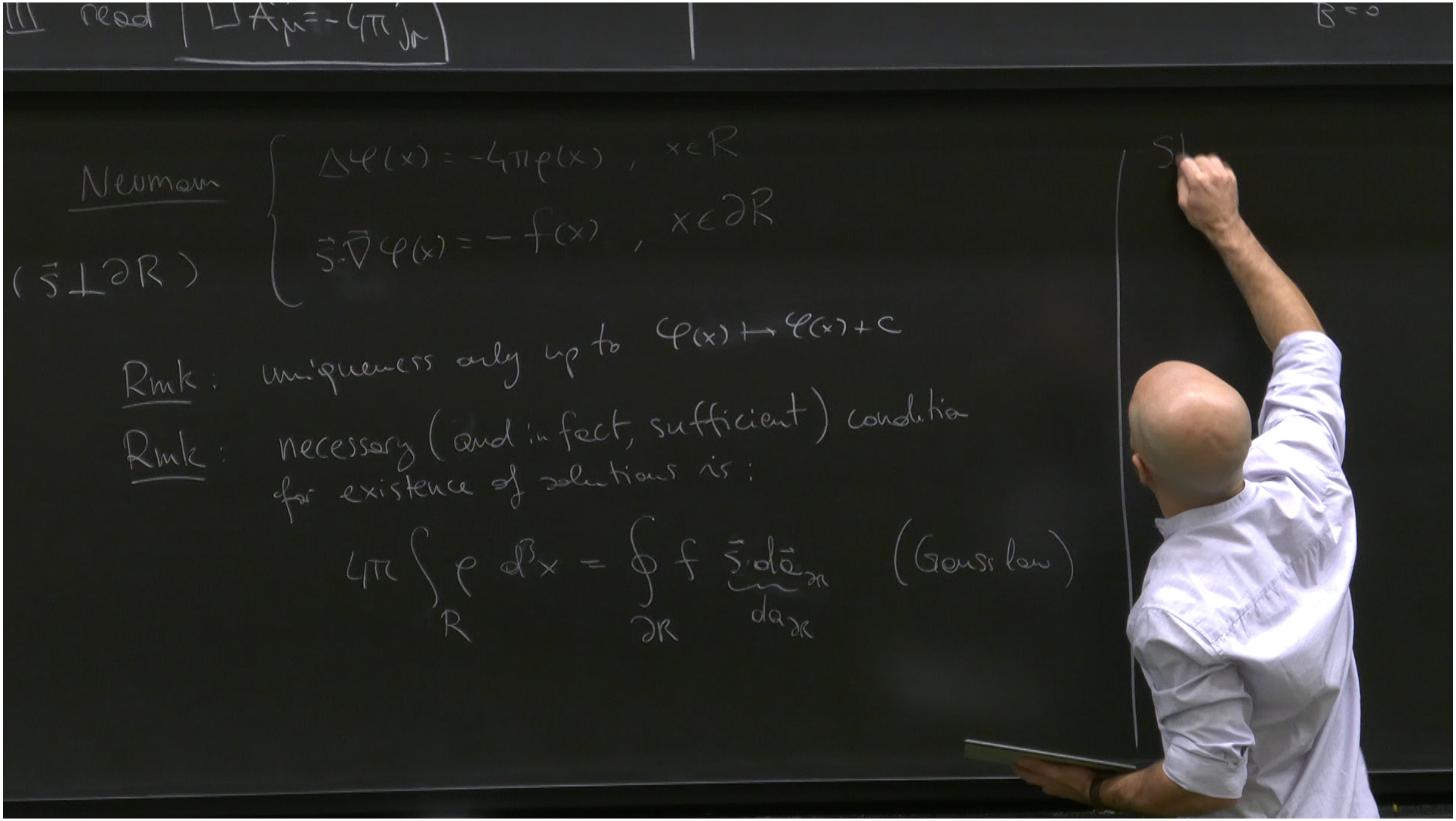
$$\begin{cases} \Delta\varphi(x) = -4\pi\rho(x)\,, & x\in R \\ \vec{s}\cdot\vec{\nabla}\varphi(x) = -f(x)\,, & x\in\partial R \end{cases}$$

$(\vec{s}\perp\partial R)$

Rmk: uniqueness only up to $\varphi(x)\mapsto\varphi(x)+c$

Rmk: necessary (and in fact, sufficient) condition for existence of solutions is:

$$4\pi\int_R \rho\, d^3x = \oint_{\partial R} f\, \underbrace{\vec{s}\cdot d\vec{\sigma}_{\partial R}}_{da_{\partial R}}$$


III need $\boxed{\Box A_\mu = -4\pi j_\mu}$
$\beta = 0$
Neumann
$\Delta\varphi(x) = -4\pi\rho(x)\,,\quad x\in R$
$\vec{s}\cdot\vec{\nabla}\varphi(x) = -f(x)\,,\quad x\in\partial R$
$(\vec{s}\perp\partial R)$
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Rmk: necessary (and in fact, sufficient) condition for existence of solutions is:
$4\pi\int_R \rho\, d^3x = \oint_{\partial R} f\, \underbrace{\vec{s}\cdot d\vec{\sigma}_{\partial R}}_{da_{\partial R}}$ (Gauss law)
S

