Title: Classical Physics Lecture - 092223

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Last time:

$$S[\gamma] = -\int_{\lambda_0}^{\lambda_1} d\lambda \left(m\sqrt{-\gamma'_\mu \gamma'^\mu} + q A_\mu(\gamma(\lambda))\gamma'^\mu \right)$$

where $A_\mu(x)$ is a fixed (non-dynamical) co-vector field

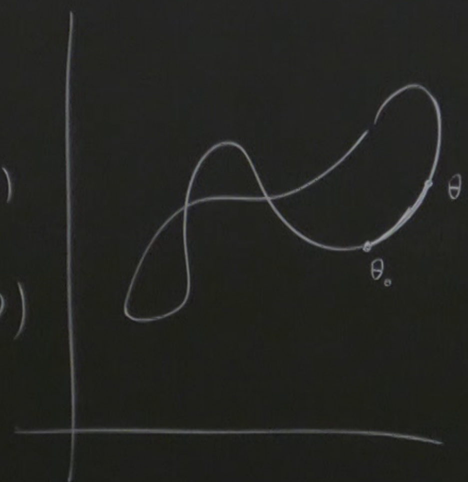
Rmk: • $S[\gamma] = -m\int_\gamma d\tau - q\int_\gamma A_\mu(\gamma)\, d\gamma^\mu$

• invariant under $\lambda \to \lambda' = f(\lambda)$ (reparametrizations)

and under Lorentz symmetry: $\begin{cases} \gamma^\mu \longmapsto (\Lambda^{-1})^\mu{}_\nu \gamma^\nu \\ A_\mu \longmapsto \Lambda^\nu{}_\mu A_\nu \end{cases}$

Q: Euler-Lagrange e.o.m. for γ?

$$\begin{cases} y = y(\theta) \\ x = x(\theta) \end{cases}$$



)

vector field

metrizations)

$\gamma^\mu \mapsto (\Lambda^{-1})^\mu{}_\nu \gamma^\nu$

$A_\mu \mapsto \Lambda^\nu{}_\mu A_\nu$

$$\frac{d}{d\lambda}\frac{\partial L}{\partial \gamma'^\mu} - \frac{\partial L}{\partial \gamma^\mu} =$$

$$= \frac{d}{d\lambda}\Big(\underbrace{\frac{m\gamma'_\mu}{\sqrt{-\gamma'_\nu\gamma'^\nu}}}_{p_\mu} + qA_\mu(\gamma)\Big) - q\gamma'^\nu \frac{\partial A_\nu}{\partial x^\mu}\Big|_{x=\gamma}$$

$$= \frac{d}{d\lambda}p_\mu + q\frac{\partial A_\mu}{\partial x^\nu}\Big|_{x=\gamma}\underbrace{\frac{d\gamma^\nu}{d\lambda}}_{\gamma'^\nu} - q\gamma'^\nu\frac{\partial A_\nu}{\partial x^\mu}\Big|_{x=\gamma}$$

$$= \frac{d}{d\lambda}p_\mu + q\gamma'^\nu(\partial_\nu A_\mu - \partial_\mu A_\nu)\big|_{x=\gamma} \hat{=} 0$$

$$\Rightarrow \dot{p}_\mu + q\dot{\gamma}^\nu(\partial_\nu A_\mu - \partial_\mu A_\nu) \hat{=} 0$$

)

vector field

(...metrizations)

$\ldots \longmapsto (\Lambda^{-1})^{\mu}{}_{\nu}\gamma^{\nu}$

$\ldots \longmapsto \Lambda^{\nu}{}_{\mu} A_{\nu}$

$$\frac{d}{d\lambda}\frac{\partial L}{\partial \gamma'^{\mu}} - \frac{\partial L}{\partial \gamma^{\mu}} =$$

$$= \frac{d}{d\lambda}\Big(\underbrace{\frac{m\gamma'_{\mu}}{\sqrt{-\gamma'_{\nu}\gamma'^{\nu}}}}_{p_{\mu}} + q A_{\mu}(\gamma) \Big) - q\gamma'^{\nu}\frac{\partial A_{\nu}}{\partial x^{\mu}}\Big|_{x=\gamma}$$

$$= \frac{d}{d\lambda} p_{\mu} + q \frac{\partial A_{\mu}}{\partial x^{\nu}}\Big|_{x=\gamma} \underbrace{\frac{d\gamma^{\nu}}{d\lambda}}_{\gamma'^{\nu}} - q\gamma'^{\nu}\frac{\partial A_{\nu}}{\partial x^{\mu}}\Big|_{x=\gamma}$$

$$= \frac{d}{d\lambda} p_{\mu} + q\gamma'^{\nu}\left(\partial_{\nu}A_{\mu} - \partial_{\mu}A_{\nu}\right)\Big|_{x=\gamma} \hat{=} 0$$

$$\Rightarrow \dot{p}_{\mu} + q u^{\nu}\left(\partial_{\nu}A_{\mu} - \partial_{\mu}A_{\nu}\right) \hat{=} 0$$