$\varphi(x) + c$

...nt) condition

$\vec{e}_{\partial R}$ (Gauss law)

$a_{\partial R}$

Sketch of pf of uniqueness

Suppose φ_1 & φ_2 solve either (D) or (N) b.v.p. (for the same ρ, V, f):

$$\Phi := \varphi_2 - \varphi_1$$

$$\Rightarrow \Delta \Phi = 0 \quad \& \text{ either } \begin{cases} \Phi = 0 & \text{at } \partial R \\ \vec{S}\cdot\vec{\nabla}\Phi = 0 & \text{at } \partial R \end{cases}$$

$\varphi(x) + c$

...nt) condition

$\vec{e}_{\partial R}$ (Gauss law)

Sketch of pf of uniqueness

Suppose φ_1 & φ_2 solve either (D) or (N) b.v.p. (for the same ρ, V, f):

$$\Phi := \varphi_2 - \varphi_1$$

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Consider:

int. by parts

$$0 \leqq \int_R |\vec{\nabla}\Phi|^2 = -\int_R \Phi \underbrace{\Delta\Phi}_{=0} + \int_{\partial R} \underbrace{\Phi\, \vec{S}\cdot\vec{\nabla}\Phi}_{=0} = 0$$

$$\Rightarrow \vec{\nabla}\Phi = 0 \text{ everywhere} \Rightarrow \Phi = \text{const.}$$

$$\Rightarrow (D)\ \Phi = \varphi_2 - \varphi_1 = 0, \quad (N)\ \Phi = \varphi_2 - \varphi_1 = c$$

☑

Green's identity

$\forall \Phi, \Psi$: $\displaystyle\int_R \Phi \, \Delta \Psi = \int_R \Psi \, \Delta \Phi + \oint_{\partial R} \left(\Phi \, \nabla_s \Psi - \Psi \, \nabla_s \Phi \right)$

$\left(\nabla_s \equiv \vec{s} \cdot \vec{\nabla} \right)$

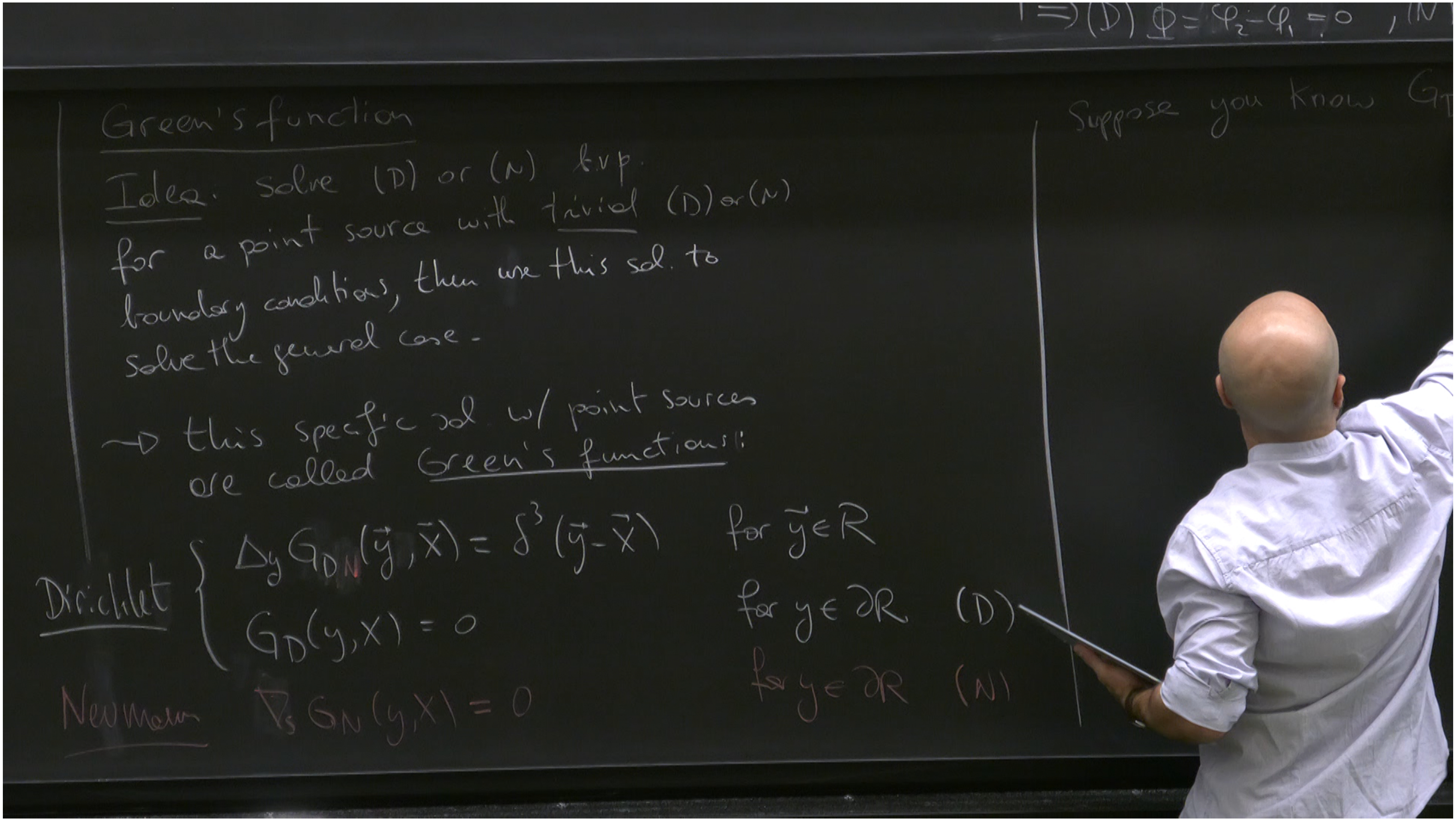
Pf : $\displaystyle\int_R \underset{(2)}{\vec{\nabla} \Phi} \cdot \underset{(1)}{\vec{\nabla} \Psi} = \ldots$ integrate by parts (1) & (2) and equate results. $\boxed{\checkmark}$

Green

$\Rightarrow (D)\ \Phi = \varphi_2 - \varphi_1 = 0 \quad , (N)$

Green's function

Idea: solve (D) or (N) b.v.p. for a point source with trivial (D) or (N) boundary conditions, then use this sol. to solve the general case.


$\Rightarrow$ (D) $\Phi = \varphi_2 - \varphi_1 = 0$, (N
Green's function
Idea: solve (D) or (N) b.v.p. for a point source with trivial (D) or (N) boundary conditions, then use this sol. to solve the general case.
$\rightarrow$ this specific sol. w/ point sources are called Green's functions:
Dirichlet
$\Delta_y G_{D,N}(\vec{y},\vec{x}) = \delta^3(\vec{y}-\vec{x})$ for $\vec{y} \in R$
$G_D(y,x) = 0$ for $y \in \partial R$ (D)
Neumann
$\nabla_s G_N(y,x) = 0$ for $y \in \partial R$ (N)
Suppose you know G

Green's identity

$\forall \Phi, \Psi$: $\int_R d^3y\, \Phi\, \Delta_y \Psi = \int_R \Psi \Delta \Phi + \oint_{\partial R} \left(\Phi \nabla_s \Psi - \Psi \nabla_s \Phi \right)$

$\left(\nabla_s \equiv \vec{s} \cdot \vec{\nabla} \right)$

Pf : $\int_R \underset{(2)}{\vec{\nabla}\Phi} \cdot \underset{(1)}{\vec{\nabla}\Psi} = \ldots$ integrate by parts (1) & (2) and equate results. $\boxed{\checkmark}$

Dirichl

Neu

...up.

...ed (D) or (N)

...s sol. to

...int sources

...unctions!:

...$\vec{x}$) for $\vec{y} \in R$

for $y \in \partial R$ (D)

for $y \in \partial R$ (N)