$\frac{\partial A_\nu}{\partial x^\mu}\Big|_{x=r}$

$\frac{\partial A_\nu}{\partial x^\alpha}\Big|_{x=r}$

$= 0$

<u>Rmk</u>

$$\frac{1}{2}\frac{d}{d\tau}(-m^2) = p^\mu \dot{p}_\mu \doteq -q\,\overbrace{\underbrace{m u^\mu\, u^\nu}_{\text{sym}}}^{p^\mu}\underbrace{(\partial_\nu A_\mu - \partial_\mu A_\nu)}_{\text{skew}} \equiv 0$$

$\mu \leftrightarrow \nu$

MAXWELL'S THEORY

(Electrodynamics)

electric charge density

I	$\vec{\nabla}\cdot\vec{E} = 4\pi\rho$		Gauss
II	$\vec{\nabla}\cdot\vec{B} = 0$		
III	$\vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t}$		Faraday
IV	$\vec{\nabla}\times\vec{B} = \frac{1}{c}\left(4\pi\vec{j} + \underline{\frac{\partial\vec{E}}{\partial t}}\right)$		Ampère-Maxwell

electric current

$$\vec{F}_L = q\left(\vec{E} + \vec{v}\times\vec{B}\right)$$ Lorentz

rge

Gauss

Faraday

Ampère-Maxwell

nt

Lorentz

div(IV) and use I :

$$0 = \frac{1}{c}\left(4\pi\, \vec{\nabla}\cdot\vec{j} + \frac{\partial}{\partial t}\underbrace{\vec{\nabla}\vec{E}}_{(\mathrm{I}):\ 4\pi\rho}\right)$$

$$\rightarrow \boxed{\partial_t \rho + \vec{\nabla}\cdot\vec{j} = 0}$$

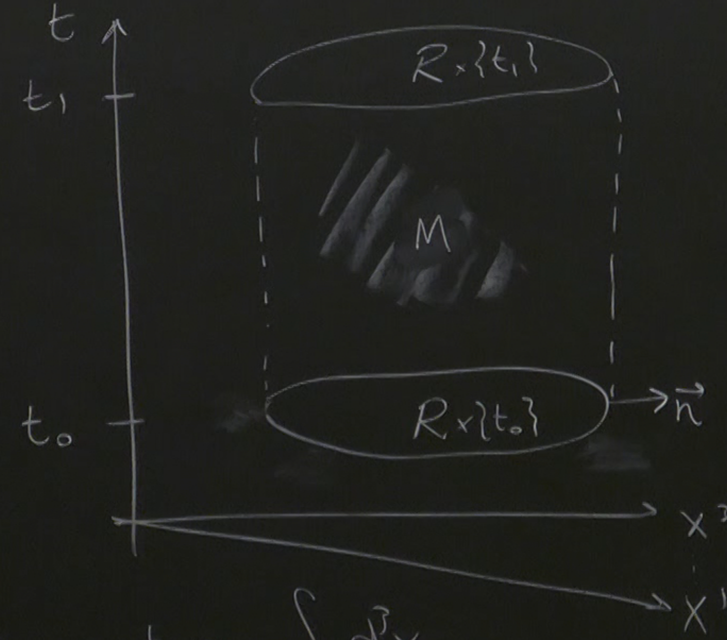
continuity eq.
(for the el. charge)

$\equiv$ charge conservation:

$$Q_R(t) = \int_R d^3x \ \rho(\vec{x},t)$$ tot. el. charge in R

$$\Phi_{\partial R}(t) = \int_{\partial R} d^2x \ \vec{n}\cdot\vec{j}(x,t)$$ outgoing flux of the el. current.

$$Q_R(t_1) - Q_R(t_0) = -\int_{t_0}^{t_1} dt \; \Phi_{\partial R}(t)$$



harge in R

Integrate cont. eq. $\int_R d^3x$

g flux of

current.

$$0 = \frac{d}{dt} Q_R + \int_R \vec{\nabla}\cdot\vec{j} = \frac{d}{dt} Q_R + \Phi_{\partial R}$$

balance eq. (charge conserv.)