Suppose you know $G_{D/N}(y,x)$,

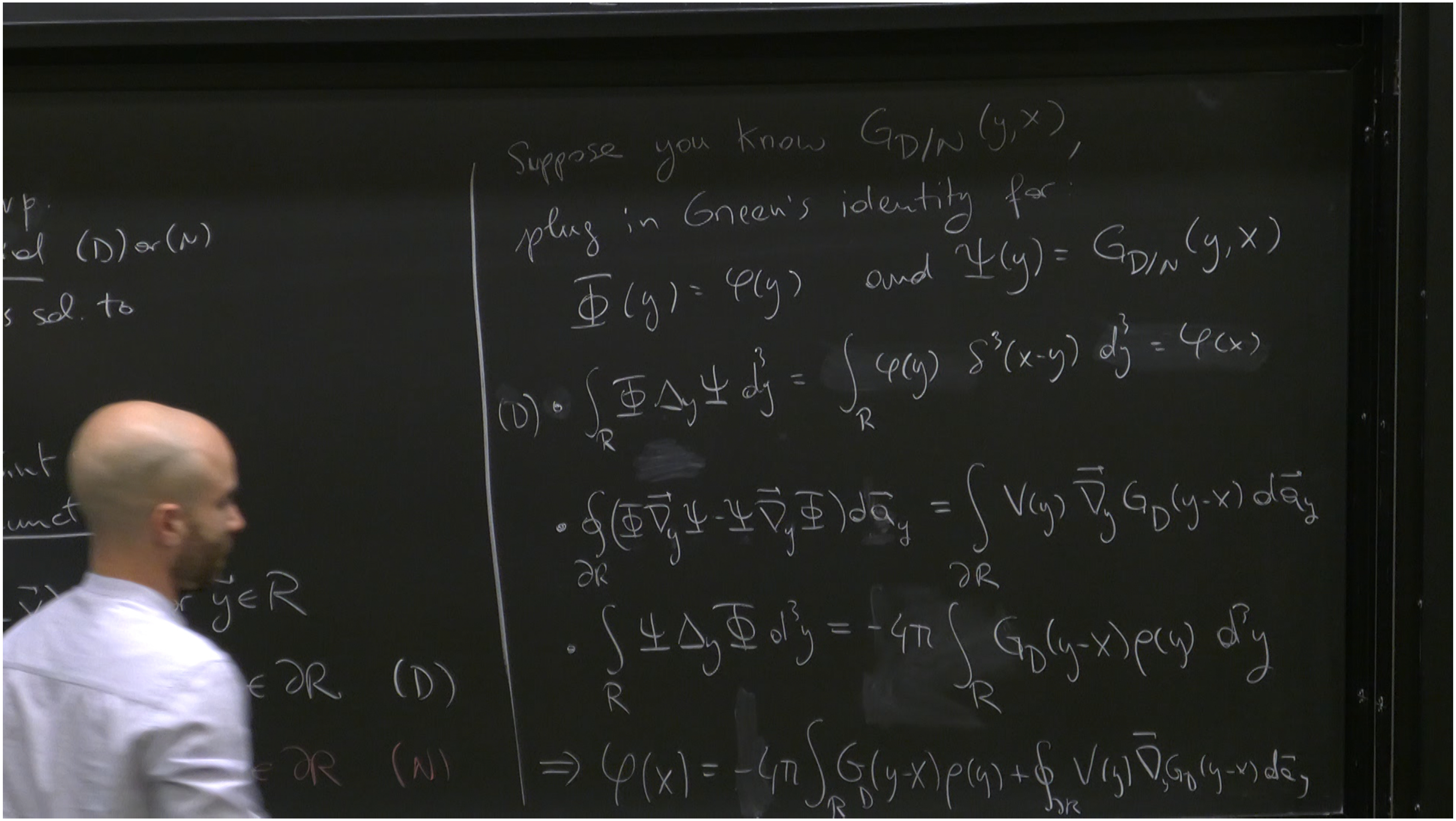
plug in Green's identity for:

$\Phi(y) = \varphi(y)$ and $\Psi(y) = G_{D/N}(y,x)$

(D) $$\int_R \Phi \Delta_y \Psi \, d^3y = \int_R \varphi(y) \, \delta^3(x-y) \, d^3y = \varphi(x)$$

$$\parallel$$

$$\oint_{\partial R} \left(\Phi \vec{\nabla}_y \Psi - \Psi \vec{\nabla}_y \Phi \right) d\vec{a}_y$$



Suppose you know $G_{D/N}(y,x)$,
plug in Green's identity for
$\Phi(y) = \varphi(y)$ and $\Psi(y) = G_{D/N}(y,x)$
(D) $\cdot \int_R \Phi \Delta_y \Psi \, d^3y = \int_R \varphi(y) \, \delta^3(x-y) \, d^3y = \varphi(x)$
$\cdot \oint_{\partial R} (\Phi \vec{\nabla}_y \Psi - \Psi \vec{\nabla}_y \Phi) \, d\vec{a}_y = \int_{\partial R} V(y) \, \vec{\nabla}_y G_D(y-x) \, d\vec{a}_y$
$\cdot \int_R \Psi \Delta_y \Phi \, d^3y = -4\pi \int_R G_D(y-x) \rho(y) \, d^3y$
$\Rightarrow \varphi(x) = -4\pi \int_R G_D(y-x) \rho(y) + \oint_{\partial R} V(y) \vec{\nabla}_y G_D(y-x) \, d\vec{a}_y$
(D) or (N)
sol. to
$\vec{y} \in R$
$\in \partial R$ (D)
$\in \partial R$ (N)

Green's function

Idea: solve (D) or (N) b.v.p. for a point source with trivial (D) or (N) boundary conditions, then use this sol. to solve the general case.

→ this specific sol. w/ point sources are called Green's functions:

Dirichlet

$$\begin{cases} \Delta_y G_{D,N}(\vec{y},\vec{x}) = \delta^3(\vec{y}-\vec{x}) & \text{for } \vec{y}\in R \\ G_D(y,x) = 0 & \text{for } y \in \partial R \quad (D) \end{cases}$$

Neumann

$$\nabla_s G_N(y,x) = \text{Area}(\partial R) \quad \text{for } y\in\partial R \quad (N)$$

Suppose you kn...

plug in Green...

$\Phi(y) = \varphi(y$...

(D) • $\int_R \Phi \Delta_y \Psi \, d^3y$...

• $\oint_{\partial R} (\Phi \vec{\nabla}_y \Psi - \Psi \vec{\nabla}$...

• $\int_R \Psi \Delta_y \Phi \, d$...