• curl of III & IV

$$0 = \vec{\nabla} \times \left(\vec{\nabla} \times \vec{E} + \frac{1}{c} \frac{\partial \vec{B}}{\partial t} \right) \quad , \quad \Delta = \vec{\nabla}^2 = \partial_x^2 + \partial_y^2 + \partial_z^2$$

(Laplacian)

$$= \vec{\nabla} (\vec{\nabla} \cdot \vec{E}) - \Delta \vec{E} + \frac{1}{c} \frac{\partial}{\partial t} \vec{\nabla} \times \vec{B}$$

$$\rightarrow \; -\frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} + \Delta \vec{E} = 4\pi \left(\vec{\nabla} \rho + \frac{\partial \vec{j}}{\partial t} \right)$$

Similarly:

$$\rightarrow \; -\frac{1}{c^2} \frac{\partial^2 \vec{B}}{\partial t^2} + \Delta \vec{B} = \frac{4\pi}{c} \vec{\nabla} \times \vec{j}$$

∂R the el. current.

$-\partial_z^2$

left hand side:

$$\Box = -\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \Delta$$ Wave operator (speed c)

(d'Alambertian)

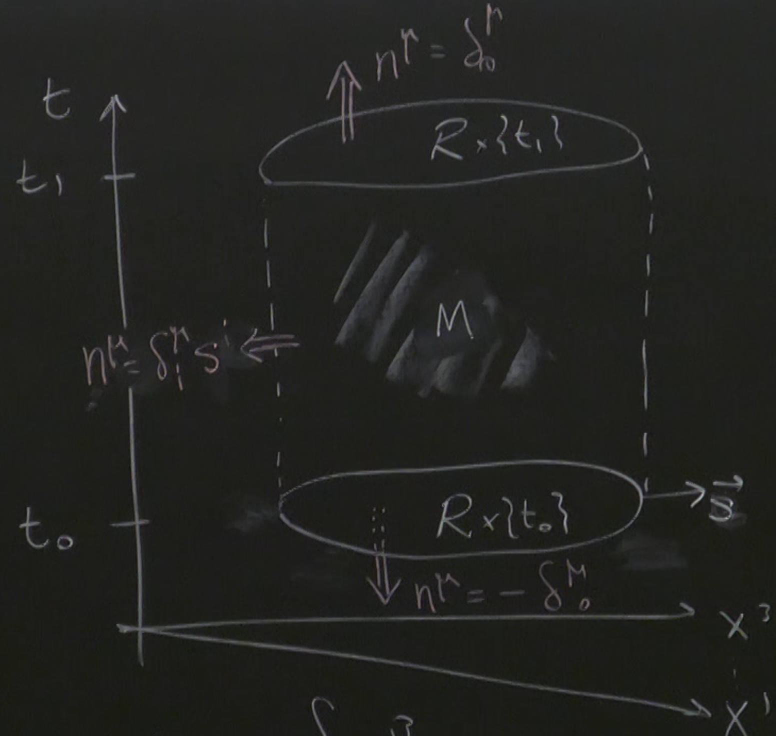
$$\vec{j} = \rho \vec{v}$$

Introduce. $\nabla_\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, \vec{\nabla}\right)$

$\Box = \eta^{\mu\nu}\nabla_\mu\nabla_\nu$ (Lorentz invariant operator)

Continuity eq can be written as:

$\nabla_\mu j^\mu = 0$ for $j^\mu = (c\rho, \vec{j})$



$$Q_R(t_1) - Q_R(t_0) = -\int_{t_0}^{t_1} dt \; \Phi_{\partial R}(t)$$

n^μ 4-normal to ∂M (outgoing wrt M)

Integrate cont. eq. $\int_R d^3x$

$$0 = \frac{d}{dt} Q_R + \int_R \vec{\nabla} \cdot \vec{j} = \frac{d}{dt} Q_R + \Phi_{\partial R}$$

balance eq. (charge conserv.)