$\Rightarrow \boxed{\varphi(x) = -4\pi ...}$

Green's identity

$\forall \Phi, \Psi$: $\displaystyle\int_R d^3y\, \Phi\, \Delta_y \Psi = \int_R \Psi \Delta \Phi + \oint_{\partial R} \left(\Phi \nabla_s \Psi - \Psi \nabla_s \Phi \right)$

$\left(\nabla_s \equiv \vec{s} \cdot \vec{\nabla} \right)$

Pf: $\displaystyle\int_R \underset{(2)}{\vec{\nabla}\Phi} \cdot \underset{(1)}{\vec{\nabla}\Psi} = \ldots$ integrate by parts (1) & (2) and equate results. ☑

$$\boxed{1 = \int_R \Delta_y G_N(y-x) = \oint_{\partial R} \nabla_s G_N(y-x)}$$

Di

Green's function

Idea: solve (D) or (N) b.v.p. for a point source with trivial (D) or (N) boundary conditions, then use this sol. to solve the general case.

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Neumann $\nabla_y G_N(y,x) = \dfrac{1}{\text{Area}(\partial R)}$ for $y\in\partial R$ (N)

Suppose you know G...

plug in Green's ide...

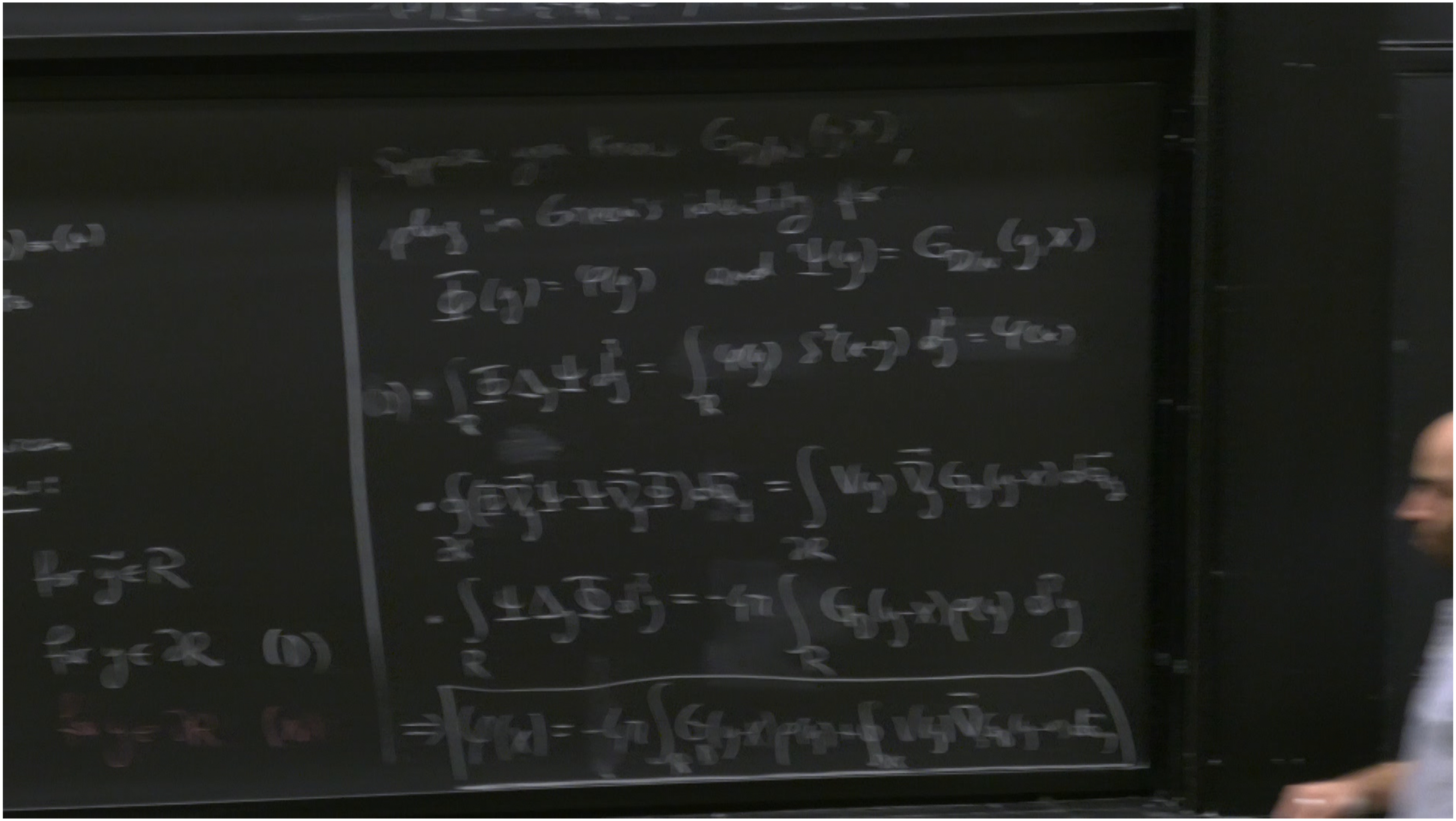
$\Phi(y) = \varphi(y)$ on ...

(D) • $\int_R \Phi \Delta_y \Psi \, d^3y = \int_R \varphi$ ...

• $\oint_{\partial R} (\Phi \vec{\nabla}_y \Psi - \Psi \vec{\nabla}_y \Phi) d\vec{a}$ ...

• $\int_R \Psi \Delta_y \Phi \, d^3y =$ ...

$\Rightarrow \boxed{\varphi(x) = -4\pi \ldots}$



How to find $G_D(y,x)$?

One strategy: "<u>method of images</u>"

<u>Rmk</u>:

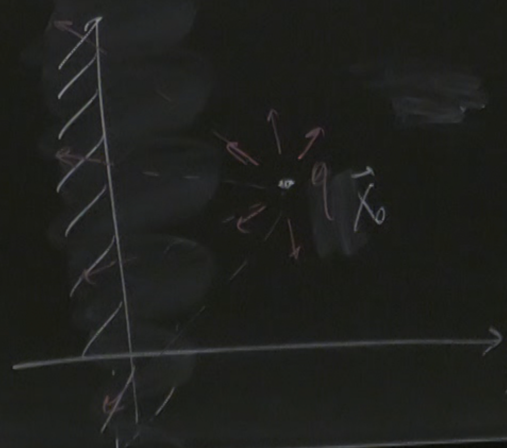
How to find $G_D(y,x)$?

One strategy: "method of images"

Rmk: $\vec{\nabla}_y G_D(y,x) \perp$ boundary $\Leftrightarrow$ $G_D(y,x) = 0 = \text{cte}$ for $y \in \partial R$.

Solving for $G_D(y,x)$ = solve for el. pot. $\varphi(y)$ for a point charge at x, with the condition that $\vec{E}(y) \perp \partial R$.

Ex: $R = \{x > 0\}$ $\quad -q \quad \bar{X}_0$



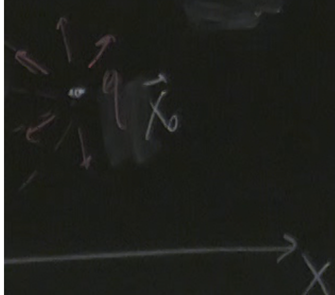
idea: $\varphi_{\text{bare}}(y) = \dfrac{q}{|\vec{X}_0 - \vec{y}|}$

$\Leftrightarrow \; G_D(y,x) = 0 = \text{const} \;$ for $y \in \partial R$.

for el. pot. $\varphi(y)$ for a

condition that $\vec{E}(y) \perp \partial R$.

idea: $\varphi_{basic}(y) = \dfrac{q}{|\vec{x}_0 - \vec{y}|}$, $\vec{E}_{basic}(y) = \dfrac{q}{|\vec{x}_0 - \vec{y}|^2} \dfrac{\vec{x}_0 - \vec{y}}{|\vec{x}_0 - \vec{y}|}$



How to find $G_D(y,x)$?

One strategy: "method of images"