...ent operator)

...-current

$$0 = \int_M \nabla_\mu j^\mu \, d^4x$$

4 dim div. thm

$$\partial M = R\times\{t_0\} \cup R\times\{t_1\} \cup \partial R\times[t_0,t_1]$$

$$= \int_{\partial M} n_\mu j^\mu \, d^3x$$

$$= \int_{R\times\{t_1\}} \overbrace{j^0}^{\rho} - \int_{R\times\{t_0\}} j^0 + \int_{\partial R\times[t_0,t_1]} s_i j^i$$

$\Uparrow n^\mu$ $\Downarrow n^\mu$

$$= Q_R(t_1) - Q_R(t_0) + \int_{t_0}^{t_1} dt \, \Phi_{\partial R}(t)$$

$\ldots S \cdot \vec{j}(x,t)$ ... the el. current. $\quad 0 = \frac{d}{dt} Q_R + \int_R \nabla \vec{j} = \frac{d}{dt} Q_R + \Phi_{\partial R}$ (charge conserv.)

... side:

$\ldots \frac{\partial^2}{\ldots t^2} + \Delta$ Wave operator (speed c)

... (...bertion)

...ce. $\nabla_\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, \vec{\nabla} \right)$

$\ldots^{\mu\nu} \nabla_\mu \nabla_\nu$ (Lorentz invariant operator)

... eq can be written as:

$\ldots = 0$ for $j^\mu = (c\rho, \vec{j})$ 4-current

$$0 = \int_M \nabla_\mu j^\mu \, d^4x$$ 4 dim div thm

$$= \int_{\partial M} n_\mu j^\mu \, d^3x \qquad \partial M = R \times \{t_0\} \cup R \times \{t_1\} \cup \partial R \times [t_0, t_1]$$

$$= \int_{R \times \{t_1\}} \overset{c\rho}{j^0} - \int_{R\times\{t_0\}} j^0 + \int_{\partial R \times [t_0, t_1]} s_i j^i$$

$\Uparrow n^\mu \qquad \Downarrow n^\mu$

$$= c\left(Q_R(t_1) - Q_R(t_0) + \int_{t_0}^{t_1} dt \, \Phi_{\partial R}(t) \right)$$

$$F_{\mu\nu} = \begin{cases} 0 & \text{if } (\mu,\nu) = (0,0) \\ -E^i & \text{if } (0,i) \\ E^i & \text{if } (i,0) \\ \epsilon^{ij}{}_k B^k & (i,j) \end{cases}$$

Maxwell: $$\begin{cases} \nabla_\mu F^{\mu\nu} = -4\pi j^\nu \\ \nabla_{[\mu} F_{\nu\rho]} = 0 \end{cases}$$

Gauss-Ampère-Maxwell (I, IV)

Faraday (II, III)

where $_{[\ldots]}$ is defined as:

$$A_{[\mu_1 \cdots \mu_p]} = \frac{1}{p!} \sum_{\sigma \in S_p} \mathrm{sign}(\sigma)\, A_{\mu_{\sigma(1)} \cdots \mu_{\sigma(p)}}$$

$\uparrow$ grp of permutation of p elements

$$A_{[\mu_1 \mu_2]} = \frac{1}{2}\left(A_{\mu_1\mu_2} - A_{\mu_2\mu_1}\right), \quad A_{(\mu_1\mu_2)} = \frac{1}{2}\left(A_{\mu_1\mu_2} + A_{\mu_2\mu_1}\right)$$

$$\epsilon_{[ijk]} = \epsilon_{ijk}, \quad \epsilon_{(ijk)} \equiv 0$$

$$A_{(\mu_1 \cdots \mu_p)} = \frac{1}{p!} \sum_{\sigma \in S_p} A_{\mu_{\sigma(1)} \cdots \mu_{\sigma(p)}}$$

$\epsilon^{ij}{}_k B^k$ (i,j)

$$A_{[\mu_1\mu_2]} = \frac{1}{2}($$

$$\epsilon_{[ijk]} = \epsilon$$

$$A_{(\mu_1\cdots\mu_p)} =$$

ll: $\begin{cases} \nabla_\mu F^{\mu\nu} = -4\pi j^\nu \\ \nabla_{[\mu} F_{\nu\rho]} = 0 \end{cases}$ Gauss-Ampère-Maxwell (I, IV)

Faraday (II, III)

$\hookrightarrow$ ex: using $F_{\mu\nu}$ skew prove $3\nabla_{[\mu}F_{\nu\rho]} = \underset{(\mu\nu\rho)}{\text{cycl}}\ \nabla_\mu F_{\nu\rho}$