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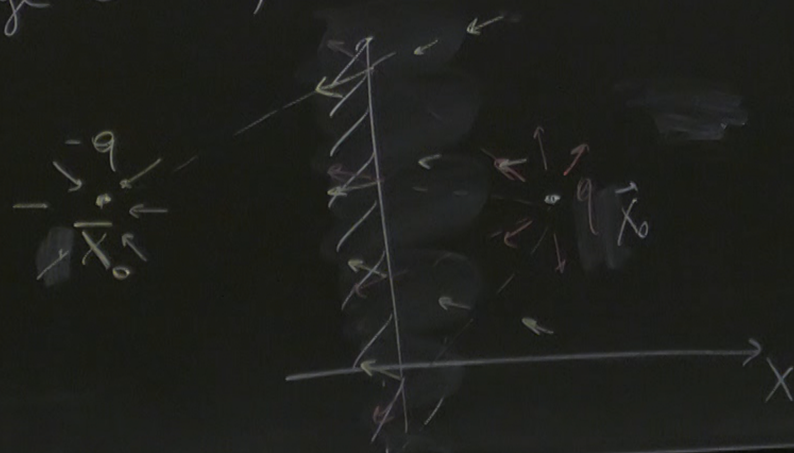
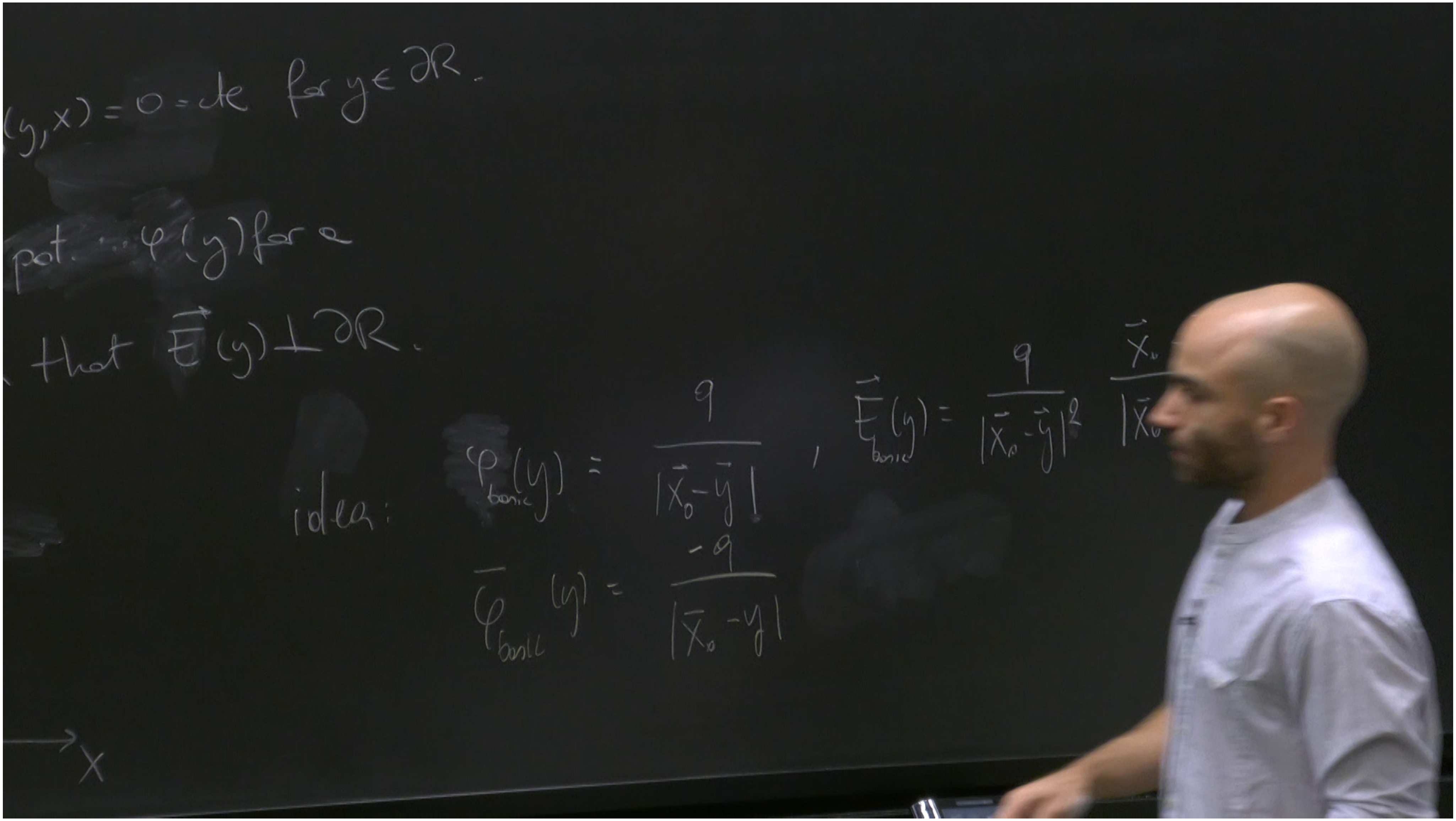


idea: $\varphi_{basic}(y) = \dfrac{q}{|\vec{x}_0 - \vec{y}|}$,

$\bar{\varphi}_{basic}(y) = \dfrac{-q}{|\bar{x}_0 - y|}$


(y,x) = 0 = cte for y ∈ ∂R.
pot. φ(y) for a
that E(y) ⊥ ∂R.
idea:
φ_basic(y) = q / |x_0 − y|
E_basic(y) = q / |x_0 − y|^2
φ̄_basic(y) = −q / |x̄_0 − y|
x

R.

I declare:

$$\varphi(y) = \varphi_{basic}(y) + \bar{\varphi}_{basic}(y)$$

check: $$\Delta\varphi(y) = \Delta\varphi_{basic} + \Delta\bar{\varphi}_{basic} = \delta^3(\underset{R}{x_0} - y) + \delta^3(\underset{R}{\bar{x}_0} - y)$$

$$\varphi_{basic}(y) = \frac{q}{|\vec{x}_0 - \vec{y}|}, \quad \vec{E}_{basic}(y) = \frac{q}{|\vec{x}_0 - \vec{y}|^2}\,\frac{\vec{x}_0 - \vec{y}}{|\vec{x}_0 - \vec{y}|}$$

$$\bar{\varphi}_{basic}(y) = \frac{-q}{|\bar{x}_0 - y|}$$

identically $\emptyset$ in R

I declare:

$$\varphi(y) = \varphi_{basic}(y) + \bar{\varphi}_{basic}(y)$$

check: $$\Delta\varphi(y) = \Delta\varphi_{basic} + \Delta\bar{\varphi}_{basic} = \delta^3(\underset{\in R}{x_0}-y) + \overbrace{\delta^3(\underset{\notin R}{\bar{x}_0}-y)} = \delta^3(x_0-y) \text{ for } y\in R$$

$\vec{E} = -\vec{\nabla}\varphi$ is normal at boundary by construction $\boxed{\checkmark}$

$$= \frac{q}{|\vec{x}_0-\vec{y}|}, \quad \vec{E}_{basic}(y) = \frac{q}{|\vec{x}_0-\vec{y}|^2}\frac{\vec{x}_0-\vec{y}}{|\vec{x}_0-\vec{y}|}$$

$$= \frac{-q}{|\bar{x}_0-y|}$$

I declare:

$$\varphi(y) = \varphi_{basic}(y) + \bar{\varphi}_{basic}(y)$$

identically $\emptyset$ in R

check: $\Delta\varphi(y) = \Delta\varphi_{basic} + \Delta\bar{\varphi}_{basic} = \underset{\in R}{\delta^3(x_0-y)} + \overbrace{\underset{\notin R}{\delta^3(\bar{x}_0-y)}} = \delta^3(x_0-y)$ for $y \in R$

$\vec{E} = -\vec{\nabla}\varphi$ is normal by construction, $\varphi(y \in \partial R) = \dfrac{q}{|x_0-y|} - \dfrac{q}{|\bar{x}_0-y|} = 0$

at boundary

$$= \frac{q}{|\vec{x}_0-\vec{y}|}, \quad \vec{E}_{basic}(y) = \frac{q}{|\vec{x}_0-\vec{y}|^2}\,\frac{\vec{x}_0-\vec{y}}{|\vec{x}_0-\vec{y}|}$$

$$y) = \frac{-q}{|\bar{x}_0-y|}$$

$G_D(y,x) =$ Coulombic pot of unit charge at $\vec{x}$ + its images charges (whose position & magnitude is determined geometrically from the shape of region R).

The wave equation

$$\Box A_\mu = -4\pi j_\mu$$

$\hookrightarrow$ time component : $\boxed{\Box \varphi = 4\pi \rho}$

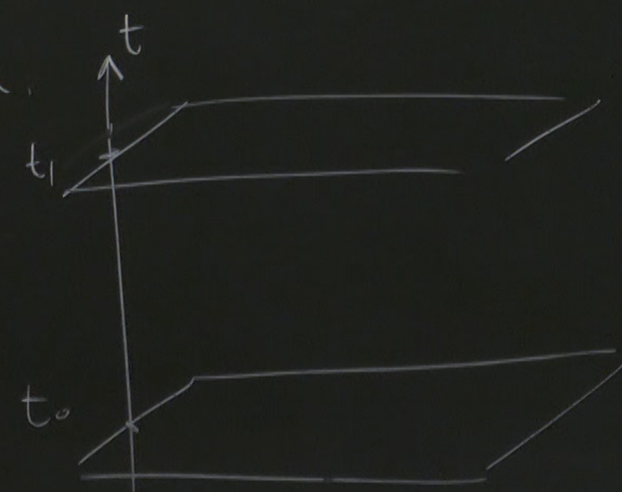
The wave equation

$\Box A_\mu^{(L)} = -4\pi j_\mu$ (Lorentz gauge)

↳ time component: $\boxed{\Box \varphi^{(L)} = 4\pi\rho}$, $\Box = \nabla_\mu \nabla^\mu = -\frac{1}{c^2}\partial_t^2 + \Delta$

Here, consider this equation over the region:

$$M = \mathbb{R}^3 \times [t_0, t_1]$$



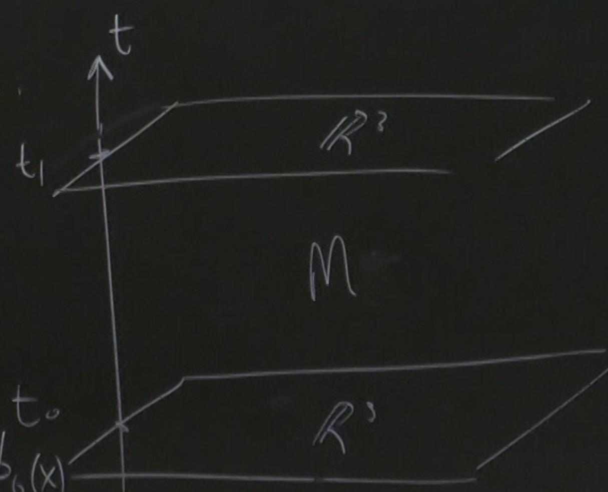
The wave equation

(Lorentz gauge)

$$\Box A_\mu^{(L)} = -4\pi j_\mu$$

↳ time component: $\boxed{\Box \varphi^{(L)} = 4\pi\rho}$, $\Box = \nabla_\mu \nabla^\mu = -\frac{1}{c^2}\partial_t^2 + \Delta$

Here, consider this equation over the region



$$M = \mathbb{R}^3 \times [t_0, t_1]$$

$\exists!$ solution to $\Box\varphi = 4\pi\rho$ at

fixed initial (final) conditions
$t_0 \to t_1$

$$\begin{cases} \varphi(t_0, x) = \phi_0(x) \\ \partial_t \varphi(t_0, x) = \pi_0(x) \end{cases}$$