From Maxwell eqs:

$$\bullet \quad 0 \equiv \underbrace{\nabla_\nu}_{\text{sym in } \mu\nu} \Big(\nabla_\mu \underbrace{F^{\mu\nu}}_{\text{skew}} \Big) \overset{\text{Maxwell}}{=} 4\pi \nabla_\nu j^\nu \quad \text{continuity eq.}$$

$$\frac{\partial^2}{\partial x^\mu \partial x^\nu} = \frac{\partial^2}{\partial x^\nu \partial x^\mu}$$

- wave equation

$$0 \underset{\text{Maxwell}}{=} \nabla^\mu \Big(\nabla_{[\mu} F_{\nu\rho]} \Big) = \frac{1}{3} \Big(\nabla^\mu \nabla_\mu F_{\nu\rho} + \nabla_\nu \nabla^\mu F_{\rho\mu} + \nabla_\rho \nabla^\mu F_{\mu\nu} \Big)$$

$\nabla_\nu j^\nu$ continuity eq.

$$\frac{1}{3}\left(\nabla^\mu \nabla_\mu F_{\nu\rho} + \nabla_\nu \nabla^\mu \overbrace{F_{\rho\mu}}^{=-F_{\mu\rho}} + \nabla_\rho \nabla^\mu F_{\mu\nu}\right)$$

$$\frac{1}{3}\left(\Box F_{\nu\rho} + 4\pi \nabla_\nu j_\rho - 4\pi \nabla_\rho j_\nu\right) = \frac{1}{3}\left(\Box F_{\nu\rho} + 8\pi \nabla_{[\nu} j_{\rho]}\right)$$

III & IV :

$$\vec{\nabla}\times\left(\vec{\nabla}\times\vec{E}+\frac{1}{c}\frac{\partial\vec{B}}{\partial t}\right)$$

$$\vec{\nabla}(\vec{\nabla}\cdot\vec{E})-\Delta\vec{E}+\frac{1}{c}\frac{\partial}{\partial t}\vec{\nabla}\times\vec{B}$$

$$-\frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2}+\Delta\vec{E}=4\pi\left(\vec{\nabla}\rho+\frac{\partial\vec{j}}{\partial t}\right)$$

$$\Box F_{\mu\nu}+8\pi\nabla_{[\mu}j_{\nu]}=0$$

...ly:

$$-\frac{1}{c^2}\frac{\partial^2\vec{B}}{\partial t^2}+\Delta\vec{B}=\frac{4\pi}{c}\vec{\nabla}\times\vec{j}$$

, $\Delta=\vec{\nabla}^2=\partial_x^2+\partial_y^2+\partial_z^2$

(Laplacian)

left hand side ...

$$\Box=-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}+ \dots$$

(d'Alambert...

Introduce:

$$\Box=\eta^{\mu\nu}\nabla \dots$$

Continuity eq ...

$$\nabla_\mu j^\mu=0$$

$8\pi\, D_{[\nu} j_{\rho]} \Big)$

Rmk

Lorentz transformations mix E & B !

- electrostatic config in ref frame K

  $B, j \equiv 0$, $\partial_t E, \partial_t \rho \equiv 0$

- in a boosted frame K' wrt K

  $\vec{j} = \rho \vec{v}$ $\quad \partial_t B = \partial_t E = \partial_t j = \partial_t \rho = 0$ (static)

  $\hookrightarrow$ Maxwell-Ampère $\rightsquigarrow$ $B \neq 0$

Rmk

$$\begin{cases} \nabla^{\mu} F_{\mu\nu} = 4\pi j_{\nu} \\ \nabla_{[\mu} F_{\nu\rho]} = 0 \end{cases}$$

make sense in all dimensions $(d+1)$

provided $F_{\mu\nu} = F_{[\mu\nu]}$

In $d \neq 3$ $\quad \#(F_{0i}) \neq \#(F_{ij})$

$\#(E^{i}) \neq \#(B^{ik})$

Only in $3+1$ d we have electromagnetic duality

$$\left(\begin{array}{c|c} & \vec{E} \\ \hline & B^{T} = -B \end{array}\right)$$

Rmk

$$\begin{cases} \nabla^\mu F_{\mu\nu} = 4\pi j_\nu \\ \nabla_{[\mu} F_{\nu\rho]} = 0 \end{cases}$$

make sense in all dimensions $(d+1)$

provided $F_{\mu\nu} = F_{[\mu\nu]}$ (skew)

In $d \neq 3$ $\quad \#(F_{0i}) \neq \#(F_{ij})$

$$\#(E^i) \neq \#(B^k), \quad d \neq \frac{d(d-1)}{2}$$

Only in 3+1 d we have electromagnetic duality in absence of sources $(j^\mu \equiv 0)$ $\quad \vec{E} \leftrightarrow \vec{B}$

dual EM tensor:

$$\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$$

$\uparrow$ completely skew symbol.

w/ $\epsilon^{0123} = 1$

$\epsilon_{\mu\nu\rho\sigma}$ is an invariant Lorentz pseudo tensor:

$$\epsilon'_{\mu'\nu'\rho'\sigma'} = \Lambda^{\mu}{}_{\mu'}\Lambda^{\nu}{}_{\nu'}\Lambda^{\rho}{}_{\rho'}\Lambda^{\sigma}{}_{\sigma'}\,\epsilon_{\mu\nu\rho\sigma}$$

$$= \det(\Lambda)\,\epsilon_{\mu'\nu'\rho'\sigma'}$$

$$= \pm\,\epsilon_{\mu'\nu'\rho'\sigma'}$$ (w/ (−) if orientation is flipped)

dual EM tensor:

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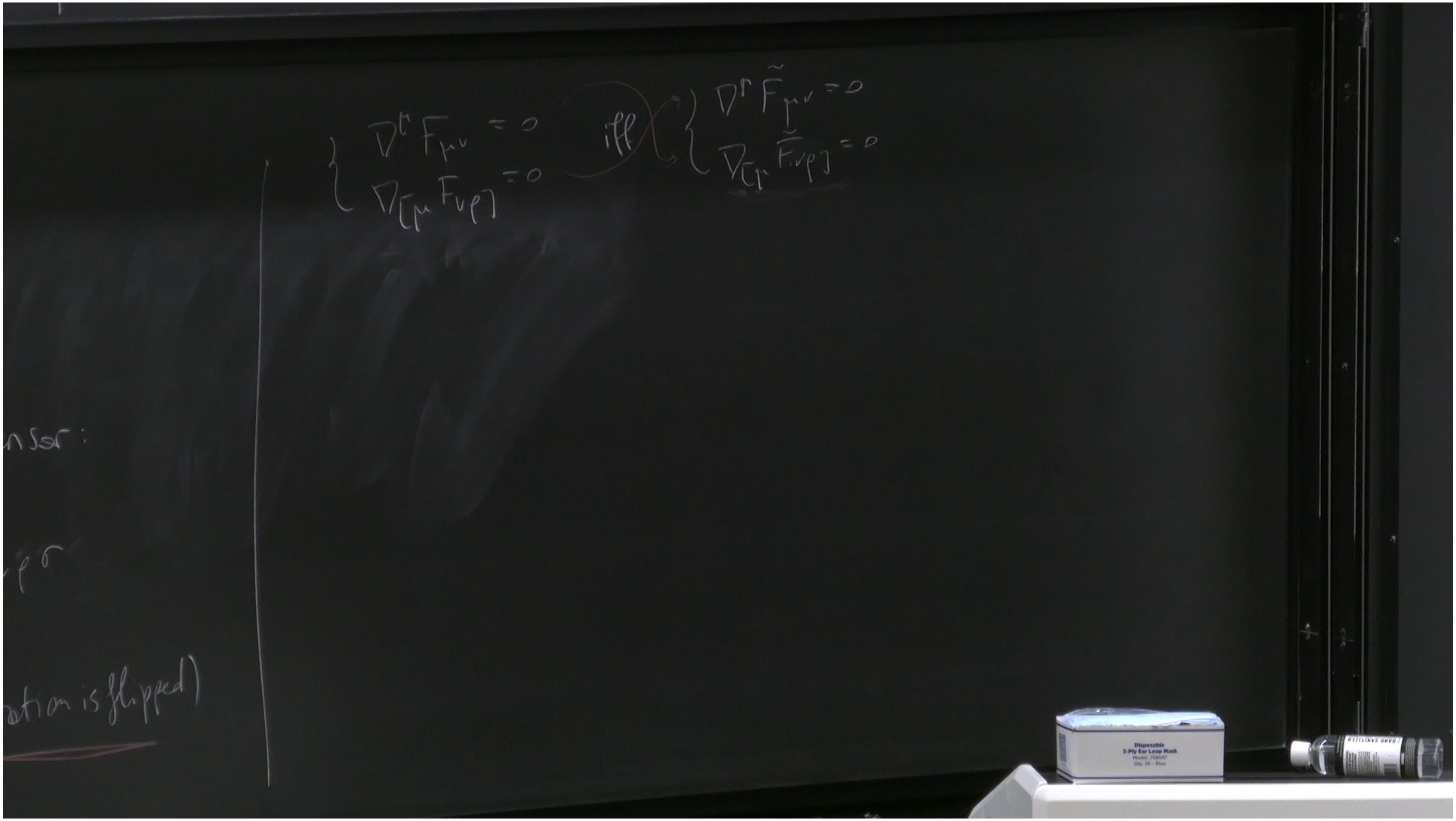
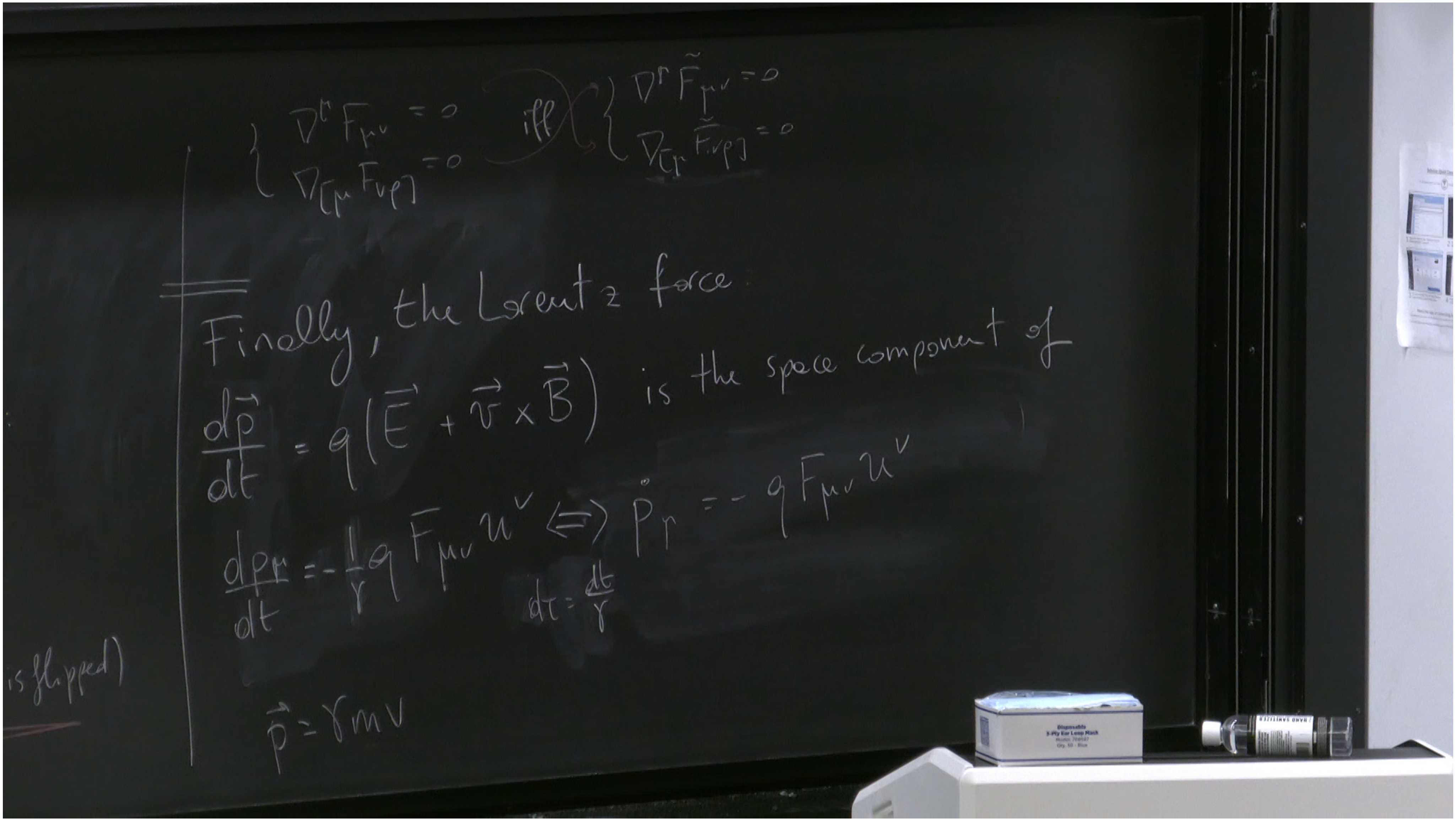

$\nabla^\mu F_{\mu\nu} = 0$
$\nabla_{[\mu} F_{\nu\rho]} = 0$
iff
$\nabla^\mu \tilde{F}_{\mu\nu} = 0$
$\nabla_{[\mu} \tilde{F}_{\nu\rho]} = 0$
nsor:
$\rho\sigma$
ation is flipped)



$\begin{cases} \nabla^\mu F_{\mu\nu} = 0 \\ \nabla_{[\mu} F_{\nu\rho]} = 0 \end{cases}$ iff $\begin{cases} \nabla^\mu \tilde{F}_{\mu\nu} = 0 \\ \nabla_{[\mu} \tilde{F}_{\nu\rho]} = 0 \end{cases}$
Finally, the Lorentz force:
$\frac{d\vec{p}}{dt} = q(\vec{E} + \vec{v} \times \vec{B})$ is the space component of
$\dot{p}_\mu = -q F_{\mu\nu} u^\nu$
$\frac{dp_\mu}{dt} = -\frac{1}{\gamma} q F_{\mu\nu} u^\nu \Leftrightarrow$
$d\tau = \frac{dt}{\gamma}$
$\vec{p} = \gamma m v$
is flipped)

Let's go back to 3+1

(II) $\vec{\nabla}\cdot\vec{B} = 0$ & (III) $\vec{\nabla}\times\vec{E} + \frac{1}{c}\frac{\partial \vec{B}}{\partial t} = 0$

↳ $\exists\, \vec{A}$ (not unique!) s.t. $\boxed{\vec{B} = \vec{\nabla}\times\vec{A}}$

plug in (III): $\vec{\nabla}\times\left(\vec{E} + \frac{1}{c}\frac{\partial \vec{A}}{\partial t}\right) = 0$

↳ $\exists\, \varphi$ (not unique) s.t. $\vec{E} + \frac{1}{c}\frac{\partial \vec{A}}{\partial t} = -\vec{\nabla}\varphi$

$$\boxed{\vec{E} = -\vec{\nabla}\varphi - \frac{1}{c}\partial_t \vec{A}}$$

The introduct of $(\vec{A}, \varphi)$ solves II & III at the price of non-uniqueness (i.e. $\vec{A} = \vec{\nabla}\xi$ $\varphi = -\frac{1}{c}\partial_t \xi$)

plug this into I & IV

$$\longrightarrow \begin{cases} \Box \vec{A} = -\frac{4\pi}{c}\vec{j} + \vec{\nabla}\Gamma \\ \Box \varphi = -4\pi\rho - \frac{1}{c}\partial_t \Gamma \end{cases}$$

where: $\Gamma = \vec{\nabla}\cdot\vec{A} + \frac{1}{c}\frac{\partial\varphi}{\partial t}$

$$\Longrightarrow \quad j^\mu = (c\rho, \vec{j})$$

$$A^\mu = (\varphi, \vec{A}) \longrightarrow \boxed{F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu}$$

$$\Gamma = \nabla_\mu A^\mu$$

Maxwell: $\boxed{\Box A^\mu = -\frac{4\pi}{c} j^\mu + \nabla^\mu \Gamma}$

$= \nabla_\mu A_\nu - \nabla_\nu A_\mu$

$\frac{\pi}{c} j^\mu + \nabla^\mu \Gamma$

gauge redundancy:

$$A_\mu \longmapsto A_\mu + \nabla_\mu \xi$$

$$F_{\mu\nu} \longmapsto F_{\mu\nu}$$

$$\Gamma \longmapsto \Gamma + \Box \xi$$

Given A_μ choose ξ_A so that $\Box \xi_A + \Gamma = 0$

$\rightarrow$ Define $A^{(L)}_\mu \equiv A_\mu + \nabla_\mu \xi_A$, $\Gamma(A^{(L)}) \equiv 0$

& Maxwell: $\boxed{\Box A^{(L)}_\mu = -\frac{4\pi}{c} j^\mu}$

("Lorentz gauge")